Quantum Ather

Volume IV:Spectral Particle Physics: From Fiber Geometry to the Standard Model.

Emergent Gauge Fields, Anomalies, RG, Flavor, and Precision Tests

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Part I

Foundations of Emergent Gauge Symmetries

Chapter 1

Internal Fiber Geometry and Gauge Isometries

1.1 The internal spectral fiber $\mathcal{F}$

Purpose and setting. Within the Quantum–Aether framework (Vol. I), the macroscopic kinematics and causal structure are encoded by the emergent effective geometry (M, g_{eff}) together with the spectral medium fields

$$\mathcal{S}(x) = \{\Phi(x),\, \rho(x),\, \eta(x),\, \chi^{\mu}(x),\, T(x)\}.$$

To organize *internal* labels and interactions of particle physics, we endow each spacetime point $x \in M$ with internal spectral coordinates $y^A(x)$ valued in a smooth, connected Riemannian manifold $(\mathcal{F}, h)$ that we call the *internal spectral fiber*. This section specifies the geometric data, derives the unique two-derivative Lagrangian compatible with reparametrizations of $\mathcal{F}$ and diffeomorphisms of M, and shows how local isometries of $(\mathcal{F}, h)$ *force* the emergence of a gauge connection.

Definition and regularity. We fix a complete Riemannian manifold

$$(\mathcal{F},\, h_{AB}), \qquad A, B = 1, \ldots, \dim \mathcal{F}, \qquad h = h_{AB}(y)\, dy^A \otimes dy^B, \quad h > 0, \tag{1.1}$$

with Levi–Civita connection ∇_A and curvature $R^A{}_{BCD}(h)$. A configuration is a section

$$y :\ (M, g_{\text{eff}}) \ \longrightarrow\ \mathcal{F}, \qquad x \ \mapsto\ y(x), \tag{1.2}$$

assumed to satisfy $y \in H^1_{\text{loc}}(M, \mathcal{F})$ so that spacetime gradients are square-integrable with respect to g_{eff}.

Leading two-derivative sigma-model. Reparametrizations $y \mapsto \tilde{y}(y)$ and diffeomorphisms of M single out the unique two-derivative scalar

$$\mathcal{L}^{(2)}_{\text{fiber}} \ =\ \frac{\kappa_y(\mathcal{S})}{2}\, g^{\mu\nu}_{\text{eff}}(x)\, h_{AB}\Big(y(x)\Big)\, \partial_\mu y^A\, \partial_\nu y^B \ -\ V_{\text{int}}(y; \mathcal{S}), \tag{1.3}$$

where $\kappa_y > 0$ is a (state-dependent) spectral stiffness and V_{int} a slow potential induced by medium couplings (e.g. ρ, η, Φ). With $\hbar = c = 1$ we have $[\kappa_y] = \text{mass}^2$, $[V_{\text{int}}] = \text{mass}^4$, while y^A and h_{AB} are dimensionless.

Euler–Lagrange equations (global G). Varying (1.3) w.r.t. y^A yields the harmonic/wave-map equation into $(\mathcal{F}, h)$:

$$\kappa_y \,\Box_{g_{\rm eff}} y^A \;+\; \kappa_y \,\Gamma^A{}_{BC}(y)\, g_{\rm eff}^{\mu\nu}\,\partial_\mu y^B\,\partial_\nu y^C \;+\; h^{AB}\,\partial_B V_{\rm int}(y;\mathcal{S}) \;=\; 0, \tag{1.4}$$

where $\Gamma^A{}_{BC}$ are Christoffel symbols of h, and $\Box_{g_{\rm eff}} = \nabla_\mu \nabla^\mu$ uses $g_{\rm eff}$.

Isometries and Killing generators. Let $G \subset \mathrm{Iso}(\mathcal{F}, h)$ be a (compact) Lie group acting by isometries $y \mapsto g\cdot y$ such that $g^* h = h$. Denote by $\{\xi_I\}$ the Killing fields generating the action:

$$\mathcal{L}_{\xi_I} h_{AB} = 0, \qquad [\xi_I, \xi_J] = f^K{}_{IJ}\,\xi_K, \qquad \xi_I = \xi_I^A(y)\,\frac{\partial}{\partial y^A}, \tag{1.5}$$

with structure constants $f^K{}_{IJ}$ of $\mathfrak{g} = \mathrm{Lie}(G)$. Global transformations $\delta y^A = \epsilon^I \xi_I^A$ leave (1.3) invariant and generate conserved Noether currents

$$J_I^\mu \;=\; \kappa_y\, h_{AB}\,\xi_I^A(y)\,\partial^\mu y^B, \qquad \nabla_\mu J_I^\mu = 0 \;\;\text{(on shell)}. \tag{1.6}$$

From global to local: necessity of a connection. Promoting $\epsilon^I \to \epsilon^I(x)$ produces the variation

$$\delta(\partial_\mu y^A) = \partial_\mu(\epsilon^I \xi_I^A) = (\partial_\mu \epsilon^I)\,\xi_I^A \;+\; \epsilon^I\,(\partial_B \xi_I^A)\,\partial_\mu y^B, \tag{1.7}$$

$$\Rightarrow\;\; \delta\mathcal{L}^{(2)}_{\rm fiber}\Big|_{\rm kin} = \kappa_y\, g_{\rm eff}^{\mu\nu}\, h_{AB}\,\partial_\mu y^A\,\xi_I^B\,\partial_\nu \epsilon^I \;+\; \cdots, \tag{1.8}$$

whose inhomogeneous term $\propto \partial_\nu \epsilon^I$ cannot be cancelled by δy alone. *Local* invariance therefore *forces* the introduction of a compensating $\mathfrak{g}$-valued one-form $\mathcal{A}_\mu = \mathcal{A}_\mu^I T_I$ with

$$\delta\mathcal{A}_\mu \;=\; -\,\partial_\mu \epsilon^I\, T_I \;+\; [\,\epsilon^J T_J,\; \mathcal{A}_\mu\,], \tag{1.9}$$

and the replacement

$$\partial_\mu y^A \;\longrightarrow\; \mathcal{D}_\mu y^A \;\equiv\; \partial_\mu y^A \;-\; \xi_I^A(y)\,\mathcal{A}_\mu^I. \tag{1.10}$$

A direct computation using $\delta y^A = \epsilon^I \xi_I^A$ and (1.9) gives

$$\delta(\mathcal{D}_\mu y^A) = \epsilon^I\,\partial_B \xi_I^A\,\mathcal{D}_\mu y^B, \tag{1.11}$$

i.e. $\mathcal{D}_\mu y^A$ transforms as a fiber vector, and the *unique* two-derivative, local, invariant Lagrangian becomes

$$\mathcal{L}^{(2)}_{\rm int} \;=\; \frac{\kappa_y(\mathcal{S})}{2}\, g_{\rm eff}^{\mu\nu}\, h_{AB}\,\mathcal{D}_\mu y^A\,\mathcal{D}_\nu y^B \;-\; V_{\rm int}(y;\mathcal{S}). \tag{1.12}$$

Curvature and Yang–Mills dynamics. Define the curvature (field strength)

$$\mathcal{F}_{\mu\nu} \;\equiv\; \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \;=\; \mathcal{F}_{\mu\nu}^I\, T_I, \tag{1.13}$$

and include the gauge-kinetic term

$$\mathcal{L}_{\rm YM} \;=\; -\frac{\kappa_G(\mathcal{S})}{4}\, g_{\rm eff}^{\mu\alpha}\, g_{\rm eff}^{\nu\beta}\,\mathrm{Tr}[\mathcal{F}_{\mu\nu}\,\mathcal{F}_{\alpha\beta}]\,, \qquad \kappa_G > 0, \tag{1.14}$$

which is generated radiatively (Sec. 1.2) by integrating out fast fiber modes or via quadratic χ^μ-response.

Bundle viewpoint. The data $(M, \mathcal{F}, G)$ define a principal G-bundle $\pi : P \to M$ and an associated bundle

$$E \;=\; P \times_G \mathcal{F} \;\longrightarrow\; M, \tag{1.15}$$

whose sections correspond to $y(x)$ modulo G-equivalence. A local section $\sigma : U \to P$ pulls back a principal connection $\omega \in \Omega^1(P, \mathfrak{g})$ to the spacetime gauge field

$$\mathcal{A} \;\equiv\; \sigma^*\omega \in \Omega^1(U, \mathfrak{g}), \qquad \mathcal{F} \;=\; \sigma^*(d\omega + \omega \wedge \omega)\,, \tag{1.16}$$

and the covariant derivative (1.10) is the standard horizontal lift of dy with respect to ω.

Equations of motion and currents (local G). Varying $\int \sqrt{-g_{\rm eff}}(\mathcal{L}^{(2)}_{\rm int} + \mathcal{L}_{\rm YM})$ w.r.t. y^A and $\mathcal{A}^I_\mu$ gives

$$\kappa_y \,\Box_{g_{\rm eff}} y^A \;+\; \kappa_y\, \Gamma^A{}_{BC}\, g^{\mu\nu}_{\rm eff}\, \mathcal{D}_\mu y^B\, \mathcal{D}_\nu y^C \;+\; h^{AB}\, \partial_B V_{\rm int} \;-\; \kappa_y\, h_{BC}\, \partial_D \xi^C_I\, \mathcal{D}_\mu y^B\, \mathcal{D}^\mu y^D\, \Xi^{AI} = 0, \tag{1.17}$$

$$\kappa_G\, D_\nu \mathcal{F}^{\nu\mu}_I \;=\; J^\mu_I \;+\; J^\mu_{I,\rm matter}, \qquad J^\mu_I \;=\; \kappa_y\, h_{AB}\, \xi^A_I(y)\, \mathcal{D}^\mu y^B, \tag{1.18}$$

with D_μ the adjoint covariant derivative and Ξ^{AI} defined by $\xi^A_I = \Xi^{AI}$. Gauge invariance implies $D_\mu J^\mu_I = 0$ on shell. The matter current $J^\mu_{I,\rm matter}$ accounts for composite sectors built from $\mathcal{S}$.

Consistency gates (S/G/TQ/D). The fiber sector must satisfy the universal gates:

(S) Stability/ghost-free: $\kappa_y > 0,\ \kappa_G > 0,\ h_{AB} > 0,$ (1.19)

(G) Hyperbolicity/causality: $\kappa_y\, h_{AB}\, g^{\mu\nu}_{\rm eff} k_\mu k_\nu > 0$ for timelike k, and YM principal symbol respec (1.20)

(TQ) Forward positivity: $c_{(yy)^2},\ c_{F^4} \;\geq 0$ in the forward limit, cf. (1.23), (1.21)

(D) Dissipation: $\eta > 0 \;\Rightarrow\; \sigma_{\rm diss} \geq 0$ with no CPT-odd linear-time terms at leading order. (1.22)

Here

$$\Delta\mathcal{L} \;\supset\; c_{(yy)^2}\Big(h_{AB}\mathcal{D}_\mu y^A \mathcal{D}_\nu y^B\Big)^2 \;+\; c_{F^4}\,{\rm Tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\Big)^2 \;+\; \cdots\,, \qquad c_{(yy)^2}, c_{F^4} \geq 0. \tag{1.23}$$

Couplings to the director χ^μ. Anisotropic but gauge-invariant kinetic mixings consistent with $g_{\rm eff}$ and parity are

$$\Delta\mathcal{L}_\chi \;=\; -\frac{1}{4}\,\delta\kappa_{IJ}{}^{\mu\nu}(\chi;\mathcal{S})\,\mathcal{F}^I_{\mu\alpha}\,\mathcal{F}^J{}_\nu{}^\alpha, \qquad \delta\kappa_{IJ}{}^{\mu\nu}\,u_\mu u_\nu = 0, \tag{1.24}$$

with u^μ the aether rest-frame. Late-time bounds (multi-messenger c_T, clocks) require $|\delta\kappa| \ll 1$ (Parts VI,VII).

Canonical examples.

- *Abelian circle:* $\mathcal{F} = S^1$, $h = d\theta^2$, $\xi = \partial_\theta$ gives
$$\mathcal{D}_\mu\theta = \partial_\mu\theta - \mathcal{A}_\mu, \qquad \mathcal{L}^{(2)} = \tfrac{\kappa_y}{2}\, g^{\mu\nu}_{\rm eff}(\partial_\mu\theta - \mathcal{A}_\mu)(\partial_\nu\theta - \mathcal{A}_\nu) - \tfrac{\kappa_G}{4} F_{\mu\nu}F^{\mu\nu}.$$
- *Three-sphere:* $\mathcal{F} = S^3 \simeq SU(2)$ with left-invariant frame; $[\xi_I, \xi_J] = \epsilon_{IJK}\xi_K$ foreshadows an $SU(2)$ gauge factor.
- *Product fibers:* $\mathcal{F} = \mathcal{F}_{\rm color} \times \mathcal{F}_{\rm weak} \times \mathcal{F}_{\rm hyper}$ with ${\rm Iso}(\mathcal{F}) \supset SU(3) \times SU(2) \times U(1)$ seed the SM group; small deformations of h encode running/mixings (Ch. 3).

Summary. The internal spectral fiber $(\mathcal{F}, h)$, its isometry group G, and the section $y(x)$ provide the geometric substrate from which *local* gauge structure emerges. Local isometries compel the covariantization (1.10) and the gauge dynamics (1.14); fiber Noether currents source the gauge fields via (1.18); and S/G/TQ/D gates (1.19)–(1.22) constrain the admissible parameter space. In Sec. 1.2 we construct the principal connection from spectral data and prove the invariance/uniqueness of (1.12).

1.2 Isometries, connections, and emergent gauge fields

Goal. We promote the global isometries of the internal spectral fiber $(\mathcal{F}, h)$ to *local* redundancies and show that this uniquely introduces a $\mathfrak{g}$-valued connection $\mathcal{A}_\mu$ on a principal G-bundle $P \to M$. We then construct $\mathcal{A}_\mu$ from fiber data, prove covariance of the fiber-covariant derivative, derive the curvature, Bianchi identities, and Ward identities, and exhibit two microscopic routes by which a Yang–Mills kinetic term $\propto \operatorname{Tr} \mathcal{F}^2$ is generated in the Quantum–Aether medium.

1. From local isometries to a unique compensating field

Noether gauging lemma (minimal coupling). Let $G \subset \mathrm{Iso}(\mathcal{F}, h)$ act on y^A via Killing fields $\xi_I = \xi_I^A(y)\partial_A$ as in (1.5). Consider local parameters $\epsilon^I(x)$ with

$$\delta_\epsilon y^A(x) = \epsilon^I(x)\, \xi_I^A\Big(y(x)\Big). \tag{1.25}$$

Then any two-derivative, local Lagrangian density $\mathcal{L}(g_{\mathrm{eff}}, h; y, \partial y)$ invariant under *global* G can be made invariant under the *local* transformation (1.25) *iff* there exists a $\mathfrak{g}$-valued one-form $\mathcal{A}_\mu = \mathcal{A}_\mu^I T_I$ transforming as

$$\delta_\epsilon \mathcal{A}_\mu \;=\; -\,\partial_\mu \epsilon^I\, T_I \;+\; [\,\epsilon^J T_J,\; \mathcal{A}_\mu\,], \tag{1.26}$$

such that $\partial_\mu y^A$ is replaced by the *fiber-covariant* derivative

$$\mathcal{D}_\mu y^A \;:=\; \partial_\mu y^A \;-\; \xi_I^A(y)\, \mathcal{A}_\mu^I. \tag{1.27}$$

Proof (sketch). Under (1.25) the variation of any kinetic term built from $h_{AB}\partial_\mu y^A \partial_\nu y^B$ contains an inhomogeneous piece $\propto \partial_\mu \epsilon^I$; introducing (1.26) and (1.27) cancels precisely these pieces and yields $\delta_\epsilon(\mathcal{D}_\mu y^A) = \epsilon^I \partial_B \xi_I^A\, \mathcal{D}_\mu y^B$, i.e. tensorial transformation on the fiber. Uniqueness at two derivatives follows from locality and dimension counting: any other admissible term differs by a total derivative or higher-derivative operators. □

Curvature, Bianchi identity, and covariance. Define the curvature

$$\mathcal{F}_{\mu\nu} \;=\; \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \;=\; \mathcal{F}_{\mu\nu}^I T_I, \tag{1.28}$$

which transforms covariantly, $\delta_\epsilon \mathcal{F}_{\mu\nu} = [\epsilon, \mathcal{F}_{\mu\nu}]$, and obeys the Bianchi identity

$$D_{[\mu}\mathcal{F}_{\nu\rho]} \;=\; 0, \qquad D_\mu = \partial_\mu + [\mathcal{A}_\mu,\; \cdot\,], \tag{1.29}$$

with D_μ the adjoint covariant derivative.

2. Bundle construction and a canonical fiber-induced connection

Principal bundle language. The triplet $(M, \mathcal{F}, G)$ defines a principal G-bundle $P \to M$ and an associated bundle $E = P \times_G \mathcal{F} \to M$ of which $y : M \to E$ is a section. An Ehresmann connection is a $\mathfrak{g}$-valued one-form ω on P splitting $TP = H \oplus V$ into horizontal/vertical subspaces; a local section $\sigma : U \to P$ pulls back $\mathcal{A} = \sigma^*\omega$, reproducing (1.27).

Horizontal lift and orthogonality projector. Let $\mathcal{O}_y := \{\xi_I(y)\}$ denote the tangent space to the G-orbit through $y \in \mathcal{F}$. The h-orthogonal projector onto $\mathcal{O}_y$ is

$$\Big(\mathbb{P}_{\parallel}\Big)^{A}{}_{B}(y) \;=\; \xi_I^A(y)\,(\mathcal{M}^{-1})^{IJ}(y)\,h_{BC}(y)\,\xi_J^C(y), \qquad \mathcal{M}_{IJ}(y) := h_{AB}(y)\,\xi_I^A(y)\,\xi_J^B(y). \tag{1.30}$$

Demanding *horizontal* fiber gradients,

$$h_{AB}\,\xi_I^A\,\mathcal{D}_\mu y^B \;=\; 0 \qquad (\text{all } I), \tag{1.31}$$

determines a *canonical, pure-gauge* connection $\mathcal{A}_\mu^{\rm pg}(y)$ by solving

$$\mathcal{M}_{IJ}(y)\,\mathcal{A}_\mu^J \;=\; h_{AB}(y)\,\xi_I^A(y)\,\partial_\mu y^B \quad\Rightarrow\quad \boxed{\mathcal{A}_\mu^{\rm pg}(y) \;=\; \Big(\mathcal{M}^{-1}\Big)_{JI}\,h_{AB}\,\xi_I^A\,\partial_\mu y^B\,T^J.} \tag{1.32}$$

By construction $h(\xi, \mathcal{D}y) = 0$, i.e. $\mathcal{D}_\mu y$ is orthogonal to group orbits. The corresponding curvature is *flat*,

$$\mathcal{F}_{\mu\nu}^{\rm pg} \;=\; 0, \tag{1.33}$$

so $\mathcal{A}^{\rm pg}$ is a local pure gauge (a pullback of Maurer–Cartan data on the orbits).

Dynamical decomposition. A general connection can be written as

$$\mathcal{A}_\mu \;=\; \mathcal{A}_\mu^{\rm pg}(y) \;+\; a_\mu, \qquad a_\mu \in \mathfrak{g}, \tag{1.34}$$

where a_μ transforms *covariantly* under local G, $a_\mu \mapsto g\,a_\mu\,g^{-1}$. Then $\mathcal{F}_{\mu\nu} = D^{\rm pg}_{[\mu}a_{\nu]} + a_{[\mu}a_{\nu]}$, with $D^{\rm pg}_\mu = \partial_\mu + [\mathcal{A}^{\rm pg}_\mu, \cdot\,]$. The *horizontal principle* (1.31) emerges dynamically when the variation of the action w.r.t. $\mathcal{A}_\mu$ is dominated by the fiber term (e.g. $\kappa_y \gg \kappa_G\,\partial^2$), while nonzero a_μ carries the physical gauge-boson degrees of freedom controlled by the Yang–Mills sector (Sec. 1.4).

Masslessness in the symmetric phase. Because $\mathcal{A}^{\rm pg}$ is pure gauge, substituting (1.32) into the fiber kinetic term eliminates any explicit quadratic mass for a_μ at leading order; gauge bosons are massless in the symmetric phase. Mass generation, when present, occurs through structural resonances (the Higgs phase; Ch. 8) or via explicit breaking in $V_{\rm int}$.

3. Gauge-invariant action, equations of motion, and Ward identity

Gauge-invariant two-derivative action. Collecting (1.27)–(1.28) with the Yang–Mills term we write

$$\mathcal{L} \;=\; \frac{\kappa_y(\mathcal{S})}{2}\,g_{\rm eff}^{\mu\nu}h_{AB}\,\mathcal{D}_\mu y^A\mathcal{D}_\nu y^B \;-\; V_{\rm int}(y;\mathcal{S}) \;-\; \frac{\kappa_G(\mathcal{S})}{4}\,g_{\rm eff}^{\mu\alpha}\,g_{\rm eff}^{\nu\beta}\,\mathrm{Tr}\Big(\mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}\Big). \tag{1.35}$$

Variation w.r.t. y^A and $\mathcal{A}_\mu$ yields

$$\kappa_y \,\Box_{g_{\rm eff}} y^A + \kappa_y \,\Gamma^A{}_{BC}\, g_{\rm eff}^{\mu\nu}\, \mathcal{D}_\mu y^B \mathcal{D}_\nu y^C + h^{AB}\,\partial_B V_{\rm int} - \kappa_y\, h_{BC}\,\partial_D \xi_I^C\, \mathcal{D}_\mu y^B \mathcal{D}^\mu y^D\, \Xi^{AI} = 0, \tag{1.36}$$

$$\kappa_G\, D_\nu \mathcal{F}_I^{\nu\mu} \;=\; J_I^\mu \;+\; J^\mu_{I,\rm matter}, \qquad J_I^\mu \;=\; \kappa_y\, h_{AB}\, \xi_I^A(y)\, \mathcal{D}^\mu y^B, \tag{1.37}$$

as in (1.17)–(1.18). Gauge invariance implies the (off-shell) Ward identity

$$D_\mu\!\left(\frac{\delta S}{\delta \mathcal{A}_\mu^I}\right) \;+\; \xi_I^A(y)\,\frac{\delta S}{\delta y^A} \;=\; 0, \tag{1.38}$$

which will be the starting point of the BRST construction in Ch. 2.

4. Microscopic origin of the Yang–Mills kinetic term

Route I: integrating out fast fiber modes (heat kernel). Split $y^A = y^A_{\rm slow} + \delta y^A$ and integrate out δy with momenta above a spectral threshold $\Lambda_{\rm spec}$, treating $\mathcal{A}_\mu$ as a slowly varying background. The quadratic fluctuation operator is a Laplace-type operator $\Delta_h = -\kappa_y\, g_{\rm eff}^{\mu\nu}\,\mathcal{D}_\mu\mathcal{D}_\nu + \cdots$. The one-loop effective action

$$\Gamma_{\rm 1L} = \frac{i}{2}\,\mathrm{Tr}\,\ln \Delta_h$$

admits a Seeley–DeWitt expansion; the a_2 coefficient generates a gauge kinetic term

$$\Gamma_{\rm 1L} \;\supset\; -\frac{1}{4}\int d^4x\,\sqrt{-g_{\rm eff}}\,\Big[\,\underbrace{\frac{\kappa_y}{24\pi^2}\,\dim(\mathcal{F})\,\ln\frac{\Lambda_{\rm spec}}{\mu}}_{=:\,\kappa_G^{\rm ind}}\,\Big]\,\mathrm{Tr}\big(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big) \;+\; \cdots, \tag{1.39}$$

up to scheme-dependent finite parts and curvature terms of $g_{\rm eff}$. Equation (1.39) shows how κ_G arises from the fiber's spectral stiffness and the density of fast modes.

Route II: spectral Kubo formula from current–current response. Alternatively, couple a classical background $\mathcal{A}_\mu$ to the conserved fiber current J_I^μ and integrate out all microscopic excitations of the medium:

$$\Delta S_{\rm eff}[\mathcal{A}] \;=\; -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\,\mathcal{A}_\mu^I(-q)\,\Pi_{IJ}^{\mu\nu}(q)\,\mathcal{A}_\nu^J(q), \qquad \Pi_{IJ}^{\mu\nu}(q) = i\int d^4x\, e^{iqx}\,\langle T J_I^\mu(x) J_J^\nu(0)\rangle_{\mathcal{S}}. \tag{1.40}$$

Ward identities enforce $q_\mu \Pi_{IJ}^{\mu\nu} = 0$. Expanding for soft q,

$$\Pi_{IJ}^{\mu\nu}(q) \;=\; \kappa_G\,\delta_{IJ}\,(q^2 g_{\rm eff}^{\mu\nu} - q^\mu q^\nu) \;+\; \mathcal{O}(q^4), \qquad \kappa_G \;=\; \frac{1}{6}\,\frac{\partial}{\partial q^2}\Pi^\mu{}_{\mu\, II}(q)\Big|_{q^2\to 0}, \tag{1.41}$$

reproducing a local $\mathrm{Tr}\,\mathcal{F}^2$ term with positive coefficient $\kappa_G \geq 0$ by spectral positivity. The medium fields $\mathcal{S}$ determine $\kappa_G(\mathcal{S})$ (running with ρ, alignment with χ^μ, etc.).

5. Examples and normalizations

Example A: $U(1)$ circle. For $\mathcal{F} = S^1$, $\theta \sim \theta + 2\pi$, $\xi = \partial_\theta$, the canonical pure-gauge connection is $\mathcal{A}_\mu^{\rm pg} = \partial_\mu\theta$, so $\mathcal{F}_{\mu\nu}^{\rm pg} = 0$ and $\mathcal{A}_\mu = \partial_\mu\theta + a_\mu$. In unitary gauge $\theta = 0$ the fiber kinetic term carries no mass for a_μ, while (1.39) or (1.41) sets the Maxwell coupling $g^{-2} \equiv \kappa_G$.

Example B: $SU(2)$ three-sphere. For $\mathcal{F} = S^3 \simeq SU(2)$ with left-invariant 1-forms L^I ($dL^I + \frac{1}{2}\epsilon^I{}_{JK} L^J \wedge L^K = 0$) and Killing fields $[\xi_I, \xi_J] = \epsilon_{IJK}\xi_K$, the matrix $\mathcal{M}_{IJ} = h_{AB}\xi_I^A \xi_J^B$ provides a canonical normalization of the weak coupling,

$$\frac{1}{g^2} \;=\; \kappa_G \mathcal{N}_{\text{weak}}, \qquad \mathcal{N}_{\text{weak}} := \frac{1}{3}\delta^{IJ} \left\langle h_{AB}\, \xi_I^A \xi_J^B \right\rangle_{\mathcal{F}}. \tag{1.42}$$

The decomposition (1.34) then reads $\mathcal{A}_\mu = (\mathcal{M}^{-1})_{JI}\, h_{AB}\, \xi_I^A \partial_\mu y^B\, T^J + a_\mu$.

6. Consistency gates specialized to the gauge sector

(S) Stability. $\kappa_G(\mathcal{S}) > 0$, $\kappa_y > 0$, and $h_{AB} > 0$ ensure positivity of the quadratic Hamiltonian and ghost-freedom.

(G) Hyperbolicity and causal cones. The principal symbol of the gauge sector is Maxwellian with respect to g_{eff}; any anisotropic corrections (from χ^μ) must satisfy smallness bounds so that the characteristic cones remain inside or coincide with the g_{eff} cone.

(TQ) Forward positivity. Wilson coefficients of induced higher-derivative operators obey forward-limit inequalities; in particular the coefficients of $(\mathcal{F}^2)^2$ and $(\mathcal{D}y)^4$ terms are nonnegative.

(D) Dissipation. Frictional or CP-odd terms linear in time derivatives are forbidden at leading order; entropy production $\sigma_{\text{diss}} \geq 0$ restricts allowed medium couplings to the gauge sector.

7. Outlook

The canonical, fiber-induced connection (1.32) supplies the *geometric* part of the gauge potential, while dynamical transverse excitations a_μ acquire a universal Yang–Mills kinetic term from the microscopic medium via (1.39) or (1.41). The off-shell identity (1.38) prepares the stage for the BRST/FP structure and Ward identities of correlation functions (Ch. 2), and the normalization prescriptions (1.42) will be used in mapping to electroweak parameters in Part III.

1.3 From χ^μ-responses to gauge boson kinematics

Aim. Having established in Sec. 1.2 that local isometries force a gauge connection $\mathcal{A}_\mu$ with curvature $\mathcal{F}_{\mu\nu}$ and a Yang–Mills term, we now determine *how the spectral medium*, through the director field χ^μ and the state variables $\mathcal{S} = \{\Phi, \rho, \eta, \chi^\mu, T\}$, renormalizes and anisotropically deforms the gauge-boson *kinematics*. We give (i) a covariant response-theory derivation of the polarization tensor, (ii) its decomposition with respect to the aether rest frame u^μ and the projected director $\hat{n}^\mu$, (iii) explicit dispersion relations (phase/group speeds, birefringence), and (iv) stability/causality/positivity gates that constrain all coefficients.

1. Spectral polarization and the Kubo map

Current–current response. Couple a classical background $\mathcal{A}_\mu$ to the conserved fiber current

$$J^\mu_I \;=\; \kappa_y\, h_{AB}\, \xi^A_I(y)\, \mathcal{D}^\mu y^B, \tag{1.43}$$

and define the retarded polarization tensor in the aether state $\mathcal{S}$:

$$\Pi^{\mu\nu}_{IJ}(x;\mathcal{S}) \;=\; i\,\theta(x^0)\,\Big\langle \big[J^\mu_I(x),\, J^\nu_J(0)\big]\Big\rangle_{\mathcal{S}}, \qquad \Pi^{\mu\nu}_{IJ}(q) \;=\; \int d^4x\, e^{iq\cdot x}\, \Pi^{\mu\nu}_{IJ}(x;\mathcal{S}). \tag{1.44}$$

Gauge invariance yields the Ward identity

$$q_\mu\, \Pi^{\mu\nu}_{IJ}(q) \;=\; 0 \;=\; \Pi^{\mu\nu}_{IJ}(q)\, q_\nu. \tag{1.45}$$

The quadratic effective action for $\mathcal{A}$ reads

$$S^{(2)}_{\rm eff}[\mathcal{A}] \;=\; -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\, \mathcal{A}^I_\mu(-q)\, \Big[\, \kappa_G\, \delta_{IJ}\, (q^2 g^{\mu\nu}_{\rm eff} - q^\mu q^\nu) \;+\; \Pi^{\mu\nu}_{IJ}(q)\, \Big]\, \mathcal{A}^J_\nu(q), \tag{1.46}$$

where the first bracket is the universal Yang–Mills kernel (Sec. 1.2) and $\Pi^{\mu\nu}_{IJ}$ encodes medium response.

Small-q expansion and locality. Analyticity at $q=0$ (gapped fast modes) implies the covariant expansion

$$\Pi^{\mu\nu}_{IJ}(q) \;=\; \alpha_{IJ}\, q^2 g^{\mu\nu}_{\rm eff} + \beta_{IJ}\, q^\mu q^\nu \;+\; \Xi^{\mu\nu\alpha\beta}_{IJ}\, q_\alpha q_\beta \;+\; \mathcal{O}(q^4), \tag{1.47}$$

with α, β numbers and $\Xi^{\mu\nu\alpha\beta}$ a rank-4 tensor built from $g^{\mu\nu}_{\rm eff}$, the aether four-velocity u^μ (unit, $u_\mu u^\mu = -1$), and the (unit) spatial director $\hat{n}^\mu$ obtained by projecting χ^μ orthogonally to u^μ:

$$\Delta^{\mu\nu} := g^{\mu\nu}_{\rm eff} + u^\mu u^\nu, \qquad \hat{n}^\mu := \frac{\Delta^\mu{}_\nu\, \chi^\nu}{\sqrt{\chi_\alpha \Delta^{\alpha\beta}\chi_\beta}}, \qquad u_\mu \hat{n}^\mu = 0, \quad \hat{n}_\mu \hat{n}^\mu = +1. \tag{1.48}$$

The Ward identity (1.45) fixes $\beta_{IJ} = -\alpha_{IJ}$ and constrains Ξ to be orthogonal to q_μ/q_ν. Thus, to quadratic order in fields and two derivatives in x, all medium effects can be represented by *gauge-invariant* quadratic forms in $\mathcal{F}_{\mu\nu}$.

2. Covariant constitutive tensor and allowed operators

Constitutive tensor. Define the most general CPT-even, quadratic, gauge-invariant Lagrangian consistent with $g_{\rm eff}$, u^μ, and $\hat{n}^\mu$:

$$\mathcal{L}^{(2)}_{\rm gauge} \;=\; -\frac{1}{4}\, Z_{IJ}\, \mathcal{F}^I_{\mu\nu}\mathcal{F}^{J\,\mu\nu} - \frac{1}{4}\, \alpha_{IJ}\, (u^\mu \mathcal{F}^I_{\mu\alpha})(u_\nu \mathcal{F}^{J\,\nu\alpha}) - \frac{1}{4}\, \beta_{IJ}\, (\hat{n}^\mu \mathcal{F}^I_{\mu\alpha})(\hat{n}_\nu \mathcal{F}^{J\,\nu\alpha}) - \frac{1}{4}\, \gamma_{IJ}\, \mathcal{T}^{\mu\nu\alpha\beta}\, \mathcal{F}^I_{\mu\nu}\mathcal{F}^J_{\alpha\beta}, \tag{1.49}$$

where $Z_{IJ} = \kappa_G\, \delta_{IJ} + \delta Z_{IJ}$ and

$$\mathcal{T}^{\mu\nu\alpha\beta} \;:=\; \frac{1}{2}\Big(\hat{n}^\mu\hat{n}^\alpha\Delta^{\nu\beta} - \hat{n}^\mu\hat{n}^\beta\Delta^{\nu\alpha} - \hat{n}^\nu\hat{n}^\alpha\Delta^{\mu\beta} + \hat{n}^\nu\hat{n}^\beta\Delta^{\mu\alpha}\Big), \qquad \Delta^{\mu\nu} = g^{\mu\nu}_{\rm eff} + u^\mu u^\nu. \tag{1.50}$$

The three matrices $(\alpha_{IJ}, \beta_{IJ}, \gamma_{IJ})$ are *even* under P and T, and descend from the low-q limit of $\Pi^{\mu\nu}_{IJ}$; CPT-odd terms (e.g. Chern–Simons-like $u_{[\mu}\hat{n}_{\nu]}\tilde{\mathcal{F}}^{\mu\nu}$) are excluded at leading order by the dissipation/thermo gate and will be discussed when explicit parity breaking is present. All indices are moved with $g_{\rm eff}$.

Rest-frame form (dielectric/magnetic tensors). In the aether rest frame $u^\mu = (1, \mathbf{0})$ and with spatial indices $i, j = 1, 2, 3$, Eq. (1.49) becomes

$$\mathcal{L}^{(2)}_{\text{gauge}}\Big|_{u=(1,0)} \;=\; \frac{1}{2}\, E^i_I\, \varepsilon^{ij}_{IJ}\, E^j_J - \frac{1}{2}\, B^i_I\, \mu^{-1\,ij}_{IJ}\, B^j_J, \tag{1.51}$$

with $E^i_I = \mathcal{F}^I_{0i}$, $B^i_I = \frac{1}{2}\epsilon^{ijk}\mathcal{F}^I_{jk}$, and the constitutive matrices

$$\varepsilon^{ij}_{IJ} = Z_{IJ}\,\delta^{ij} + \alpha_{IJ}\,\delta^{ij} + \beta_{IJ}\,\hat{n}^i\hat{n}^j + \gamma_{IJ}\,(\delta^{ij} - \hat{n}^i\hat{n}^j), \tag{1.52}$$

$$\mu^{-1\,ij}_{IJ} = Z_{IJ}\,\delta^{ij} + \gamma_{IJ}\,(\delta^{ij} - \hat{n}^i\hat{n}^j), \tag{1.53}$$

up to equivalent rearrangements between (α, β, γ) that leave (1.49) invariant.[1]

3. Plane-wave analysis and dispersion relations

Gauge fixing and transverse sector. Impose a covariant R_ξ gauge, $(\partial_\mu \mathcal{A}^\mu)^I = 0$, with ξ finite. For a plane wave $\mathcal{A}^I_\mu(x) = \text{Re}\{\,\varepsilon^I_\mu\, e^{-ik\cdot x}\,\}$ the equations of motion from (1.46)+(1.49) reduce to

$$\Big[\; Z_{IJ}(k^2 g^{\mu\nu}_{\text{eff}} - k^\mu k^\nu) \;+\; \Sigma^{\mu\nu}_{IJ}(k;\mathcal{S})\;\Big]\varepsilon^J_\nu \;=\; 0, \qquad \Sigma^{\mu\nu}_{IJ} \;=\; \frac{\partial^2}{\partial\mathcal{A}^I_\mu\,\partial\mathcal{A}^J_\nu}\int d^4x\,\Delta\mathcal{L}\Big|_{\mathcal{A}\to 0}, \tag{1.54}$$

with $\Delta\mathcal{L} \equiv \mathcal{L}^{(2)}_{\text{gauge}} + (\kappa_G - \frac{1}{4}Z)\,\text{Tr}\,\mathcal{F}^2$. Physical polarizations are transverse to k^μ; longitudinal modes are gauge artifacts. Let θ be the angle between the spatial wavevector $\vec{k}$ and $\hat{\vec{n}}$.

Birefringent transverse speeds (to first order in anisotropy). To leading order in the small anisotropies $\{\alpha, \beta, \gamma, \delta Z\}$, the two transverse polarizations have dispersion

$$\omega^2 \;=\; c^2_{T,\pm}(\theta)\,|\vec{k}|^2 + \mathcal{O}\Big(|\vec{k}|^4/M^2\Big), \qquad c^2_{T,\pm}(\theta) \;=\; 1 + \delta c_\perp + \frac{\delta c_\parallel - \delta c_\perp}{2}\Big(1 \pm \sqrt{1 - \mathcal{E}\,\sin^2\theta}\Big), \tag{1.55}$$

where (in a gauge basis that diagonalizes Z_{IJ})

$$\delta c_\perp \;\equiv\; \frac{\gamma}{Z}, \qquad \delta c_\parallel \;\equiv\; \frac{\beta}{Z}, \qquad \mathcal{E} \;\equiv\; \frac{(\beta-\gamma)^2}{Z^2} \;+\; \mathcal{O}(\alpha/Z), \tag{1.56}$$

with an understood adjoint-index structure (projected onto the propagating eigenmodes). Equation (1.55) displays *birefringence* (+ and −) controlled by the misalignment between $\vec{k}$ and $\hat{\vec{n}}$. In the isotropic limit $\alpha = \beta = \gamma = 0$ both speeds reduce to unity in g_{eff} units.

Group velocity and hyperbolicity. The group velocity $v_g = \partial\omega/\partial|\vec{k}|$ equals $c_{T,\pm}(\theta)$ at this order. The gauge-sector hyperbolicity gate requires

$$0 \;<\; c^2_{T,\pm}(\theta) \;\le 1 \quad \text{for all angles } \theta, \qquad \Longrightarrow \qquad Z > 0,\;\; \gamma > -Z,\;\; \beta > -Z,\;\; |\beta - \gamma| \ll Z. \tag{1.57}$$

In practice, late-time constraints enforce $|c^2_{T,\pm} - 1| \ll 1$; see Parts VI and VII for quantitative bounds.

[1] Equations (1.52)–(1.53) are one convenient parameterization; other equivalent bases are related by linear transformations that preserve (1.49).

4. Group-space mixing and canonical normalization

Kinetic mixing matrix. The constitutive tensors carry adjoint indices. In a generic basis there is kinetic mixing between different gauge factors (and generators). Let

$$\mathbb{Z} \equiv (Z_{IJ}), \quad \boldsymbol{\alpha} \equiv (\alpha_{IJ}), \quad \boldsymbol{\beta} \equiv (\beta_{IJ}), \quad \boldsymbol{\gamma} \equiv (\gamma_{IJ}). \tag{1.58}$$

Define the canonically normalized fields by a positive-definite square root $\mathbb{Z}^{1/2}$:

$$\mathcal{A}'_\mu \;=\; \mathbb{Z}^{1/2}\,\mathcal{A}_\mu, \qquad \mathcal{F}'_{\mu\nu} \;=\; \mathbb{Z}^{1/2}\,\mathcal{F}_{\mu\nu}\,\mathbb{Z}^{1/2}, \tag{1.59}$$

which maps $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) \to (\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}})$ by conjugation with $\mathbb{Z}^{-1/2}$. Dispersion is then governed by the eigenvalues of $(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\gamma}})$ in the propagating subspace. This is the appropriate starting point for electroweak mixing (Part III).

5. Positivity, sum rules, and dissipation

Spectral positivity and Kramers–Kronig. In frequency space the retarded kernel obeys

$$\mathrm{Im}\;\Pi^{\mu\nu}_{IJ}(\omega,\vec{q}) \;\geq\; 0 \quad (\omega > 0), \qquad \Pi^{\mu\nu}_{IJ}(\omega,\vec{q}) = \frac{2}{\pi}\,\mathcal{P}\!\int_0^\infty d\omega'\,\frac{\omega'\;\mathrm{Im}\;\Pi^{\mu\nu}_{IJ}(\omega',\vec{q})}{\omega'^2-\omega^2}, \tag{1.60}$$

so that the low-frequency coefficients (α, β, γ) are *nonnegative* combinations of spectral weights, consistent with the forward-limit bounds of Part VI. Dissipation (finite $\eta > 0$) enters as $\mathrm{Im}\;\Pi \neq 0$ and produces an attenuation length $L_{\mathrm{att}} = (2\;\mathrm{Im}\;\omega)^{-1}$ for the gauge waves; stability requires $\mathrm{Im}\;\omega \leq 0$.

6. Worked example: uniaxial medium aligned with $\hat{n}$

Setup. Assume a uniaxial response with small anisotropy parameters $\delta_\parallel, \delta_\perp$:

$$\varepsilon^{ij}_{IJ} = \delta_{IJ}\Big[\,(1+\delta_\perp)\,(\delta^{ij}-\hat{n}^i\hat{n}^j) + (1+\delta_\parallel)\,\hat{n}^i\hat{n}^j\,\Big], \qquad \mu^{-1\,ij}_{IJ} = \delta_{IJ}\,\Big[\,1+\tilde{\delta}_\perp + (\tilde{\delta}_\parallel - \tilde{\delta}_\perp)\hat{n}^i\hat{n}^j\,\Big]. \tag{1.61}$$

Solving Maxwell's equations in Fourier space for a wave with $\vec{k}$ at an angle θ to $\hat{\vec{n}}$ yields two transverse branches with speeds

$$c^2_{T,1}(\theta) \;=\; \frac{1+\tilde{\delta}_\perp}{1+\delta_\perp} \;+\; \mathcal{O}(\delta^2), \qquad c^2_{T,2}(\theta) \;=\; \frac{1+\tilde{\delta}_\perp\cos^2\theta+\tilde{\delta}_\parallel\sin^2\theta}{1+\delta_\perp\cos^2\theta+\delta_\parallel\sin^2\theta} \;+\; \mathcal{O}(\delta^2), \tag{1.62}$$

reproducing (1.55) upon the identifications $\gamma/Z \leftrightarrow \tilde{\delta}_\perp - \delta_\perp$ and $\beta/Z \leftrightarrow \tilde{\delta}_\parallel - \delta_\parallel$.

7. Gates specialized to χ^μ-deformed kinematics

(S) Stability. Positive energy density demands ε^{ij}_{IJ} positive-definite and $\mu^{-1\,ij}_{IJ}$ positive-definite:

$$\forall\,\vec{E},\vec{B} \;\neq 0: \quad E^i_I\,\varepsilon^{ij}_{IJ}\,E^j_J > 0, \qquad B^i_I\,\mu^{-1\,ij}_{IJ}\,B^j_J > 0. \tag{1.63}$$

(G) Causality/hyperbolicity. Characteristic cones of the gauge sector must not exceed the g_{eff} cone:

$$0 < c^2_{T,\pm}(\theta) \leq 1 \;\text{ for all }\theta \quad\Rightarrow\quad \Big\|\alpha,\beta,\gamma,\delta Z\Big\| \;\ll Z. \tag{1.64}$$

(TQ) Forward positivity. Coefficients of $(\mathcal{F}^2)^2$, $(\mathcal{D}y)^4$, and mixed terms generated upon integrating out fast modes must satisfy the forward-limit inequalities of Part VI; in particular the quadratic corrections in (1.49) arise from *nonnegative* spectral integrals.

(D) Dissipation. Entropy production $\sigma_{\rm diss} \geq 0$ bounds any CP-odd transport; at two derivatives, CPT-odd, linear-in-time terms are excluded. Attenuation is allowed but small: $|\,\mathrm{Im}\,\omega| \ll |\,\mathrm{Re}\,\omega|$ in all regimes probed so far.

8. Summary and interface to later parts

The director field χ^μ organizes the medium's *anisotropic* response and imprints controlled, gauge-invariant deformations of the gauge-boson kinematics via the constitutive tensors (ε, μ^{-1}). At low momenta, the entire effect is captured by the CPT-even operators of (1.49), which lead to birefringent but hyperbolic wave propagation with speeds (1.55) constrained by the S/G/TQ/D gates. Group-space kinetic mixing is canonically removed by (1.59) and will be crucial for the electroweak sector (Part III). The quantitative values of $(Z, \alpha, \beta, \gamma)$ are computable from current–current spectra via the Kubo map (**??**)–(1.60), closing the circle between microscopic spectral physics and macroscopic gauge-boson propagation.

1.4 Spectral covariant derivative and field strengths

Objective. We formalize the *spectral covariant derivative* that acts on fields valued in the internal fiber $(\mathcal{F}, h)$ and on matter in arbitrary G-representations, derive its commutators, and relate all curvatures to (i) the Riemann tensor of $(\mathcal{F}, h)$ and (ii) the gauge field strength $\mathcal{F}_{\mu\nu}$ introduced in Sec. 1.2. This provides the differential-calculus backbone for Ward/BRST identities (Ch. 2) and for the RG/phenomenology mapping (Chs. 3, 9).

1. Conventions and basic objects

Indices and metrics. Greek indices $\mu, \nu, \ldots$ live on spacetime $(M, g_{\rm eff})$; capital Latin $A, B, \ldots$ on the fiber $(\mathcal{F}, h)$; adjoint indices $I, J, \ldots$ for $\mathfrak{g} = \mathrm{Lie}(G)$. We employ the covariant derivative

$$\nabla_\mu \text{ built from } g_{\rm eff}, \qquad \nabla_A \text{ (Levi–Civita) built from } h_{AB}, \qquad D_\mu = \partial_\mu + [\mathcal{A}_\mu, \cdot\,] \text{ on adjoint tensors.} \tag{1.65}$$

The Killing fields $\xi_I = \xi_I^A(y)\,\partial_A$ generate the G-action on $\mathcal{F}$ and satisfy (1.5). The fiber-covariant derivative acting on the *map* $y : M \to \mathcal{F}$ is

$$\mathcal{D}_\mu y^A \;=\; \partial_\mu y^A \;-\; \xi_I^A(y)\,\mathcal{A}_\mu^I, \tag{1.66}$$

as in (1.27), and transforms tensorially under local G.

2. Horizontal differential on the pullback bundle

Horizontal derivative d_H. Let $E = P \times_G \mathcal{F} \to M$ be the associated bundle and $y : M \to E$ a section. For any smooth function $f : \mathcal{F} \to \mathbb{R}$, define the *horizontal* exterior derivative along y by

$$(d_H f)(x) \;:=\; \Big(\partial_\mu f\big(y(x)\big) - \mathcal{A}_\mu^I(x)\,\mathcal{L}_{\xi_I} f\big(y(x)\big)\Big)\,dx^\mu \;=\; \partial_A f(y)\,\mathcal{D}_\mu y^A\,dx^\mu, \tag{1.67}$$

where $\mathcal{L}_{\xi_I}$ is the Lie derivative on $\mathcal{F}$. Equation (1.67) shows that d_H is the pullback of the Ehresmann horizontal differential induced by the principal connection ω with $\mathcal{A} = \sigma^*\omega$.

Curvature as obstruction to horizontality. Acting twice,

$$d_H^2 f \;=\; -\frac{1}{2}\,\mathcal{F}^I_{\mu\nu}\,(\mathcal{L}_{\xi_I} f)\,dx^\mu \wedge dx^\nu, \tag{1.68}$$

so the gauge curvature $\mathcal{F}$ measures the failure of horizontal integrability: $d_H^2 = 0$ iff $\mathcal{F} = 0$ (pure gauge).

3. Spectral covariant derivative on fiber tensors

Definition on $T\mathcal{F}$-valued fields. Let $V^A(y)$ be a fiber vector field pulled back along y. The *spectral covariant derivative* combines (i) the pullback of the Levi–Civita connection of h and (ii) the vertical G-action generated by ξ_I:

$$\boxed{\mathbb{D}_\mu V^A \;:=\; \partial_\mu V^A \;+\; \Gamma^A{}_{BC}(y)\,\partial_\mu y^B\,V^C \;-\; (\nabla_B \xi_I^A)(y)\,V^B\,\mathcal{A}^I_\mu\,.} \tag{1.69}$$

Here $\Gamma^A{}_{BC}$ are Christoffels of h, and $\nabla_B \xi_I^A$ is the fiber-covariant derivative of the Killing field. Equation (1.69) ensures that $\mathbb{D}_\mu V^A$ transforms as a fiber vector under local G and as a spacetime vector under diffeomorphisms of M. Lower indices are treated with h_{AB} so that $\mathbb{D}_\mu h_{AB} = 0$.

Action on general fiber tensors. For a (p,q) tensor $T^{A_1\cdots A_p}{}_{B_1\cdots B_q}(y)$,

$$\mathbb{D}_\mu T^{A_1\cdots A_p}{}_{B_1\cdots B_q} = \partial_\mu T^{A_1\cdots A_p}{}_{B_1\cdots B_q} + \sum_{r=1}^{p} \Gamma^{A_r}{}_{CD}\,\partial_\mu y^C\,T^{A_1\cdots D\cdots A_p}{}_{B_1\cdots B_q} - \sum_{s=1}^{q} \Gamma^D{}_{B_s C}\,\partial_\mu y^C\,T^{A_1\cdots A_p}{}_{B_1\cdots D\cdots B_q} - \mathcal{L}^{(\xi)}_{\mathcal{A}_\mu}\,T^{A_1\cdots A_p}{}_{B_1\cdots B_q}, \tag{1.70}$$

where $\mathcal{L}^{(\xi)}_{\mathcal{A}_\mu}$ is the Lie derivative along $\mathcal{A}^I_\mu \xi_I$ acting on fiber indices.

Commutators and curvatures. A standard computation gives, for any $V^A(y)$,

$$[\mathbb{D}_\mu, \mathbb{D}_\nu]\,V^A \;=\; R^A{}_{BCD}(h)\,\partial_\mu y^B\,\partial_\nu y^C\,V^D \;-\; \mathcal{F}^I_{\mu\nu}\,(\nabla_B \xi_I^A)\,V^B, \tag{1.71}$$

and, as a special case, for the coordinates y^A themselves,

$$[\mathbb{D}_\mu, \mathbb{D}_\nu]\,y^A \;=\; -\,\mathcal{F}^I_{\mu\nu}\,\xi_I^A(y). \tag{1.72}$$

Equations (1.71)–(1.72) express that the total curvature experienced by fiber tensors is the sum of the pulled-back Riemann curvature of $(\mathcal{F}, h)$ and the vertical curvature generated by the gauge field.

4. Covariant derivative on matter in a G-representation

Representation data. Let $\psi(x)$ transform in some finite-dimensional (possibly reducible) representation R of G with Hermitian generators $(T_I^{(R)})_r{}^s$, $[T_I^{(R)}, T_J^{(R)}] = i f^K{}_{IJ} T_K^{(R)}$. The spectral covariant derivative on ψ is

$$\mathcal{D}^{(R)}_\mu \psi \;=\; \nabla_\mu \psi \;-\; i\,\mathcal{A}^I_\mu\,T_I^{(R)}\,\psi, \tag{1.73}$$

and the associated field strength in R is $\mathcal{F}^{(R)}_{\mu\nu} = \mathcal{F}^I_{\mu\nu}\,T_I^{(R)}$. The Bianchi identity reads $D_{[\mu}\mathcal{F}_{\nu\rho]} = 0$ in any representation.

Coupled Lagrangians and currents. For complex scalars ϕ and Dirac spinors Ψ in R,

$$\mathcal{L}_\phi = g_{\text{eff}}^{\mu\nu}\,(\mathcal{D}_\mu^{(R)}\phi)^\dagger(\mathcal{D}_\nu^{(R)}\phi) - V(\phi;y;\mathcal{S}), \tag{1.74}$$

$$\mathcal{L}_\Psi = \bar\Psi\, i\gamma^\mu e_\mu{}^a(g_{\text{eff}})\left(\partial_a + \tfrac{1}{4}\omega_{abc}\gamma^{bc} - i\,\mathcal{A}_a^I T_I^{(R)}\right)\Psi, \tag{1.75}$$

with $e_\mu{}^a$ the vierbein and ω_{abc} the spin connection of g_{eff}. Variation w.r.t. $\mathcal{A}_\mu$ yields the total adjoint current

$$J_I^\mu \;=\; \kappa_y\, h_{AB}\,\xi_I^A\,\mathcal{D}^\mu y^B + J_{I,\phi}^\mu + J_{I,\Psi}^\mu, \qquad J_{I,\phi}^\mu = i\,(\mathcal{D}^{(R)\mu}\phi)^\dagger T_I^{(R)}\phi + \text{h.c.}, \quad J_{I,\Psi}^\mu = \bar\Psi\gamma^\mu T_I^{(R)}\Psi. \tag{1.76}$$

The Yang–Mills equation is $\kappa_G D_\nu \mathcal{F}_I^{\nu\mu} = J_I^\mu$ as in (1.37).

5. Energy–momentum tensor and gauge invariance

Stress tensor. Varying the two-derivative action $\mathcal{L}_{\text{int}}^{(2)} + \mathcal{L}_{\text{YM}}$ w.r.t. $g_{\text{eff}}^{\mu\nu}$ gives

$$T_{\mu\nu}^{(y)} = \kappa_y\Big(h_{AB}\,\mathcal{D}_\mu y^A\,\mathcal{D}_\nu y^B - \tfrac{1}{2}\,g_{\text{eff}\,\mu\nu}\,h_{AB}\,\mathcal{D}_\alpha y^A\,\mathcal{D}^\alpha y^B\Big) - g_{\text{eff}\,\mu\nu}\,V_{\text{int}}, \tag{1.77}$$

$$T_{\mu\nu}^{(\mathcal{A})} = \kappa_G\,\mathrm{Tr}\Big(\mathcal{F}_{\mu\alpha}\mathcal{F}_\nu{}^\alpha - \tfrac{1}{4}\,g_{\text{eff}\,\mu\nu}\,\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}\Big). \tag{1.78}$$

Gauge invariance and the equations of motion imply $\nabla^\mu(T_{\mu\nu}^{(y)} + T_{\mu\nu}^{(\mathcal{A})}) = 0$ (up to medium-exchange terms when κ_y, κ_G depend on $\mathcal{S}$ explicitly).

6. Differential identities

Bianchi identities. Besides $D_{[\mu}\mathcal{F}_{\nu\rho]} = 0$, the pulled-back fiber curvature satisfies

$$\nabla_{[\mu}\Big(R^A{}_{BCD}(h)\,\partial_{\nu]}y^B\Big) = R^A{}_{BCD}\,\nabla_{[\mu}\partial_{\nu]}y^B \;=\; -\tfrac{1}{2}\,R^A{}_{BCD}\,\mathcal{F}_{\mu\nu}^I\,\xi_I^B, \tag{1.79}$$

so the total curvature algebra closes on (1.71).

Ward identity (off shell). For the action $S[y,\mathcal{A},\dots]$ built from $\mathbb{D}_\mu$ and $\mathrm{Tr}\,\mathcal{F}^2$,

$$D_\mu\left(\frac{\delta S}{\delta \mathcal{A}_\mu^I}\right) + \xi_I^A(y)\,\frac{\delta S}{\delta y^A} = 0, \tag{1.80}$$

the same as (1.38), to be elevated to BRST language in Ch. 2.

7. Examples and checks

$U(1)$ **fiber.** For $\mathcal{F} = S^1$ with $\xi = \partial_\theta$, $\nabla_B\xi^A = 0$, $R^A{}_{BCD}(h) = 0$, and (1.71) reduces to $[\mathbb{D}_\mu, \mathbb{D}_\nu]V = 0$ on fiber scalars, while $[\mathbb{D}_\mu, \mathbb{D}_\nu]\theta = -F_{\mu\nu}$ as in (1.72).

$SU(2)$ **three-sphere.** For $\mathcal{F} = S^3 \simeq SU(2)$ with left-invariant frame, $\nabla_B\xi_I^A = \epsilon_{IJK}\,E^{JA}E^K{}_B$ (in an orthonormal basis E^{IA}), so the second term in (1.71) is proportional to $\mathcal{F}_{\mu\nu}$ in the adjoint, as expected for non-Abelian transport on group orbits.

8. Summary

The spectral covariant derivative $\mathbb{D}_\mu$ furnishes a unified calculus on the pullback bundle $y^*(T\mathcal{F})$ and on matter representations: it (i) preserves h_{AB}, (ii) collapses to the horizontal differential d_H on fiber scalars, (iii) produces commutators (1.71) that separate fiber Riemann curvature from the gauge curvature $\mathcal{F}_{\mu\nu}$, and (iv) yields conserved currents and a gauge-invariant stress tensor. These structures will be repeatedly invoked in the anomaly and BRST analysis (Part II) and in constructing electroweak/color sectors from fiber isometries (Parts III, IV).

1.5 Spectral calibration and the case for a seven-dimensional fiber

Purpose. We add a final section to Ch. 1 that (i) fixes canonical normalizations and furnishes *predictive anchors* already at the foundational level, and (ii) justifies the *dimensionality* and *field content* of the spectral medium used throughout Vols. I–III: why a *seven*-dimensional internal fiber is singled out, and why the structural set

$$\mathcal{S}(x) = \{\rho(x),\, \eta(x),\, \chi^\mu(x),\, T(x)\} \quad \text{with a phase field } \Phi(x)$$

is both *minimal* and *sufficient* for emergent gauge physics. In doing so we connect the calculus of Secs. 1.1–1.4 to concrete normalizations and to geometric identities unique to seven dimensions.

1. Canonical normalization and coupling maps

Adjoint metric and Killing norms. Let $G \subset \mathrm{Iso}(\mathcal{F}, h)$ act by Killing fields $\xi_I = \xi_I^A \partial_A$. Define the fiber *Killing metric*

$$\mathcal{M}_{IJ}(y) \;:=\; h_{AB}(y)\, \xi_I^A(y)\, \xi_J^B(y), \qquad \langle \mathcal{M}_{IJ} \rangle_{\mathcal{F}} \;:=\; \frac{1}{\mathrm{Vol}(\mathcal{F})} \int_{\mathcal{F}} d\mathrm{vol}_h\, \mathcal{M}_{IJ}(y), \tag{1.81}$$

and choose a basis where $\langle \mathcal{M}_{IJ} \rangle_{\mathcal{F}} = \delta_{IJ}\, \mathcal{N}_I$ is diagonal. The Yang–Mills kinetic term (1.14) then fixes the *canonical couplings* by

$$\frac{1}{g_I^2(\mu)} \;:=\; \kappa_G(\mu; \mathcal{S})\, \mathcal{N}_I. \tag{1.82}$$

For product fibers with $G = G_1 \times G_2 \times \cdots$, the block-diagonal $\langle \mathcal{M} \rangle$ yields the familiar multi-coupling structure. Any small deformation of h induces correlated shifts of the $\mathcal{N}_I$ and hence of $g_I(\mu)$.

Electroweak angle from geometry. For $SU(2)_L \times U(1)_Y$, let the diagonal entries be $\mathcal{N}_2$ and $\mathcal{N}_1$. The weak mixing angle at the normalization scale μ is

$$\tan\theta_W(\mu) \;=\; \frac{g'(\mu)}{g(\mu)} \;=\; \sqrt{\frac{\mathcal{N}_2}{\mathcal{N}_1}}, \qquad \sin^2\theta_W(\mu) \;=\; \frac{1}{1 + \mathcal{N}_1/\mathcal{N}_2}. \tag{1.83}$$

Equation (1.83) is a *geometric* identity (independent of gauge fixing), to be evolved by the spectral RG (Ch. 3) from a high scale down to electroweak scales.

Constitutive tensors and canonical fields. Writing the quadratic gauge Lagrangian as in (1.49), define $\mathbb{Z} \equiv (Z_{IJ})$ and the canonically normalized fields

$$\mathcal{A}'_\mu \;=\; \mathbb{Z}^{1/2}\,\mathcal{A}_\mu, \qquad \mathcal{F}'_{\mu\nu} \;=\; \mathbb{Z}^{1/2}\,\mathcal{F}_{\mu\nu}\,\mathbb{Z}^{1/2}, \tag{1.84}$$

so that isotropic pieces are absorbed into $g_I(\mu)$ while anisotropies (α, β, γ) become small, Hermitian deformations in the primed basis. This is the starting point for precision fits (Part VII).

2. Predictive anchors at leading order

Oblique parameters. At two derivatives the medium only renormalizes transverse propagation; the leading new-physics imprints in the electroweak sector can be organized as

$$S \;\propto\; \frac{\partial}{\partial q^2}\Big[\Pi_{3B}(q^2)\Big]_{q^2=0}, \qquad T \;\propto\; \Pi_{11}(0)-\Pi_{33}(0), \qquad U \;\propto\; \frac{\partial}{\partial q^2}\Big[\Pi_{11}(q^2)-\Pi_{33}(q^2)\Big]_{q^2=0}, \tag{1.85}$$

and translate into linear combinations of (α, β, γ) after the canonical map (1.84). The custodial gate (Ch. 8) constrains $T \approx 0$ at tree level.

Transverse speeds and birefringence. Gauge-boson phase/group speeds obey (1.55) with $|c^2_{T,\pm} - 1| \ll 1$ enforced by hyperbolicity and by late-time bounds (Part VI). This yields immediate *null tests* (vanishing limits) and small correlated deviations.

3. Why $\dim \mathcal{F} = 7$: stable three-forms, cross products, and G_2-structure

Stable 3-forms in seven dimensions. In $\mathbb{R}^7$ there exist *stable* 3-forms φ whose GL(7)-orbit is open; such a φ determines a Riemannian metric h and a volume form via the algebraic identity

$$(i_X\varphi) \wedge (i_Y\varphi) \wedge \varphi \;=\; 6\,h(X,Y)\,\mathrm{vol}_h \qquad (\forall\, X, Y \in T\mathcal{F}), \tag{1.86}$$

and its Hodge dual $*\varphi$. The stabilizer of φ in GL(7) is the exceptional group G_2 (dimension 14). Thus a G_2-*structure* on $\mathcal{F}$ is equivalently specified by $(\varphi, *\varphi)$ and induces h.

Vector cross product and uniqueness in $n = 7$. A bilinear, alternating *vector cross product* $\times : T\mathcal{F} \times T\mathcal{F} \to T\mathcal{F}$ satisfying

$$h(X\times Y, X\times Y) \;=\; h(X,X)\,h(Y,Y)-h(X,Y)^2, \qquad X\times(X\times Y) = -\,h(X,X)\,Y+h(X,Y)\,X, \tag{1.87}$$

exists only for $n = 3$ and $n = 7$. In $n = 7$ it is encoded by φ through $h(X \times Y, Z) = \varphi(X, Y, Z)$. The *spectral* medium requires such a nondegenerate trilinear form to close the response algebra of $(\rho, \eta, \chi^\mu, T)$ consistently with isotropy gates; hence $n = 7$ is the minimal dimension allowing this structure.

Embedding of gauge isometries. G_2 admits $SU(3)$ as a maximal subgroup; the decomposition of G_2 representations under $SU(3)$ naturally supplies the *color* sector seeds, while the remaining weak/hypercharge factors arise from additional isometries or from product fibers with small mixing (Parts III, IV). Crucially, the *manifold* dimension 7 need not equal $\dim(\mathrm{Iso}\,\mathcal{F})$ (e.g. S^n has $\dim \mathrm{Iso} = n(n+1)/2$), permitting rich gauge algebras on a low-dimensional $\mathcal{F}$.

Decomposition of forms (useful identities). On a G_2-manifold,

$$\Lambda^2 \;=\; \Lambda^2_7 \;\oplus\; \Lambda^2_{14}, \qquad \Lambda^3 \;=\; \Lambda^3_1 \;\oplus\; \Lambda^3_7 \;\oplus\; \Lambda^3_{27}, \tag{1.88}$$

where $\Lambda^2_{14} \cong \mathfrak{g}_2$ and $\Lambda^2_7 \cong T\mathcal{F}$. This decomposition matches the split of gauge-curvature 2-forms into "adjoint-like" and "vector-like" pieces under G_2, providing a convenient bookkeeping for anisotropies (Sec. 1.3) and for WZW/CS terms (Part II).

4. Why four structural fields + Φ

Minimality and completeness. The spectral state requires exactly the following slow fields to parameterize kinematics, dissipation, and anisotropy at leading order:

$$\text{density } \rho(x), \quad \text{viscous modulus } \eta(x), \quad \text{director } \chi^\mu(x) \text{ (unit, time-like projection fixed by } u^\mu), \quad \text{intern} \tag{1.89}$$

supplemented by a *phase* $\Phi(x)$ that encodes the spectral order (Vol. I). This set is *minimal*: removing any element violates at least one gate (S/G/TQ/D) or spoils covariance of the constitutive response; it is *sufficient*: higher-order responses can be expanded in their gradients.

Constructing a G_2-compatible 3-form. Project χ^μ onto $\mathcal{F}$ through the section y and use the pullback frame $\{e^A\}$ to form a normalized 3-form

$$\varphi \;:=\; c_1(\rho,\eta,T)\; e^{123} \;+\; c_2(\rho,\eta,T) \sum_{\mathrm{cyc}} e^1 \wedge (e^2 \wedge e^3 - e^4 \wedge e^5 - e^6 \wedge e^7) \;+\; \cdots, \tag{1.90}$$

with coefficients $c_{1,2} > 0$ fixed by (ρ,η,T) and the ellipsis denoting the unique G_2-invariant completion aligned with the projected director. The stability conditions $\varphi \wedge *\varphi > 0$ and (1.86) then determine h and select a G_2-structure compatible with the medium.

Phase field and emergent $U(1)$. The phase Φ arises as the argument of a complex order parameter $\Psi = f(\rho,\eta,T)\, e^{i\Phi}$ built from the structural set. Local rephasings $\Phi \to \Phi + \alpha(x)$ source an emergent $U(1)$ (hypercharge seed) via the minimal replacement $\partial_\mu \Phi \to \partial_\mu \Phi - \mathcal{B}_\mu$. In this sense, Φ is *built on* (ρ,η,χ^μ,T) and closes the set needed for Abelian and non-Abelian gauge emergence.

5. Gates and falsifiability of the 7D choice

Stability and hyperbolicity. The G_2-compatible φ enforces positivity of h and controls anisotropies so that $c^2_{T,\pm} \le 1$. Violations (e.g. loss of stability of φ) signal breakdown of the effective description.

Quantum/analyticity gates. Forward-limit bounds (Part VI) applied to the $(\mathcal{F}^2)^2$ sector tie (α, β, γ) to nonnegative spectral densities. If a lower-dimensional $\mathcal{F}$ is adopted, the required trilinear structure fails and the same bounds cannot be satisfied simultaneously with isotropy, providing a *kill criterion* against $n \neq 7$.

Phenomenological anchors. Eqs. (1.82)–(1.83) and the birefringence bounds from Sec. 1.3 furnish immediate, globally fit-able relations. Any attempt to replace $(\rho, \eta, \chi^\mu, T)$ by a smaller set generically violates at least one anchor or gate.

6. Summary

We have (i) fixed canonical normalizations of gauge fields and couplings from fiber geometry, (ii) supplied predictive anchors already at leading order, and (iii) justified the *seven*-dimensional choice for the internal fiber via stable 3-forms and the unique vector cross product in $n = 7$, naturally aligning with a G_2-structure. The structural set $\{\rho, \eta, \chi^\mu, T\}$ with a phase Φ is both minimal and sufficient for emergent gauge dynamics. This closes the foundational chapter and interfaces cleanly with anomalies/BRST (Part II) and with electroweak/color embeddings (Parts III, IV).

Chapter 2

Ward Identities, BRST, and FP Ghosts as Redundancies

2.1 Spectral origin of Ward identities

Scope. In Secs. 1.1–1.4 we established that local isometries of the internal fiber $(\mathcal{F}, h)$ necessitate a G-connection $\mathcal{A}_\mu$ and yield the off-shell classical identity

$$D_\mu\left(\frac{\delta S}{\delta \mathcal{A}_\mu^I}\right) \;+\; \xi_I^A(y)\,\frac{\delta S}{\delta y^A} \;+\; i\left(\frac{\delta S}{\delta \psi}\,T_I\psi - \bar\psi\,T_I\,\frac{\delta S}{\delta \bar\psi}\right) = 0, \tag{2.1}$$

for any action $S[y, \mathcal{A}, \phi, \psi; \mathcal{S}]$ built from the spectral covariant calculus. The aim here is to *derive the full quantum Ward identities*—for connected and 1PI functionals—directly from the *spectral* origin of gauge redundancy, clarify their background-field form, and list immediate corollaries (transversality, charge conservation, vertex constraints). Anomalous terms (measure Jacobians) will be handled systematically in Part II.

1. Spectral gauge transformations and Noether identity

Local isometries as field redefinitions. Under a local G-reparametrization with parameters $\epsilon^I(x)$,

$$\delta_\epsilon y^A = \epsilon^I \xi_I^A(y), \qquad \delta_\epsilon \mathcal{A}_\mu = -D_\mu \epsilon \equiv -\partial_\mu \epsilon - [\mathcal{A}_\mu, \epsilon], \qquad \delta_\epsilon \psi = i\,\epsilon^I T_I \psi, \tag{2.2}$$

and similarly for complex scalars ϕ in a representation R. Invariance of the *classical* Lagrangian under (2.2) yields (2.1), which encodes the Noether current conservation $D_\mu J_I^\mu = 0$ on shell.

2. Path integral and connected-functional Ward identity

Generating functional. Introduce sources J_I^μ for $\mathcal{A}_\mu^I$, K_A for y^A, $J_\phi, J_\phi^\dagger$ for $\phi, \phi^\dagger$, and Grassmann sources $\bar\eta, \eta$ for $\psi, \bar\psi$. With the spectral state $\mathcal{S} = \{\Phi, \rho, \eta, \chi^\mu, T\}$ treated as background,

$$Z[J, K, \dots] \;=\; \int \mathcal{D}\mathcal{A}\,\mathcal{D}y\,\mathcal{D}\phi\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\;\exp\Big\{\, iS[y, \mathcal{A}, \phi, \psi; \mathcal{S}] + i\int d^4x\,\Big(J_I^\mu \mathcal{A}_\mu^I + K_A y^A + J_\phi^\dagger \phi + \phi^\dagger J_\phi + \bar\eta\psi + \bar\psi\eta \tag{2.3}$$

Perform the change of variables corresponding to (2.2). Assuming a gauge-invariant measure (no anomaly, see below), $\delta Z = 0$ gives, after integrating by parts and using arbitrariness of $\epsilon^I(x)$,

$$\begin{aligned} 0 = \int \mathcal{D}\Phi\; e^{i(\cdots)} \Big[& - J_I^\mu D_\mu \epsilon^I + K_A\, \xi_I^A \epsilon^I + i(J_\phi^\dagger T_I \phi - \phi^\dagger T_I J_\phi)\epsilon^I + i(\bar\eta T_I \psi - \bar\psi T_I \eta)\epsilon^I \Big] \\ \Rightarrow \quad & D_\mu^x \frac{\delta W}{\delta J_\mu^I(x)} - K_A(x)\, \xi_I^A\Big(\frac{\delta W}{\delta K}\Big) + i\Big(J_\phi^\dagger T_I \frac{\delta W}{\delta J_\phi^\dagger} - \frac{\delta W}{\delta J_\phi} T_I J_\phi\Big) \\ & + i\Big(\bar\eta T_I \frac{\delta W}{\delta \bar\eta} - \frac{\delta W}{\delta \eta} T_I \eta\Big) = \mathcal{A}_I(x), \end{aligned} \tag{2.4}$$

where $W = -i \ln Z$ and $\mathcal{A}_I(x)$ denotes a possible local anomaly density (zero if the measure is invariant). Equation (2.4) is the *connected* Ward identity in the spectral setting.

Momentum-space corollaries. Differentiating (2.4) twice w.r.t. J and setting sources to zero gives the transversality of the gauge polarization,

$$q_\mu\, \Pi_{IJ}^{\mu\nu}(q) = 0, \qquad \Pi_{IJ}^{\mu\nu}(q) \equiv \frac{\delta^2 W}{\delta J_\mu^I\, \delta J_\nu^J}\Big|_0, \tag{2.5}$$

valid even in the presence of spectral anisotropies (Sec. 1.3). Mixed differentiations yield charge-conservation relations for matter correlators.

3. 1PI effective action and background-field identity

Legendre transform. Define classical fields by $\mathcal{A}_c^{I,\mu} = \delta W/\delta J_\mu^I$, $y_c^A = \delta W/\delta K_A$, etc., and the 1PI effective action $\Gamma[\mathcal{A}_c, y_c, \phi_c, \psi_c] = W - \int (J\cdot\mathcal{A}_c + K\cdot y_c + \cdots)$. Then (2.4) becomes

$$D_\mu\Big(\frac{\delta\Gamma}{\delta \mathcal{A}_{c\,\mu}^I}\Big) + \xi_I^A(y_c)\,\frac{\delta\Gamma}{\delta y_c^A} + i\left(\frac{\delta\Gamma}{\delta\psi_c}\,T_I\psi_c - \bar\psi_c\, T_I\,\frac{\delta\Gamma}{\delta\bar\psi_c}\right) = \mathcal{A}_I, \tag{2.6}$$

the *master Ward identity*. When $\mathcal{A}_I = 0$ it expresses exact gauge redundancy of Γ.

Background-field method. Split $\mathcal{A} = \bar{\mathcal{A}} + a$ and couple sources so that $\bar{\mathcal{A}}$ transforms as a connection while a transforms covariantly. Gauge fixing can then be chosen to preserve *background* gauge invariance, leading to the background Ward identity

$$\bar D_\mu\Big(\frac{\delta\Gamma}{\delta \bar{\mathcal{A}}_\mu^I}\Big) + \xi_I^A(y_c)\,\frac{\delta\Gamma}{\delta y_c^A} + i\left(\frac{\delta\Gamma}{\delta\psi_c}\,T_I\psi_c - \bar\psi_c\, T_I\,\frac{\delta\Gamma}{\delta\bar\psi_c}\right) = \mathcal{A}_I, \tag{2.7}$$

with $\bar D_\mu$ built from $\bar{\mathcal{A}}_\mu$. This form is preserved by renormalization and is the natural entry point to spectral RG (Ch. 3).

4. Spectral origin of the measure and anomaly placeholder

Measure from the medium. Formally, the functional measure $\mathcal{D}\Phi$ inherits the spectral inner products fixed by (g_{eff}, h) and by the constitutive tensors of Sec. 1.3. In the absence of chiral matter, the Jacobian of (2.2) is unity, $\mathcal{A}_I = 0$. With chiral fermions, the Jacobian may produce a local functional $\mathcal{A}_I(x)$ proportional to the anomaly polynomial; see Part II. In all anomaly-free spectra one has $\mathcal{A}_I = 0$ and (2.6) holds exactly.

5. Immediate consequences

Transversality and constitutive deformations. Equations (2.5) and (2.6) imply that, even with χ^μ-induced anisotropy, the inverse propagator maintains the projector structure

$$\left[\Delta^{-1}\right]^{\mu\nu}_{IJ}(q) \;=\; Z_{IJ}\,(q^2 g^{\mu\nu}_{\text{eff}} - q^\mu q^\nu) \;+\; \Pi^{\mu\nu}_{IJ}(q), \qquad q_\mu \left[\Delta^{-1}\right]^{\mu\nu}_{IJ}(q) = 0, \tag{2.8}$$

constraining dispersion relations (Sec. 1.3) and forbidding gauge-boson masses in the symmetric phase.

Vertex constraints (tree-level form). Differentiating (2.6) w.r.t. $\mathcal{A}$ yields the non-Abelian Ward identity for the three-vertex at tree level,

$$q^\mu\,\Gamma^{IJK}_{\mu\alpha\beta}(q,p,r) \;=\; f^I{}_{LM}\,\Gamma^{LK}_{\alpha\beta}(p)\,\delta^{MJ} \;-\; f^I{}_{LM}\,\Gamma^{LJ}_{\alpha\beta}(r)\,\delta^{MK}, \tag{2.9}$$

with momenta $q+p+r=0$ and $\Gamma_{\alpha\beta}$ the 1PI two-point function. In Sec. 2.2 this generalizes to the *Slavnov–Taylor* identity via BRST.

Gauss law and physical states. In canonical language, the time-component of (2.1) is the Gauss constraint; quantization enforces $G^I|\text{phys}\rangle = 0$ for the Gauss generators G^I. Section 2.2 will encode this constraint cohomologically via BRST.

6. Summary and bridge to BRST

Classical spectral redundancy (2.2) yields the master Ward identities (2.4) and (2.6). These enforce transversality, constrain vertices, and guarantee gauge-boson masslessness prior to symmetry breaking. To quantize while keeping the redundancy explicit, one introduces gauge fixing and ghost fields; the resulting BRST symmetry packages all Ward identities into a single *Zinn–Justin equation*, to be derived in Sec. 2.2. Possible anomalous terms $\mathcal{A}_I$ are postponed to the dedicated anomaly analysis (Part II).

2.2 Effective BRST symmetry and gauge-fixing functionals

Aim. We construct the *effective* BRST symmetry for the spectral gauge theory developed in Secs. 1.1–1.4 and use it to define gauge-fixing functionals adapted to the spectral medium. The outcome is a BRST-exact gauge-fixing+ghost sector that (i) preserves background covariance, (ii) diagonalizes the quadratic action by cancelling fiber–gauge mixing, and (iii) yields the Slavnov–Taylor (Zinn–Justin) functional identity controlling renormalization.

1. BRST algebra in the spectral setting

Field content and transformations. Introduce the ghost $c^I(x)$ (Grassmann, adjoint), the antighost $\bar{c}_I(x)$, and the Nakanishi–Lautrup multiplier $b_I(x)$. With $D_\mu = \partial_\mu + [\mathcal{A}_\mu, \cdot]$,

the BRST differential s acts by

$$s\,\mathcal{A}^I_\mu = (D_\mu c)^I \;=\; \partial_\mu c^I + f^I{}_{JK}\,\mathcal{A}^J_\mu c^K, \tag{2.10}$$

$$s\,c^I = -\tfrac{1}{2}\,f^I{}_{JK}\,c^J c^K, \tag{2.11}$$

$$s\,\bar{c}_I = b_I, \qquad s\,b_I = 0, \tag{2.12}$$

$$s\,y^A = c^I\,\xi^A_I(y), \tag{2.13}$$

$$s\,\phi = i\,c^I T^{(R)}_I\,\phi, \qquad s\,\psi = i\,c^I T^{(R)}_I\,\psi, \qquad s\,\bar{\psi} = -\,i\,\bar{\psi}\,c^I T^{(R)}_I. \tag{2.14}$$

Using $[\xi_I,\xi_J]^A = f^K{}_{IJ}\,\xi^A_K$ and the Jacobi identity, one verifies *nilpotency* $s^2 = 0$ on all fields:

$$s^2\mathcal{A}_\mu = 0, \quad s^2 c = 0, \quad s^2\bar{c} = 0, \quad s^2 y^A = 0, \quad s^2\phi = 0, \quad s^2\psi = 0. \tag{2.15}$$

The BRST variations (2.13) are the spectral lift of the fiber isometries to field space.

Ghost number and gradings. Assign $\mathrm{gh}(\mathcal{A}, y, \phi, \psi, \bar{\psi}, b) = 0$, $\mathrm{gh}(c) = +1$, $\mathrm{gh}(\bar{c}) = -1$, so that s raises ghost number by $+1$. The spectral covariant calculus (Sec. 1.4) commutes with s on covariant building blocks.

2. BRST-exact gauge fixing: general construction

Gauge-fixing fermion. Let $\mathcal{F}^I[\mathcal{A}, y; \bar{\mathcal{A}}]$ be a (background-covariant) gauge functional of ghost number 0, where we allow a background connection $\bar{\mathcal{A}}_\mu$ for the background-field method. Define the *gauge-fixing fermion* (ghost number -1)

$$\Psi \;=\; \int d^4x\sqrt{-g_{\mathrm{eff}}}\,\Big[\,\bar{c}_I\,\mathcal{F}^I[\mathcal{A}, y; \bar{\mathcal{A}}] \;-\; \frac{\xi}{2}\,\bar{c}_I\,b^I\,\Big], \tag{2.16}$$

with dimensionless gauge parameter $\xi > 0$. The BRST-exact gauge-fixing+ghost action is

$$S_{\mathrm{gf+gh}} \;=\; s\,\Psi = \int d^4x\sqrt{-g_{\mathrm{eff}}}\,\Big[\,b_I\,\mathcal{F}^I - \frac{\xi}{2}\,b_I b^I \;-\; \bar{c}_I\,\mathcal{M}^I{}_J\,c^J\,\Big], \tag{2.17}$$

$$\mathcal{M}^I{}_J := \left.\frac{\delta\mathcal{F}^I}{\delta\alpha^J}\right|_{\delta_\alpha\mathcal{A}=D\alpha,\;\delta_\alpha y=\alpha^K\xi_K(y)} = \frac{\delta\mathcal{F}^I}{\delta\mathcal{A}^K_\mu}\,D_\mu\delta^K{}_J \;+\; \frac{\delta\mathcal{F}^I}{\delta y^A}\,\xi^A_J(y), \tag{2.18}$$

the *Faddeev–Popov* operator. Eliminating b_I yields $(2\xi)^{-1}\int\sqrt{-g_{\mathrm{eff}}}\,\mathcal{F}_I\mathcal{F}^I$ plus the ghost determinant $\det\mathcal{M}$.

Background covariance. If $\mathcal{F}^I$ transforms as an adjoint scalar under background transformations ($\delta\bar{\mathcal{A}}_\mu = -\bar{D}_\mu\epsilon$, $\delta a_\mu = [\epsilon, a_\mu]$, etc.), then $S_{\mathrm{gf+gh}}$ respects *background* gauge invariance and the background Ward identity (2.7) holds.

3. Two canonical families of gauges

(A) Background-covariant R_ξ (minimal) gauge. Split $\mathcal{A}_\mu = \bar{\mathcal{A}}_\mu + a_\mu$. The minimal background gauge is

$$\mathcal{F}^I_{\mathrm{min}} \;=\; \bar{D}^\mu a^I_\mu, \tag{2.19}$$

with $\bar{D}_\mu = \partial_\mu + [\bar{\mathcal{A}}_\mu, \cdot]$. Then

$$S^{\mathrm{min}}_{\mathrm{gf+gh}} \;=\; \int d^4x\sqrt{-g_{\mathrm{eff}}}\,\left[\,\frac{1}{2\xi}\,(\bar{D}^\mu a^I_\mu)^2 \;-\; \bar{c}_I\,\bar{D}^\mu D_\mu c^I\,\right], \tag{2.20}$$

and the quadratic gauge sector is diagonal in $(a, \bar{c}, c)$ but may retain mixing with y through the fiber kinetic term.

(B) Spectral *'t Hooft* gauge (mixing-free at quadratic order). The leading mixing between a_μ and δy^A arises from the expansion of $\mathcal{L}^{(2)}_{\rm int} = \frac{\kappa_y}{2} g^{\mu\nu} h_{AB} \mathcal{D}_\mu y^A \mathcal{D}_\nu y^B$ around a background $(\bar{y}, \bar{\mathcal{A}})$:

$$\mathcal{L}^{(2)}_{\rm int} \supset -\kappa_y\, g^{\mu\nu} h_{AB}(\bar{y})\, \xi^A_I(\bar{y})\, \partial_\mu \delta y^B\, a^I_\nu \;+\; \cdots \tag{2.21}$$

To cancel (2.21) at the quadratic level, choose the *spectral 't Hooft* gauge

$$\boxed{\mathcal{F}^I_{\rm spec} \;=\; \bar{D}^\mu a^I_\mu \;-\; \lambda\, \mathcal{M}^{IJ}(\bar{y})\, h_{AB}(\bar{y})\, \xi^A_J(\bar{y})\, \delta y^B, \qquad \mathcal{M}_{IJ}(\bar{y}) := h_{AB}\, \xi^A_I\, \xi^B_J \;\; \text{(cf. (1.30))},} \tag{2.22}$$

with dimensionless λ fixed by the quadratic action. A convenient normalization is

$$\lambda \;=\; \sqrt{\frac{\kappa_y}{\kappa_G}}\,, \tag{2.23}$$

which yields a diagonal quadratic form in $(a, \delta y)$ after integration by parts, while preserving background covariance. The corresponding ghost operator is

$$\mathcal{M}^I{}_J\Big|_{\rm spec} \;=\; \bar{D}^\mu D_\mu\, \delta^I{}_J \;-\; \lambda\, \mathcal{M}^{IK}(\bar{y})\, h_{AB}(\bar{y})\, \xi^A_K(\bar{y})\, \xi^B_J(\bar{y}) \;+\; \cdots\,, \tag{2.24}$$

where the ellipsis denotes terms with at most one background field insertion (curvature of $\bar{\mathcal{A}}$ and derivatives of $\bar{y}$).

Remarks.

- In phases with symmetry breaking (Ch. 8), (2.22) reduces to the usual 't Hooft gauge where the combination proportional to the would-be Goldstones cancels the a–Goldstone mixing.
- The family $\mathcal{F}^I = \bar{D}\cdot a^I - \lambda\, \mathsf{M}^I{}_A(\bar{y})\, \delta y^A$ with any M proportional to $\mathcal{M}^{IJ} h_{AB} \xi^B_J$ preserves the spectral horizontal principle (Sec. 1.2) at quadratic order.

4. BRST sources and the Zinn–Justin (Slavnov–Taylor) identity

External sources (antifields). Introduce sources $K^{I\mu}$, K_A, K_ϕ, K_ψ, $K_{\bar\psi}$, and L_I coupled to BRST variations:

$$S_{\rm ext} \;=\; \int d^4x \sqrt{-g_{\rm eff}} \Big[K^{I\mu}\, s\mathcal{A}^I_\mu + K_A\, sy^A + K_\phi\, s\phi + K_\psi\, s\psi + K_{\bar\psi}\, s\bar\psi + \tfrac{1}{2} L_I\, sc^I \Big]. \tag{2.25}$$

The full classical action is $S_{\rm cl} = S + S_{\rm gf+gh} + S_{\rm ext}$, with S the gauge-invariant spectral action of Sec. 1.4.

Slavnov functional. Define the Slavnov–Taylor functional

$$\begin{aligned} \mathcal{S}(\Gamma) \;:=\; \int d^4x \sqrt{-g_{\rm eff}} \Big[&\frac{\delta\Gamma}{\delta K^{I\mu}} \frac{\delta\Gamma}{\delta \mathcal{A}^I_\mu} + \frac{\delta\Gamma}{\delta K_A} \frac{\delta\Gamma}{\delta y^A} + \frac{\delta\Gamma}{\delta K_\phi} \frac{\delta\Gamma}{\delta \phi} + \frac{\delta\Gamma}{\delta K_\psi} \frac{\delta\Gamma}{\delta \psi} + \frac{\delta\Gamma}{\delta K_{\bar\psi}} \frac{\delta\Gamma}{\delta \bar\psi} \\ &+ b_I \frac{\delta\Gamma}{\delta \bar{c}_I} + \frac{\delta\Gamma}{\delta L_I} \frac{\delta\Gamma}{\delta c^I} \Big], \end{aligned} \tag{2.26}$$

where Γ is the 1PI functional. BRST invariance implies the *Zinn–Justin equation*

$$\boxed{\mathcal{S}(\Gamma) \;=\; 0} \tag{2.27}$$

to all orders in perturbation theory (modulo anomalies discussed in Part II). Linearizing (2.27) around the classical action gives the tree-level Slavnov identity; its functional derivatives reproduce all Ward/Slavnov–Taylor relations among 1PI vertices.

5. Gauge-parameter and background dependence (Nielsen identities)

Gauge-parameter flow. Consider the ξ-dependence of Γ for fixed renormalization scheme. Differentiating $S_{\mathrm{gf+gh}} = s\Psi$ with respect to ξ and using $s^2 = 0$ yields the *Nielsen identity*

$$\frac{\partial \Gamma}{\partial \xi} \;=\; \mathcal{S}(\Gamma)\,\Xi[\Phi;\xi], \qquad \Xi := \int d^4x \sqrt{-g_{\mathrm{eff}}}\,\Big(-\tfrac{1}{2}\,\bar{c}_I\,\bar{c}^I \Big), \tag{2.28}$$

schematically written in terms of a local functional Ξ. Since the r.h.s. is Slavnov-exact, on the *BRST cohomology* (physical sector) $\partial_\xi \Gamma = 0$: *physical* observables and S-matrix elements are ξ-independent.

Background identity. In background gauges, variations of $\bar{\mathcal{A}}_\mu$ produce a background Ward identity that is compatible with (2.27). Renormalization can be chosen to preserve background covariance, simplifying the spectral RG analysis (Ch. 3).

6. Quadratic action and propagators (spectral 't Hooft gauge)

Diagonalization. Expanding the total BRST-fixed action to quadratic order around $(\bar{y}, \bar{\mathcal{A}})$ and using (2.22)–(2.23), one finds

$$S^{(2)} = \int d^4x \sqrt{-g_{\mathrm{eff}}} \Big[\tfrac{\kappa_G}{2}\, a^I_\mu\, \mathcal{K}^{\mu\nu}_{IJ}\, a^J_\nu + \tfrac{\kappa_y}{2}\, \delta y^A\, \mathcal{H}_{AB}\, \delta y^B + \frac{1}{2\xi}\,(\bar{D}^\mu a^I_\mu)^2 - \bar{c}_I\, \bar{D}^\mu D_\mu c^I + \cdots \Big], \tag{2.29}$$

with $\mathcal{K}^{\mu\nu}_{IJ}$ the standard background-covariant Maxwell/YM operator and $\mathcal{H}_{AB}$ the fiber Hessian; crucially, the cross term $a\cdot\partial\delta y$ cancels. The dots include curvature and anisotropy insertions controlled by Sec. 1.3.

Propagators. In momentum space (flat g_{eff} and trivial background for illustration), the gauge propagator in R_ξ reads

$$\langle a^I_\mu a^J_\nu \rangle(q) \;=\; \frac{\delta^{IJ}}{\kappa_G}\,\frac{1}{q^2}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2} \right), \qquad \langle c^I \bar{c}_J \rangle(q) \;=\; \frac{\delta^I{}_J}{q^2}\,, \tag{2.30}$$

up to the spectral renormalizations of Sec. 1.3. These provide the starting point for loop computations in background-covariant form.

7. Summary

We have constructed the BRST algebra (2.10)–(2.14), defined BRST-exact gauge-fixing+ghost actions through the fermion (2.16), and presented two canonical gauge families: the minimal background R_ξ gauge (2.19) and the *spectral 't Hooft* gauge (2.22) that removes fiber–gauge mixing at quadratic order (with normalization (2.23)). Coupling external BRST sources leads to the Zinn–Justin functional identity (2.27), while Nielsen identities (2.28) guarantee gauge-parameter independence of physical observables. These structures underpin the bookkeeping interpretation of Faddeev–Popov ghosts (Sec. 2.3) and the unitarity/gauge-independence checks (Sec. 2.4).

2.3 Faddeev–Popov as bookkeeping of redundancy

Aim. We show that the Faddeev–Popov (FP) construction is nothing but a precise *bookkeeping* of the gauge redundancy generated by local fiber isometries: it factors out the volume of gauge orbits in field space and implements a well-defined transverse "slice" (the gauge condition). We derive the FP determinant from first principles, represent it by ghosts, connect it to the BRST-exact gauge-fixing sector of Sec. 2.2, and discuss background covariance, Abelian decoupling, and Gribov issues in the spectral setting.

1. Gauge orbits and factorization of the path integral

Configuration space and orbits. Let $\mathcal{C}$ be the space of field configurations $\Phi \equiv (\mathcal{A}_\mu, y, \phi, \psi, \bar\psi)$ endowed with the spectral inner products induced by (g_{eff}, h) (Sec. 1.3). The gauge group $\mathcal{G}$ of local isometries acts freely (perturbatively) on $\mathcal{C}$:

$$\Phi \;\mapsto\; \Phi^\alpha \equiv \exp\Big(\alpha^I \mathcal{R}_I\Big)\,\Phi, \qquad \mathcal{R}_I \cdot (\mathcal{A}_\mu, y, \phi, \psi) = \Big(D_\mu \delta^K{}_I,\ \xi_I(y),\ iT_I\phi,\ iT_I\psi\Big), \tag{2.31}$$

with $\alpha^I(x)$ compactly supported and $\mathcal{R}_I$ the (field-space) generators. The naive path integral

$$Z \;=\; \int_{\mathcal{C}} \mathcal{D}\Phi\ \exp\Big(iS[\Phi;\mathcal{S}]\Big), \tag{2.32}$$

overcounts by the (infinite) volume of gauge orbits $\mathcal{O}_\Phi = \{\Phi^\alpha\}$.

Gauge slice and unity insertion. Pick a gauge functional $\mathcal{F}^I[\Phi;\bar{\mathcal{A}}]$ (ghost number 0), possibly background-covariant, such that each orbit intersects the slice $\mathcal{F}^I = 0$ once in a neighborhood (Gribov issues aside). Insert

$$1 \;=\; \Delta_{\mathcal{F}}[\Phi] \int \mathcal{D}\alpha\ \prod_x \delta\Big(\mathcal{F}^I[\Phi^\alpha;\bar{\mathcal{A}}](x)\Big), \tag{2.33}$$

into (2.32). Changing variables $\Phi \mapsto \Phi^\alpha$ and using gauge invariance of S, the $\mathcal{D}\alpha$ integral factorizes as the volume $\mathrm{Vol}(\mathcal{G})$, leaving

$$\frac{Z}{\mathrm{Vol}(\mathcal{G})} \;=\; \int_{\mathcal{C}} \mathcal{D}\Phi\ \Delta_{\mathcal{F}}[\Phi]\ \prod_x \delta\Big(\mathcal{F}^I[\Phi;\bar{\mathcal{A}}](x)\Big)\ \exp\Big(iS[\Phi;\mathcal{S}]\Big). \tag{2.34}$$

Thus $\Delta_{\mathcal{F}}$ is the *Jacobian* measuring the angle between the orbit and the slice.

2. FP determinant from the Jacobian

Linearization along the orbit. Under an infinitesimal gauge variation $\delta_\alpha\Phi$, the gauge functional changes by

$$\delta_\alpha \mathcal{F}^I \;=\; \left.\frac{\delta \mathcal{F}^I}{\delta \Phi}\right|_{\Phi} \cdot (\mathcal{R}_J \cdot \Phi)\; \alpha^J \;=\; \mathcal{M}^I{}_J[\Phi;\bar{\mathcal{A}}]\; \alpha^J, \tag{2.35}$$

with the FP operator

$$\boxed{\mathcal{M}^I{}_J[\Phi;\bar{\mathcal{A}}] \;:=\; \frac{\delta \mathcal{F}^I}{\delta \mathcal{A}^K_\mu}\, D_\mu \delta^K{}_J \;+\; \frac{\delta \mathcal{F}^I}{\delta y^A}\, \xi^A_J(y) \;+\; \frac{\delta \mathcal{F}^I}{\delta \phi}\, (iT_J\phi) \;+\; \frac{\delta \mathcal{F}^I}{\delta \psi}\, (iT_J\psi) \;+\cdots} \tag{2.36}$$

(cf. (2.18)). The Jacobian of the map $\alpha \mapsto \mathcal{F}$ is $\det \mathcal{M}$, hence

$$\Delta_{\mathcal{F}}[\Phi] \;=\; \det \mathcal{M}[\Phi;\bar{\mathcal{A}}]. \tag{2.37}$$

Ghost representation. Introduce Grassmann ghosts $(\bar{c}_I, c^I)$ to exponentiate the determinant:

$$\det \mathcal{M} \;=\; \int \mathcal{D}\bar{c}\,\mathcal{D}c\; \exp\!\Big\{\, i \int d^4x \sqrt{-g_{\text{eff}}}\; \bar{c}_I\, \mathcal{M}^I{}_J\, c^J \Big\}. \tag{2.38}$$

Equation (2.38) defines the *ghost action* $S_{\text{gh}} = \int \sqrt{-g_{\text{eff}}}\, \bar{c}\, \mathcal{M}\, c$.

3. Equivalence to the BRST-exact gauge-fixing sector

Adding the gauge condition. Enforce the slice softly by a Gaussian weight (Nakanishi–Lautrup field b_I):

$$1 \;\propto\; \int \mathcal{D}b\; \exp\!\Big\{\, i \int d^4x \sqrt{-g_{\text{eff}}}\; \Big(b_I \mathcal{F}^I - \tfrac{\xi}{2} b_I b^I \Big) \Big\}. \tag{2.39}$$

Combining (2.38) and (2.39) with (2.34) yields the BRST-exact sector (2.17) upon identifying the gauge-fixing fermion Ψ in (2.16). Thus *FP and BRST are equivalent descriptions*: the former as a Jacobian, the latter as a cohomological symmetry generated by s with $S_{\text{gf+gh}} = s\Psi$.

Gauge-independence (bookkeeping). Consider a smooth family $\mathcal{F}_t$ of gauges. The derivative of the fixed path integral w.r.t. t is a BRST-exact insertion (Nielsen-type identity of Sec. 2.2), which vanishes on BRST-closed observables. Hence physical predictions do not depend on the bookkeeping choice $\mathcal{F}$.

4. Spectral gauges and explicit FP operators

(A) Minimal background R_ξ gauge. For $\mathcal{F}^I_{\min} = \bar{D}^\mu a^I_\mu$ (Eq. (2.19)), one finds

$$\mathcal{M}^I{}_J\Big|_{\min} \;=\; \bar{D}^\mu D_\mu\, \delta^I{}_J, \qquad \mathcal{L}^{\min}_{\text{gh}} \;=\; -\,\bar{c}_I\, \bar{D}^\mu D_\mu c^I, \tag{2.40}$$

so ghosts couple only to the gauge field (through the covariant derivatives). In the Abelian case, $\bar{D}_\mu D_\mu = \partial^2$, and ghosts *decouple* entirely (their determinant is a field-independent constant).

(B) Spectral 't Hooft gauge. For $\mathcal{F}^I_{\rm spec} = \bar{D}^\mu a^I_\mu - \lambda\, \mathcal{M}^{IJ}(\bar{y})\, h_{AB}(\bar{y})\, \xi^A_J(\bar{y})\, \delta y^B$ (Eq. (2.22)),

$$\mathcal{M}^I{}_J\Big|_{\rm spec} \;=\; \bar{D}^\mu D_\mu\, \delta^I{}_J \;-\; \lambda\, \mathcal{M}^{IK}(\bar{y})\, h_{AB}(\bar{y})\, \xi^A_K(\bar{y})\, \xi^B_J(\bar{y}) \;+\; \cdots\,, \tag{2.41}$$

where the ellipsis includes insertions of $\bar{\mathcal{F}}_{\mu\nu}$, $\nabla\bar{y}$, and higher-order fluctuations. At quadratic order the ghost Lagrangian is

$$\mathcal{L}^{\rm spec}_{\rm gh} \;=\; -\,\bar{c}_I\, \bar{D}^\mu D_\mu c^I \;+\; \lambda\, \bar{c}_I\, \mathcal{M}^{IK}(\bar{y})\, h_{AB}(\bar{y})\, \xi^A_K(\bar{y})\, \xi^B_J(\bar{y})\, c^J \;+\; \cdots\,, \tag{2.42}$$

which preserves background covariance. Ghost–a_μ three-vertices arise from expanding D_μ; ghost–δy couplings appear beyond quadratic order when $\xi^A_J(y)$ is expanded around $\bar{y}$.

5. Geometric meaning of the FP operator

Angle between the slice and the orbit. Let $\mathcal{S}_\Phi := \{\Psi \in T_\Phi\mathcal{C} \,|\, \delta\mathcal{F}^I[\Phi;\bar{\mathcal{A}}]\cdot\Psi = 0\}$ be the tangent to the slice, and $\mathcal{O}_\Phi := \{\mathcal{R}_J\cdot\Phi\}$ the orbit directions. With the spectral inner product $\langle\cdot,\cdot\rangle$,

$$\mathcal{M}^I{}_J \;=\; \left\langle \frac{\delta\mathcal{F}^I}{\delta\Phi}\,,\, \mathcal{R}_J\cdot\Phi \right\rangle, \tag{2.43}$$

so $\det\mathcal{M}$ measures the local "sine of the angle" between $\mathcal{S}_\Phi$ and $\mathcal{O}_\Phi$. In the bundle picture, $\mathcal{M}$ computes the Jacobian of projecting field variations onto the vertical (gauge) directions, cf. the projector $\mathbb{P}_\parallel$ in (1.30).

6. Abelian decoupling and non-Abelian interactions

Abelian case. For $G = U(1)$ and $\mathcal{F} = \partial\cdot a$ (or its background version), $\mathcal{M} = -\partial^2$ is field-independent; ghosts decouple to *all* orders. In the spectral 't Hooft gauge the extra term in (2.41) reduces to a background constant if $\xi = \partial_\theta$ is covariantly constant on $\mathcal{F}$, so ghosts remain non-dynamical.

Non-Abelian case. For G non-Abelian, $\mathcal{M}$ contains the adjoint covariant Laplacian $\bar{D}^\mu D_\mu$; consequently ghosts carry propagators $\sim (q^2)^{-1}$ and couple to a_μ via $\bar{c}\, f\, a\, \partial c$ vertices (and to background fields). These interactions are crucial for (i) unitarity cancellations and (ii) maintaining the Slavnov–Taylor identities at loop level.

7. Gribov copies and the spectral gauges

Local vs global gauge fixing. The FP construction is *local* in configuration space: globally, gauge slices may intersect an orbit more than once (Gribov copies). In Euclidean signature with positive-definite inner products, define the *first Gribov region* Ω by the positivity of the FP operator, $\mathcal{M} > 0$. Restricting the functional integral to Ω removes near-zero-mode instabilities.

Spectral remarks. The spectral 't Hooft family (2.22) can be tied to minimizing a convex functional (e.g. $\|a\|^2$ plus a fiber term), which disfavors multiple intersections perturbatively. Nonperturbative implications (horizons, confinement) are addressed in the strong-sector chapter (Ch. 12).

8. Practical Feynman rules (quadratic level)

Propagators. From (2.29) and (2.40) one reads, in flat background,

$$\langle a^I_\mu a^J_\nu\rangle(q) \;=\; \frac{\delta^{IJ}}{\kappa_G}\,\frac{1}{q^2}\left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right), \qquad \langle c^I\bar c_J\rangle(q) \;=\; \frac{\delta^I{}_J}{q^2}, \qquad \langle \delta y^A \delta y^B\rangle(q) \;=\; \frac{(\mathcal{H}^{-1})^{AB}}{\kappa_y}, \tag{2.44}$$

where $\mathcal{H}$ is the fiber Hessian (Sec. 2.2).

Vertices. Expanding $\bar c\,\bar D^\mu D_\mu c$ gives the standard ghost–gluon vertex $f^I{}_{JK}\,q_\mu$ (with q the ghost momentum), while spectral background insertions produce controlled corrections fixed by Sec. 1.3. Ghost–δy vertices appear at higher order from expanding $\xi^A_J(y)$ around $\bar y$.

9. Summary

The FP determinant is the Jacobian of the change of variables from gauge parameters to gauge conditions; it factors out the orbit volume and renders the integral finite. Its Grassmann representation defines the ghost sector, which in turn is *equivalent* to the BRST-exact gauge-fixing action built from a gauge-fixing fermion. In Abelian theories ghosts decouple; in non-Abelian theories they mediate essential unitarity and renormalization constraints. The spectral gauges (minimal background and spectral 't Hooft) yield background-covariant FP operators and diagonalize the quadratic action, while Gribov subtleties are confined to nonperturbative domains discussed later. This establishes FP ghosts as a rigorous bookkeeping device for redundancy, completing the bridge to the unitarity and gauge-independence checks of Sec. 2.4.

2.4 Unitarity and gauge-independence checks

Aim. We establish (i) *unitarity* of the S–matrix in the covariantly quantized spectral gauge theory and (ii) *gauge-independence* of physical observables. The key tools are the BRST symmetry and the Zinn–Justin identity (2.27), together with LSZ reduction, Källén–Lehmann representations adapted to $g_{\rm eff}$, and Nielsen-type identities for gauge parameters. Throughout we assume anomaly-free matter content (Part II handles anomalies).

1. BRST cohomology and the physical Hilbert space

BRST charge and cohomology. BRST invariance of the gauge-fixed classical action $S_{\rm cl}$ implies a conserved current j^μ_B and a nilpotent charge

$$Q_B \;:=\; \int_{\Sigma_t} d\Sigma_\mu\, j^\mu_B, \qquad Q_B^2=0, \qquad s\Phi \;=\; i\,[Q_B,\Phi\}\,, \tag{2.45}$$

with graded commutator $[\ ,\ \}$ and Σ_t a Cauchy slice of $(M, g_{\rm eff})$. The *physical* state space is the BRST cohomology at ghost number 0,

$$\mathcal{H}_{\rm phys} \;:=\; \ker Q_B\big|_{\rm gh=0} \,\Big/\, \operatorname{im} Q_B\big|_{\rm gh=-1}. \tag{2.46}$$

Vectors of the form $Q_B|\chi\rangle$ are *null*: they have zero norm and decouple from all matrix elements between physical states (Kugo–Ojima quartet mechanism).

S preserves cohomology. BRST invariance to all orders ((2.27)) implies $[Q_B, S] = 0$ in the Heisenberg picture, hence

$$S : \ \mathcal{H}_{\text{phys}} \ \longrightarrow \ \mathcal{H}_{\text{phys}}. \tag{2.47}$$

This is the algebraic backbone of unitarity and gauge independence: unphysical polarizations and ghosts only populate BRST doublets that do not contribute to S–matrix elements between cohomology classes.

2. Optical theorem on the BRST cohomology

Statement. With $S = 1 + iT$ and $S^\dagger S = 1$,

$$2 \ \text{Im} \ \mathcal{M}_{fi} \ = \ \sum_{n \in \mathcal{H}_{\text{phys}}} \int d\Pi_n \ \mathcal{M}^*_{fn} \, \mathcal{M}_{ni}, \qquad \mathcal{M}_{fi} \ := \ \langle f|T|i\rangle, \tag{2.48}$$

where the sum/integral is over a complete set of *physical* intermediate states only.

Derivation (sketch). Insert the identity on Fock space as $1 = \mathbb{P}_{\text{phys}} + \{Q_B, \mathcal{K}\}$, with $\mathbb{P}_{\text{phys}}$ the projector onto $\mathcal{H}_{\text{phys}}$ and $\mathcal{K}$ a contracting homotopy. Since $[Q_B, T] = 0$ and $\langle f|\{Q_B, \cdot\}|i\rangle = 0$ for $|i\rangle, |f\rangle \in \mathcal{H}_{\text{phys}}$, only $\mathbb{P}_{\text{phys}}$ survives, yielding (2.48). Thus absorptive parts come from physical on-shell cuts (Cutkosky rules) with no leakage into unphysical sectors.

3. Källén–Lehmann representation and positivity

Transverse decomposition. In an R_ξ-type gauge (Sec. 2.2), the (Fourier-space) two-point function of a gauge boson in the symmetric phase admits the decomposition

$$\Delta^{\mu\nu}_{IJ}(q) \ = \ \Pi^{\mu\nu}_T(q) \, \delta_{IJ} \, \Delta_T(q^2) \ + \ \xi \, \Pi^{\mu\nu}_L(q) \, \delta_{IJ} \, \frac{1}{q^2 + i0} \, , \tag{2.49}$$

with projectors $\Pi^{\mu\nu}_T = g^{\mu\nu}_{\text{eff}} - q^\mu q^\nu / q^2$ and $\Pi^{\mu\nu}_L = q^\mu q^\nu / q^2$. The *BRST* structure ensures that only the transverse part couples to conserved external currents, while the longitudinal piece cancels against ghost/Goldstone exchanges in all physical amplitudes.

Spectral form and positivity. The transverse scalar function obeys a Källén–Lehmann representation

$$\Delta_T(q^2) \ = \ \int_0^\infty d\mu^2 \ \frac{\rho_T(\mu^2)}{q^2 - \mu^2 + i0}, \qquad \rho_T(\mu^2) \ \geq \ 0, \tag{2.50}$$

where ρ_T is positive by spectral positivity on $\mathcal{H}_{\text{phys}}$. In the spectral medium, the role of q^2 is played by the appropriate quadratic form determined by g_{eff} and the constitutive tensors of Sec. 1.3; hyperbolicity and the (TQ) gate guarantee the same analytic structure and positivity.

4. Gauge-parameter independence: Nielsen identities on shell

Master identity. Let Γ be the 1PI functional satisfying the Zinn–Justin equation (2.27). For a gauge parameter ξ, the Nielsen identity reads schematically

$$\frac{\partial \Gamma}{\partial \xi} \;=\; \mathcal{S}(\Gamma)\cdot \Xi[\Phi;\xi], \tag{2.51}$$

with a local functional Ξ (cf. (2.28)). Taking functional derivatives and evaluating on shell ($\delta\Gamma/\delta\Phi = 0$) makes the r.h.s. vanish between BRST-closed external states, so *all* physical S–matrix elements are ξ-independent.

Pole masses and residues. For a stable particle with propagator denominator $\Delta_T^{-1}(q^2) \equiv q^2 - \Sigma_T(q^2)$, the pole m^2 solves $\Delta_T^{-1}(m^2) = 0$. Differentiating w.r.t. ξ and using (2.51) gives

$$\frac{\partial m^2}{\partial \xi} \;=\; 0, \qquad \frac{\partial}{\partial \xi}\,\mathrm{Res}\,\Delta_T(q^2)\Big|_{q^2=m^2} \;=\; 0, \tag{2.52}$$

so physical masses and LSZ residues are gauge independent (similarly for composite bound-state poles defined by Bethe–Salpeter equations consistent with (2.27)).

5. Cancellation of unphysical contributions (quartet mechanism)

External legs. Amplitudes with at least one external state of the form $Q_B|\chi\rangle$ vanish:

$$\langle f|S|Q_B\chi\rangle \;=\; \langle f|[S,Q_B]|\chi\rangle \;=\; 0. \tag{2.53}$$

Longitudinal polarizations and ghost states organize into BRST quartets and decouple from $\mathcal{H}_{\mathrm{phys}}$.

Internal lines. Any ξ-dependence of internal propagators appears in a BRST-exact form whose contributions cancel among sets of diagrams related by Slavnov–Taylor identities. At fixed order, this follows from differentiating Feynman rules w.r.t. ξ and using (2.27); nonperturbatively it is encoded in the cohomological projection used in (2.48).

6. Worked checks

(A) Transversality of the vacuum polarization. From the connected Ward identity (2.5), the 1PI two-point function is transverse:

$$q_\mu\,\Gamma^{\mu\nu}_{IJ}(q) \;=\; 0. \tag{2.54}$$

Hence only $\Pi_T^{\mu\nu}$ enters physical observables; birefringent deformations from Sec. 1.3 modify dispersion but preserve transversality and positivity of ρ_T.

(B) Gauge independence of a $2\to 2$ amplitude. Consider $W^aW^b \to W^cW^d$ in a phase with symmetry breaking (Ch. 8) or its unbroken analogue with adjoint vectors. The ξ-dependent pieces of gauge and Goldstone propagators, together with ghost loops, cancel in the on-shell amplitude by Slavnov–Taylor identities inherited from (2.27). The remaining result depends only on transverse form factors and physical parameters (masses, couplings) whose values are ξ-independent by (2.52).

7. Consistency with S/G/TQ/D gates

(S) Stability. Positivity of the spectral density $\rho_T \geq 0$ and of the Hamiltonian quadratic form in the BRST cohomology ensures no negative-norm leakage into $\mathcal{H}_{\rm phys}$.

(G) Hyperbolicity/causality. The constitutive deformations of Sec. 1.3 keep the transverse poles inside the $g_{\rm eff}$ light-cone and maintain microcausality; Cutkosky cuts obey the same causal support.

(TQ) Forward positivity. Forward-limit dispersion relations applied to transverse amplitudes yield the standard positivity bounds on higher-derivative Wilson coefficients. Since only $\mathcal{H}_{\rm phys}$ contributes to absorptive parts, the bounds are unaffected by gauge fixing.

(D) Dissipation. Allowed imaginary parts of retarded correlators correspond to positive total cross sections via (2.48); entropy production constraints (Vol. I) exclude CPT-odd, first-derivative instabilities in the gauge sector at leading order.

8. Summary

BRST invariance promotes gauge redundancy to a cohomological structure in which (i) the S–matrix acts on $\mathcal{H}_{\rm phys}$, (ii) the optical theorem (2.48) involves only physical intermediate states, and (iii) physical masses, residues, and S–matrix elements are independent of gauge-fixing choices by Nielsen identities (2.51)–(2.52). Longitudinal and ghost contributions cancel by the quartet mechanism; birefringent deformations from the spectral medium modify only the *transverse* kinematics and remain compatible with S/G/TQ/D gates. This completes the foundational BRST/FP chapter and prepares the anomaly analysis of Part II.

Chapter 3

Spectral RG and Coupling Unification Windows

3.1 Running of $g_{1,2,3}$ from fiber geometry

Aim. We derive the renormalization-group (RG) running of the gauge couplings $g_I(\mu)$ ($I = 1, 2, 3$ for hypercharge, weak, and color in the SM limit) *from* the spectral construction of the gauge sector developed in Ch. 1. Two inputs determine the flow: (i) universal quantum fluctuations of gauge/matter fields (group-theory controlled), and (ii) the *fiber geometry* through the normalization and mild kinetic mixing of the gauge kinetic term. We present: the background-field derivation of β-functions, the decomposition into *geometric* vs. *dynamical* running, a matrix form allowing kinetic mixing, and practical RG equations for $g_I(\mu)$ and $\sin^2\theta_W(\mu)$.

1. Definitions and background-field setup

Geometric normalization (UV anchor). From Secs. 1.2–1.5, the canonically normalized Yang–Mills sector at a spectral matching scale $\Lambda_{\rm spec}$ is

$$\mathcal{L}_{\rm YM} \;=\; -\frac{1}{4}\, Z_{IJ}(\Lambda_{\rm spec})\; \mathcal{F}^I_{\mu\nu}\mathcal{F}^{J\,\mu\nu}, \qquad Z_{IJ} \;=\; \kappa_G(\Lambda_{\rm spec};\mathcal{S})\,\langle \mathcal{M}_{IJ}\rangle_{\mathcal{F}} \;+\; \delta Z_{IJ}, \tag{3.1}$$

with $\mathcal{M}_{IJ} = h_{AB}\,\xi^A_I \xi^B_J$ the Killing-norm matrix on the fiber and δZ_{IJ} small anisotropy-induced corrections (Sec. 1.3). In a basis where Z_{IJ} is (nearly) diagonal, we define

$$\frac{1}{g_I^2(\Lambda_{\rm spec})} \;:=\; Z_{II}(\Lambda_{\rm spec}) \;\equiv\; \kappa_G(\Lambda_{\rm spec})\,\mathcal{N}_I \;+\; \delta Z_{II}. \tag{3.2}$$

Background-field method and scheme. We employ dimensional regularization in $d = 4 - 2\epsilon$ and minimal subtraction ($\overline{\rm MS}$), in *background-covariant* gauge (Sec. 2.2). Background gauge invariance implies the simple renormalization identity

$$Z_g\, Z_{\mathcal{A}}^{1/2} \;=\; 1, \tag{3.3}$$

so that the β-functions are extracted from the coefficient of $\ln\mu$ in the background two-point function (vacuum polarization).

2. One-loop running: dynamical piece

Universal coefficient. Let the light spectrum at scale μ contain gauge bosons of G_I, Weyl fermions in representations R_f and complex scalars in R_s. The one-loop *dynamical* contribution to the running of the *inverse* couplings is

$$\mu\frac{d}{d\mu}\left(\frac{1}{g_I^2}\right)\Bigg|_{\rm dyn} = \frac{1}{8\pi^2}\left[\frac{11}{3}C_2(G_I) - \frac{4}{3}\sum_{f\in\text{light}}T_I(R_f) - \frac{1}{6}\sum_{s\in\text{light}}T_I(R_s)\right] + \mathcal{O}(g^2), \quad (3.4)$$

with $C_2(G_I)$ the quadratic Casimir in the adjoint and $T_I(R)$ the Dynkin index defined by $\mathrm{Tr}_R(T_I^a T_I^b) = T_I(R)\,\delta^{ab}$. Equation (3.4) is identical to the standard Yang–Mills result; the *spectral* specifics enter through the list of *light* composites (thresholds; Sec. 3.2) and through the geometric piece below.

Spectral occupancy functions. Instead of fixed integers, the effective counts of light modes are *spectral occupancy* functions

$$\nu_f^{(R)}(\mu;\mathcal{S}) \in [0,\,\mathrm{mult}_f^{(R)}], \qquad \nu_s^{(R)}(\mu;\mathcal{S}) \in [0,\,\mathrm{mult}_s^{(R)}], \quad (3.5)$$

encoding smooth decoupling across spectral thresholds $m(\mathcal{S})$. Concretely,

$$\sum_{f\in\text{light}}T_I(R_f) \longrightarrow \sum_R T_I(R)\,\nu_f^{(R)}(\mu;\mathcal{S}), \qquad \sum_{s\in\text{light}}T_I(R_s) \longrightarrow \sum_R T_I(R)\,\nu_s^{(R)}(\mu;\mathcal{S}),$$

with $\nu\to 0$ for $\mu\ll m$ and $\nu\to\mathrm{mult}$ for $\mu\gg m$.

3. Geometric piece: running from fiber renormalization

Decomposition. Since $g_I^{-2} = \kappa_G\,\mathcal{N}_I + \delta Z_{II}$ (Eq. (3.2)), the total running splits as

$$\mu\frac{d}{d\mu}\left(\frac{1}{g_I^2}\right) = \underbrace{\mathcal{N}_I\,\mu\frac{d\kappa_G}{d\mu}}_{\text{“dilatonic” (universal)}} + \underbrace{\kappa_G\,\mu\frac{d\mathcal{N}_I}{d\mu}}_{\text{fiber-geometry}} + \underbrace{\mu\frac{d\,\delta Z_{II}}{d\mu}}_{\text{anisotropy}} + \left(\mu\frac{d}{d\mu}\frac{1}{g_I^2}\right)_{\rm dyn}. \quad (3.6)$$

The first term arises from loops that renormalize the overall gauge stiffness (e.g. integrating fast fiber modes; Sec. 1.2); the second from renormalization of the Killing norms $\mathcal{N}_I = \langle\mathcal{M}_{II}\rangle_{\mathcal{F}}$ via the y-sector; the third from small χ^μ-induced anisotropies (Sec. 1.3). At one loop in a Lorentz-invariant scheme, $\mu\,d\delta Z_{II}/d\mu$ contributes only to finite redefinitions of wave functions and *not* to the universal log of $\mathcal{F}^2$; we therefore keep it subleading and return in Sec. 3.3.

Background-field heat-kernel estimate. Using the heat-kernel route of Sec. 1.2, integrating out fast fiber fluctuations generates

$$\mu\frac{d\kappa_G}{d\mu} = \frac{\alpha_{\rm spec}}{8\pi^2}\dim(\mathcal{F}) + \cdots, \qquad \mu\frac{d\mathcal{N}_I}{d\mu} = \frac{1}{8\pi^2}\left(\beta^{\rm fib}\right)_I + \cdots, \quad (3.7)$$

where $\alpha_{\rm spec} > 0$ is a calculable constant fixed by the microscopic dispersion of fiber modes, and $(\beta^{\rm fib})_I$ depends on the curvature of $(\mathcal{F}, h)$ and on response of the Killing frame ξ_I under coarse-graining (both computable in concrete fiber models; App. 24.4). Dots denote higher-derivative suppressed terms.

4. Matrix form and kinetic mixing

General kinetic term. Allow an off-diagonal Z_{IJ} (small kinetic mixing in group space). Define the symmetric coupling matrix $\mathbb{G}^{-2} := \mathbb{Z} \equiv (Z_{IJ})$. The matrix RG equation reads

$$\mu \frac{d}{d\mu}\mathbb{Z} \;=\; \frac{1}{8\pi^2}\Big[\,\frac{11}{3}\,\mathbb{C}_A \;-\; \frac{4}{3}\sum_R \mathbb{T}(R)\,\nu_f^{(R)} \;-\; \frac{1}{6}\sum_R \mathbb{T}(R)\,\nu_s^{(R)}\Big] \;+\; \underbrace{\mu\frac{d}{d\mu}\Big(\kappa_G\,\langle\mathcal{M}\rangle_{\mathcal{F}}\Big)}_{\text{geometric}}, \tag{3.8}$$

where $(\mathbb{C}_A)_{IJ} = C_2(G_I)\,\delta_{IJ}$ for simple factors (and zero for $U(1)$), while $\mathbb{T}(R)$ is diagonal with entries $T_I(R)$ for fields charged under G_I (off-diagonals arise only for multi-$U(1)$ or explicit fiber-induced mixing). Canonical couplings are obtained by diagonalizing $\mathbb{Z}$ at each scale.

Comments. For a single $U(1)$, (3.8) reduces to the familiar Abelian running with possible kinetic mixing to other $U(1)$'s. For $SU(2)\times U(1)$, the running of the mixing angle can be expressed directly (below).

5. Practical RG equations

One-loop β functions (component form). Combining (3.4) and (3.6), the couplings obey

$$\mu\frac{d}{d\mu}\left(\frac{1}{g_I^2}\right) = \frac{1}{8\pi^2}\left[\frac{11}{3}C_2(G_I) - \frac{4}{3}\sum_R T_I(R)\,\nu_f^{(R)}(\mu;\mathcal{S}) - \frac{1}{6}\sum_R T_I(R)\,\nu_s^{(R)}(\mu;\mathcal{S})\right] + \mathcal{N}_I\,\mu\frac{d\kappa_G}{d\mu} + \kappa_G\,\mu\frac{d\mathcal{N}}{d\mu} \tag{3.9}$$

to leading order in small anisotropies and kinetic mixing. In the *SM limit* (standard field content and negligible geometric running), (3.9) reproduces the known one-loop coefficients.

Electroweak mixing angle. With $g \equiv g_2$, $g' \equiv g_1$, $\sin^2\theta_W = g'^2/(g^2+g'^2)$, one finds

$$\mu\frac{d}{d\mu}\sin^2\theta_W = \frac{g^2 g'^2}{(g^2+g'^2)^2}\left[\mu\frac{d}{d\mu}\left(\frac{1}{g'^2}\right) - \mu\frac{d}{d\mu}\left(\frac{1}{g^2}\right)\right], \tag{3.10}$$

with the r.h.s. supplied by (3.9) for $I = 1, 2$. At the spectral matching scale,

$$\sin^2\theta_W(\Lambda_{\text{spec}}) \;=\; \frac{1}{1+\mathcal{N}_1/\mathcal{N}_2} \;+\; \mathcal{O}(\delta Z), \tag{3.11}$$

cf. Eq. (1.83), then evolved down using (3.10).

Callan–Symanzik form. For any n-point 1PI vertex $\Gamma^{(n)}$, the spectral Callan–Symanzik equation reads

$$\left[\,\mu\frac{\partial}{\partial\mu} + \sum_I \beta_{g_I}\frac{\partial}{\partial g_I} - \sum_\Phi n_\Phi\,\gamma_\Phi\,\right]\Gamma^{(n)}(\{p\};\{g_I\},\mu;\mathcal{S}) \;=\; 0, \qquad \beta_{g_I} \;=\; -\frac{g_I^3}{16\pi^2}\,b_I \;+\; \cdots, \tag{3.12}$$

with b_I read off from the square bracket of (3.9) plus geometric pieces translated via $\mu\,d(1/g_I^2) = -(2/g_I^3)\beta_{g_I}$.

6. Worked limits and checks

SM-like limit (consistency check). If (i) $\mu\, d\kappa_G/d\mu = \mu\, d\mathcal{N}_I/d\mu = 0$, (ii) ν's are sharp steps at particle masses, then (3.9) collapses to the standard one-loop running. This provides a nontrivial cross-check of the spectral construction.

Spectral unification anchor. If the fiber is close to a symmetric point where $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 \equiv \mathcal{N}$ at $\Lambda_{\rm spec}$, then

$$\frac{1}{g_1^2} : \frac{1}{g_2^2} : \frac{1}{g_3^2}\Big|_{\Lambda_{\rm spec}} \;=\; 1:1:1 \;+\; \mathcal{O}(\delta Z,\ \text{geometric splittings}), \tag{3.13}$$

i.e. the geometric initial condition is *unified* up to controlled deformations. Downward running then tests (near-)unification windows (Sec. 3.4).

Insensitivity to mild anisotropy. CPT-even, two-derivative anisotropies (Sec. 1.3) do not alter the universal $\ln\mu$ coefficient of $\mathcal{F}^2$ at one loop; they renormalize higher-dimensional operators and finite parts. Thus (3.9) remains valid to leading-log accuracy under the S/G/TQ/D gates.

7. Summary

The running of $g_{1,2,3}$ in the spectral framework is governed by a *sum* of a universal, group-theory controlled dynamical piece (3.4) and a *geometric* piece from the fiber-induced normalization (3.1). The master equation (3.9) packages both, with smooth threshold functions $\nu(\mu;\mathcal{S})$ capturing decoupling of spectral composites and with optional kinetic mixing handled by the matrix flow (3.8). Geometric relations such as (3.11) serve as UV anchors propagated to low energies via (3.10). These ingredients set the stage for a detailed treatment of thresholds and universality classes (Sec. 3.2) and for assessing (near-)unification (Sec. 3.4).

3.2 Thresholds, decoupling, and universality classes

Aim. We turn the master flow (3.9) into a *predictive* RG by (i) specifying how heavy degrees of freedom (DoF) decouple in the spectral medium, (ii) giving matching conditions for gauge couplings (and kinetic mixing) across thresholds, and (iii) organizing qualitatively distinct RG patterns into *universality classes* governed by fiber geometry and symmetry. Throughout we use the background-field scheme of Sec. 3.1, which guarantees gauge-covariant renormalization and simple matching.

1. Spectral scales and heavy spectrum

Intrinsic scales. The medium provides *spectral* scales $\Lambda_{\rm spec}$ (geometric matching), and dynamical mass scales $M_i(\mathcal{S})$ for composites tied to $(\rho, \eta, \chi^\mu, T)$ and the fiber geometry $(\mathcal{F}, h)$:

$$M_i \;\equiv\; M_i\Big(\rho, \eta, T;\, \mathrm{curv}(\mathcal{F}),\ \|\xi\|,\ \cdots\Big), \qquad \Lambda_{\rm IR} \;\ll\; M_i \;\ll\; \Lambda_{\rm spec}\,. \tag{3.14}$$

We denote by $\mathcal{H}$ the set of heavy multiplets to be integrated out at a renormalization scale $\mu \simeq M_i$.

2. Decoupling theorem in the spectral medium

Appelquist–Carazzone (spectral form). Let $\mathcal{O}$ be a gauge-invariant observable built from fields of momentum $p \sim \mu \ll M$. Integrating out a heavy multiplet H of mass M yields an EFT with Lagrangian

$$\mathcal{L}_{\rm EFT} \;=\; \mathcal{L}_{\rm light} \;+\; \sum_{d>4} \frac{c_d(\mu)}{M^{d-4}}\,\mathcal{O}_d, \tag{3.15}$$

and

$$\mathcal{O}(\mu)\Big|_{\rm full} - \mathcal{O}(\mu)\Big|_{\rm EFT} \;=\; \mathcal{O}\!\left(\frac{\mu^2}{M^2}\right), \qquad \text{(for fixed } g_{\rm eff},\ \text{anisotropies obeying Sec. 1.3).} \tag{3.16}$$

Hence heavy states affect light observables only through (i) *threshold* renormalization of relevant/marginal couplings, and (ii) higher-dimension operators suppressed by M.

3. Smooth thresholds via spectral occupancy

Occupancy functions. As introduced in (3.5), the effective counts of active DoF are *smooth* functions $\nu(\mu;\mathcal{S})$. A convenient, scheme-compatible parameterization is

$$\nu_x^{(R)}(\mu;\mathcal{S}) \;=\; \mathrm{mult}_x^{(R)}\;\vartheta_n\!\left(\frac{\mu}{M_{x,R}}\right), \qquad \vartheta_n(z) \;:=\; \frac{z^{2n}}{1+z^{2n}} \;=\; 1-\frac{1}{1+z^{2n}}\,, \tag{3.17}$$

with $x = f, s$ (fermion/scalar) and $n \geq 1$ setting how sharp the crossover is.[1] Inserting (3.17) into (3.9) implements *smooth* decoupling without spoiling background Ward identities.

4. Matching conditions at a hard threshold

Definition. When a sharp treatment is preferred, choose matching scale $\mu = M$ and split the theory into *above* (+) and *below* (−) regimes. The background-field scheme yields simple relations for the inverse couplings:

$$\frac{1}{g_I^2(\mu)}\Big|_- = \frac{1}{g_I^2(\mu)}\Big|_+ \;-\; \frac{b_I^{(H)}}{8\pi^2}\,\ln\frac{\mu}{M} \;+\; \Delta_I^{\rm geom} \;+\; \Delta_I^{\rm aniso} \;+\; \mathcal{O}\!\left(\frac{\mu^2}{M^2}\right), \tag{3.18}$$

$$b_I^{(H)} := \frac{4}{3}\sum_{f\in H} T_I(R_f) \;+\; \frac{1}{6}\sum_{s\in H} T_I(R_s) \;-\; \frac{11}{3}\sum_{v\in H} C_2(G_I)\,, \tag{3.19}$$

where H is the set of heavy multiplets crossing out at $\mu = M$ (including possible heavy vectors v in broken phases).[2] The *geometric* shift

$$\Delta_I^{\rm geom} \;=\; \Big[\kappa_G\,\mathcal{N}_I\Big]_+^-, \tag{3.20}$$

accounts for abrupt changes in the fiber normalization (e.g. if the coarse-grained $\langle \mathcal{M}_{II}\rangle_{\mathcal{F}}$ jumps at the matching surface). The *anisotropy* piece $\Delta_I^{\rm aniso}$ collects finite renormalizations of the CPT-even tensors in Sec. 1.3; to leading logs it can be neglected for isotropic flows.

[1] Heat-kernel derivations give equivalent forms: $\vartheta(z) = 1 - \mathrm{E}_1(z^2)/\mathrm{E}_1(0^+)$ or $\vartheta(z) = 1 - e^{-z^2}\sum_{k=0}^{n-1} z^{2k}/k!$. All admissible ϑ share the limits $\vartheta \to 0$ ($z \ll 1$) and $\vartheta \to 1$ ($z \gg 1$).

[2] For unbroken simple G_I, heavy vectors do not occur; they appear when integrating out massive gauge bosons below a breaking scale.

Matrix form with kinetic mixing. For a kinetic matrix $\mathbb{Z} = \mathbb{G}^{-2}$ (Sec. 3.1),

$$\mathbb{Z}_- \;=\; \mathbb{Z}_+ - \frac{1}{8\pi^2}\,\mathbb{B}^{(H)}\,\ln\frac{\mu}{M} + \Delta^{\text{geom}} + \Delta^{\text{aniso}}, \qquad \left(\mathbb{B}^{(H)}\right)_{IJ} = \delta_{IJ}\, b_I^{(H)} \text{ (+ mixing entries for multi-}U(1 \tag{3.21}$$

Diagonalize $\mathbb{Z}_\pm$ at each step to extract canonical g_I and mixing angles.

Electroweak mixing angle. A heavy set H induces a step in $\sin^2\theta_W$ at $\mu = M$,

$$\sin^2\theta_W\Big|_- \;=\; \sin^2\theta_W\Big|_+ \;+\; \frac{g^2 g'^2}{(g^2+g'^2)^2}\left[\frac{b_1^{(H)}}{8\pi^2}\ln\frac{\mu}{M} \;-\; \frac{b_2^{(H)}}{8\pi^2}\ln\frac{\mu}{M} \;+\; \Delta_1^{\text{geom}} - \Delta_2^{\text{geom}}\right], \tag{3.22}$$

plus small anisotropy corrections.

5. Worked threshold examples

(A) Vectorlike fermion multiplet. Add N_Ψ Dirac fermions in representation R of G_I with mass M_Ψ (from spectral binding). Then

$$b_I^{(H)} \;=\; \frac{4}{3}\,N_\Psi\,T_I(R), \qquad \Delta_I^{\text{geom}} \approx 0 \quad (\text{if } \langle \mathcal{M}_{II}\rangle_{\mathcal{F}} \text{ varies slowly at } M_\Psi). \tag{3.23}$$

This produces a *downward* kink in $1/g_I^2$ below M_Ψ, flattening asymptotic freedom (or accelerating Abelian screening).

(B) Scalar multiplet. A complex scalar in R with mass M_ϕ contributes $b_I^{(H)} = \frac{1}{6}T_I(R)$, typically a milder effect than fermions.

(C) Composite spectral resonance. A narrow composite vector in the adjoint (mass M_V) from the fiber sector—e.g. a Φ–χ^μ bound mode—acts as a heavy gauge boson below M_V, giving $b_I^{(H)} = -\frac{11}{3}C_2(G_I)$ in (3.19), i.e. an *upward* kink in $1/g_I^2$ (stronger asymptotic freedom) when integrated out. Its presence also shifts κ_G by a finite amount computable in the heat-kernel scheme, thus a nonzero Δ_I^{geom}.

(D) Smooth vs. sharp. Using ϑ_n with $n \geq 2$ in (3.17) reproduces the sharp matching formulas upon the replacement $\ln(\mu/M) \to \int d\ln\mu\,\partial_\mu\vartheta_n$, while avoiding artificial kinks in numerical evolution.

6. Universality classes of spectral RG flows

Definition (phenomenological). Two flows $\{g_I(\mu)\}$ and $\{\tilde{g}_I(\mu)\}$ belong to the same *universality class* if there exists a reparameterization of scales $\mu \to f(\mu)$ and a smooth homotopy of fiber data $(\mathcal{F}, h; \mathcal{S}) \to (\tilde{\mathcal{F}}, \tilde{h}; \tilde{\mathcal{S}})$ such that:

(i) The *dynamical* coefficients $b_I(\mu)$ of (3.4) are identical functions of the active representations (same light spectrum up to decoupling),

(ii) The *geometric* contributions $(\kappa_G \mathcal{N}_I)(\mu)$ differ only by an overall smooth rescaling and subleading $\mathcal{O}(\mu^2/\Lambda_{\text{spec}}^2)$,

(iii) All S/G/TQ/D gates are preserved along the homotopy.

Prototypical classes.

Class U (Unified-anchored). $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3$ at $\Lambda_{\rm spec}$ up to $\mathcal{O}(\delta Z)$. Flows mimic GUT-like boundary conditions without introducing a new simple group; deviations are driven by thresholds and mild geometric splitting.

Class S (Split-anchored). $\mathcal{N}_I$ differ at $\Lambda_{\rm spec}$ by order-one factors set by fiber isometries; nevertheless, correlated thresholds (e.g. entire $SU(5)$-like multiplets of vectorlike matter) drive *near*-unification in an intermediate window.

Class A (Anisotropy-tolerant). Small CPT-even anisotropies of Sec. 1.3 are present; they do not change one-loop logs but renormalize finite parts and higher-dimension operators. Precision constraints bound them yet allow the same leading RG class as U or S.

Fixed points and walking. In non-Abelian sectors with many light flavors, the two-loop β may admit an IR Banks–Zaks fixed point. In the spectral framework, the location of such a point depends only on the *dynamical* part (field content) to leading order; geometric pieces shift the overall normalization but not the existence of the fixed point, hence it is a universality property.

7. Practical RG algorithm with thresholds

Step-by-step.

1. Choose $\Lambda_{\rm spec}$ and fix $\mathcal{N}_I$ from fiber geometry; set $g_I(\Lambda_{\rm spec})$ via (3.2).
2. List heavy multiplets H with masses $M_i(\mathcal{S})$; pick either sharp matching at $\mu = M_i$ (use (3.18)–(3.21)) or smooth occupancies ϑ_n (use (3.17) in (3.9)).
3. Evolve down using the matrix flow (3.8); when needed, re-diagonalize $\mathbb{Z}$ to extract canonical g_I and $\sin^2\theta_W$ via (3.10).
4. Check S/G/TQ/D gates at each step (hyperbolicity, positivity bounds, small anisotropies).
5. Identify the universality class (U/S/A) by comparing (b_I) patterns and geometric anchors; proceed to unification analysis (Sec. 3.4).

8. Summary

Heavy spectral states decouple smoothly, encoded by occupancy functions (3.17), or sharply via matching formulas (3.18)–(3.21). Geometric shifts $\Delta_I^{\rm geom}$ capture changes in fiber normalization, while anisotropy effects enter as finite, constrained deformations. RG patterns cluster into a few universality classes determined by (i) the light spectrum (dynamical logs) and (ii) the fiber anchor $(\mathcal{N}_I, \kappa_G)$. These ingredients furnish a controlled pipeline from spectral microphysics to coupling evolution, paving the way to assess (near-)unification windows and phenomenological constraints in Sec. 3.4.

3.3 Positivity/causality gates along the flow

Aim. We give a *flow-compatible* formulation of the S/G/TQ/D gates for the spectral gauge sector: (S) stability (no ghosts/gradient instabilities), (G) hyperbolicity/causality (light-cone inclusion), (TQ) unitarity/analyticity (forward-limit positivity), and (D) dissipative admissibility (signs of absorptive parts). We show how these conditions are preserved under (i) RG running of g_I and kinetic mixing, (ii) smooth/sharp thresholds, and (iii) small anisotropies from χ^μ (Sec. 1.3). The outcome is a set of *algebraic* and *dispersive* inequalities that define an RG-stable admissible region in parameter space.

1. Principal symbol, cones, and hyperbolicity

Principal symbol. With the CPT-even, two-derivative constitutive deformations of Sec. 1.3, the quadratic gauge Lagrangian in momentum space reads (suppressing adjoint indices)

$$\mathcal{L}^{(2)} \;=\; -\tfrac{1}{2}\, A_\mu\, \mathcal{K}^{\mu\nu}(q)\; A_\nu\,, \qquad \mathcal{K}^{\mu\nu}(q) \;=\; Z\,(q^2 g_{\rm eff}^{\mu\nu} - q^\mu q^\nu) \;+\; \widehat{\alpha}^{\mu\nu}(q;\mathcal{S}), \tag{3.24}$$

where $Z>0$ (or a positive-definite matrix Z_{IJ}) is the kinetic normalization, and $\widehat{\alpha}^{\mu\nu}$ collects the small anisotropic tensors proportional to (α,β,γ) of Eq. (1.49). The *principal symbol* is

$$\mathscr{P}(n) \;:=\; \Pi_T^{\mu\nu}(n)\,\mathcal{K}_{\mu\nu}(n)\,, \qquad \Pi_T^{\mu\nu}(n) := g_{\rm eff}^{\mu\nu} - \frac{n^\mu n^\nu}{n^2}, \tag{3.25}$$

evaluated on covectors n_μ transverse to gauge redundancy.

Hyperbolicity/causal cones (gate G). *Strict hyperbolicity* requires that for any space-like n_μ (w.r.t. $g_{\rm eff}$) the eigenvalues of $\mathscr{P}(n)$ be strictly positive. Equivalently, the signal (group) speeds $c_{T,\pm}^2$ of Eq. (1.55) satisfy

$$0 \;<\; c_{T,-}^2 \;\le\; c_{T,+}^2 \;\le\; 1 \qquad \text{for all directions}\,, \tag{3.26}$$

i.e. the gauge cone lies *inside* the $g_{\rm eff}$ cone. In the isotropic limit $\widehat{\alpha}^{\mu\nu}\to 0$, $c_{T,\pm}^2\to 1$.

Cone-preservation lemma. *If* $Z_{IJ}(\mu)$ is positive-definite and $\|\widehat{\alpha}\|(\mu)\le\varepsilon(\mu)$ with $\varepsilon(\mu)<Z_{\min}(\mu)$, *then* the inclusion (3.26) holds. Moreover, under an infinitesimal RG step $\mu\to\mu e^{d\ell}$ generated by smooth β-functions, the map

$$(Z,\widehat{\alpha}) \;\mapsto\; (Z+\delta Z,\; \widehat{\alpha}+\delta\widehat{\alpha})$$

preserves strict positivity of $\mathscr{P}$ provided $Z_{\min}+\delta Z_{\min} > \varepsilon+\delta\varepsilon$. Since δZ arises from wave-function renormalization and threshold matching (Sec. 3.2), and $\delta\widehat{\alpha}$ from higher-dimension, finite renormalizations, the inequality is maintained by continuity for sufficiently small $d\ell$.

2. Stability and reflection positivity

Hamiltonian positivity (gate S). The gauge-field Hamiltonian density extracted from (1.49) is quadratic in electric/magnetic fields,

$$\mathcal{H} \;=\; \tfrac{1}{2}\,(E^i,\; B^i)\begin{pmatrix} \boldsymbol{\epsilon} & \boldsymbol{\zeta} \\ \boldsymbol{\zeta}^\top & \boldsymbol{\mu}^{-1}\end{pmatrix}\begin{pmatrix} E^i \\ B^i\end{pmatrix}, \qquad \boldsymbol{\epsilon},\; \boldsymbol{\mu}^{-1}\succ 0, \quad \boldsymbol{\epsilon}-\boldsymbol{\zeta}\,\boldsymbol{\mu}\,\boldsymbol{\zeta}^\top\succ 0, \tag{3.27}$$

where the 3×3 blocks are functions of (Z,α,β,γ) and adjoint indices (Schur complement conditions). *Gate S* demands the 6×6 matrix be positive-definite at all μ.

Euclidean reflection positivity. In the Euclidean continuation on $(M, g_{\rm eff})$, the two-point function admits a Källén–Lehmann representation whose spectral measure is nonnegative on the BRST cohomology (Sec. 2.4), ensuring reflection positivity if and only if (3.27) holds. Thus (S) $\Leftrightarrow$ positivity of the Euclidean measure for transverse modes.

3. Analyticity/forward-limit positivity (gate TQ)

Forward amplitude. Consider $2 \to 2$ scattering of transversely polarized gauge bosons in a fixed adjoint channel $\mathcal{R}$ and fixed external polarization projector Π_T. The (crossing-symmetric) forward amplitude $\mathcal{M}_{\mathcal{R}}(s) \equiv \mathcal{M}(s, t{=}0)$ obeys, under standard assumptions (causality, locality, mass gap or sufficient subtractions):

$$\mathcal{M}''_{\mathcal{R}}(0) \;=\; \frac{2}{\pi}\int_{s_{\rm th}}^{\infty} \frac{ds}{s^3}\; \mathrm{Im}\;\mathcal{M}_{\mathcal{R}}(s) \;\geq\; 0, \tag{3.28}$$

by the optical theorem ($\mathrm{Im}\,\mathcal{M} \propto$ total cross section ≥ 0). Matching to the low-energy EFT gives quadratic constraints on Wilson matrices of $(\mathcal{F}^2)^2$ operators:

$$\boxed{\mathbf{C}_{\mathcal{R}} \;\succeq\; 0\,, \qquad \big[\mathbf{C}_{\mathcal{R}}\big]_{(IJ)(KL)} \text{ from } \;\Delta\mathcal{L} \supset \tfrac{1}{\Lambda^4}\, C^{IJ,KL}(\mathcal{F}^I_{\mu\nu}\mathcal{F}^{J\,\mu\nu})(\mathcal{F}^K_{\rho\sigma}\mathcal{F}^{L\,\rho\sigma})\;.} \tag{3.29}$$

In bases with kinetic mixing, (3.29) means *all* principal minors of $\mathbf{C}$ are nonnegative.

Sum-rule form. Equation (3.28) yields RG-invariant, positive sum rules:

$$\mathbf{v}^{\top}\mathbf{C}_{\mathcal{R}}\,\mathbf{v} \;=\; \frac{1}{\pi}\int_{s_{\rm th}}^{\infty}\frac{ds}{s^3}\;\sigma_{\mathcal{R}}(s;\mathbf{v}) \;\geq\; 0 \quad\Longrightarrow\quad \mathbf{C}_{\mathcal{R}} \succeq 0, \tag{3.30}$$

for any polarization/vector $\mathbf{v}$ in group space. Since the r.h.s. is a *positive* integral, RG evolution (which redistributes spectral weight but keeps $\sigma \geq 0$) preserves (3.29).

4. Dissipation gate (D) and causal Kramers–Kronig

Retarded correlators. For a conserved adjoint current J^μ_I, the retarded two-point function obeys Kramers–Kronig relations,

$$\mathrm{Re}\;\Pi^{\mu\nu}_{IJ}(\omega,\mathbf{q}) - \mathrm{Re}\;\Pi^{\mu\nu}_{IJ}(0,\mathbf{q}) = \frac{2}{\pi}\,\mathrm{P}\!\int_0^{\infty} d\omega'\;\omega'\;\frac{\mathrm{Im}\;\Pi^{\mu\nu}_{IJ}(\omega',\mathbf{q})}{\omega'^2-\omega^2}. \tag{3.31}$$

Gate D demands $\omega^{-1}\,\mathrm{Im}\;\Pi^{\rm ret} \geq 0$ for $\omega > 0$ (positive spectral densities), which is equivalent to nonnegative total cross sections via the optical theorem. Medium-induced dissipative coefficients (conductivities à la Kubo) are therefore ≥ 0 and RG-stable.

5. Threshold matching respects positivity

Heavy states and EFT coefficients. Integrating out a heavy multiplet H at mass M shifts the $(\mathcal{F}^2)^2$ Wilson matrix by

$$\Delta\mathbf{C}_{\mathcal{R}} \;=\; \frac{1}{\pi}\int_{s_{\rm th}}^{\infty}\frac{ds}{s^3}\;\Delta\sigma^{(H)}_{\mathcal{R}}(s) \;\succeq\; 0, \tag{3.32}$$

because the extra contribution to the total cross section $\Delta\sigma^{(H)} \geq 0$. Hence *matching moves* $\mathbf{C}$ *deeper* into the positive cone, never out of it. This holds for vectorlike fermions/scalars and for narrow composite resonances (Sec. 3.2).

Kinetic normalization. Sharp matching of Z_{IJ} (Eq. (3.21)) preserves positive-definiteness because the radiative correction to the background two-point function adds a Hermitian matrix with positive real part to $\mathbb{Z}$ in a background-covariant scheme.

6. Anisotropies and microcausality bounds

Small-χ^μ regime. Write $\widehat{\alpha}^{\mu\nu}$ in the rest frame u^μ of the medium and decompose into irreducible, CPT-even pieces (cf. Sec. 1.3). The dispersion relations give (no sum on $\pm$)

$$c^2_{T,\pm}(\hat{\mathbf{q}}) \;=\; 1 \;+\; \delta c^2_{T,\pm}(\hat{\mathbf{q}}), \qquad \delta c^2_{T,\pm} \;=\; \mathcal{O}\!\left(\frac{\|\widehat{\alpha}\|}{Z}\right). \tag{3.33}$$

Gate G imposes $\delta c^2_{T,\pm} \le 0$ (subluminal cone) and $|\delta c^2_{T,+} - \delta c^2_{T,-}| = \mathcal{O}(\|\widehat{\alpha}\|/Z)$ (controlled birefringence). Under RG, $\|\widehat{\alpha}\|$ renormalizes only by finite, higher-derivative pieces at one loop; thus if (3.33) holds at μ_0, it continues to hold for μ in a neighborhood.

Energy-positivity constraints. In the 6×6 block form (3.27), the Schur conditions imply, to first order in anisotropies,

$$\delta\boldsymbol{\epsilon} \;\succ\; 0, \qquad \delta\boldsymbol{\mu}^{-1} \;\succ\; 0, \qquad \|\boldsymbol{\zeta}\| \;\le\; \sqrt{\|\delta\boldsymbol{\epsilon}\|\,\|\delta\boldsymbol{\mu}^{-1}\|}, \tag{3.34}$$

with norms taken in the adjoint Hilbert structure. These inequalities are RG-stable under smooth running.

7. RG-stable admissible region and practical checklist

Convexity and stability. The set

$$\mathfrak{A} \;:=\; \Big\{(Z_{IJ},\ \widehat{\alpha},\ \mathbf{C}) \;\Big|\; Z \succ 0,\ \mathscr{P} > 0,\ \mathbf{C} \succeq 0,\ (3.34)\ \text{hold}\Big\}$$

is a *convex cone*. Steps of RG running and threshold matching act as (i) positive rescalings of Z plus Hermitian additions, (ii) finite shifts of $\widehat{\alpha}$ bounded by (3.33), and (iii) positive-semidefinite updates of $\mathbf{C}$ per (3.32). Therefore $\mathfrak{A}$ is *invariant* under the flow to leading-log accuracy.

Checklist (per RG step $\mu \to \mu e^{d\ell}$).

1. Diagonalize $\mathbb{Z}$: verify $Z_{\min} > 0$ and $Z_{\min} > \|\widehat{\alpha}\|$.
2. Compute $c^2_{T,\pm}(\hat{\mathbf{q}})$: check $0 < c^2_{T,-} \le c^2_{T,+} \le 1$.
3. Update $\mathbf{C}$ (or its eigenvalues): verify $\mathbf{C} \succeq 0$.
4. If crossing a threshold, apply (3.21) and (3.32); recheck items 1–3.
5. Monitor Euclidean reflection positivity by testing (3.27) (Schur conditions (3.34)).

8. Summary

Causality and unitarity constraints can be organized into algebraic (SPD/Schur) and dispersive (sum-rule) inequalities that define a convex, RG-invariant admissible set for the spectral gauge sector. Hyperbolicity is enforced by cone inclusion (3.26); Hamiltonian/Euclidean positivity by (3.27); forward-limit analyticity by $\mathbf{C} \succeq 0$ in (3.29); and dissipative consistency by Kramers–Kronig (3.31). Threshold matching and smooth decoupling preserve these properties, ensuring that once the gates are satisfied at a scale μ_0, they remain satisfied along the spectral RG flow.

3.4 Prospects for (near-)unification and constraints

Aim. We synthesize the ingredients of Secs. 3.1–3.3 to (i) define quantitative criteria for *(near-)unification* of gauge couplings in the spectral framework, (ii) derive analytic estimates for putative unification scales including geometric and threshold effects, (iii) outline a practical scanning/fit pipeline, and (iv) state consistency constraints (perturbativity, Landau-pole avoidance, positivity/causality gates, precision data). The guiding idea is that the *UV anchor* from fiber geometry (Sec. 1.5)—notably $\mathcal{N}_I$ and the relation (3.11)—can seed unification patterns without invoking a simple GUT group, while thresholds from spectral composites steer the low-energy values.

1. Quantifying (near-)unification

Pairwise distances and tolerance. Define the (dimensionless) pairwise distances among inverse couplings

$$\Delta_{ij}(\mu) \;:=\; \frac{\left|\, g_i^{-2}(\mu) - g_j^{-2}(\mu) \,\right|}{\overline{g^{-2}}(\mu)}\,, \qquad \overline{g^{-2}}(\mu) \;:=\; \tfrac{1}{3}\sum_{k=1}^{3} g_k^{-2}(\mu)\,, \tag{3.35}$$

and the unification score

$$\epsilon_{\mathrm{unif}}(\mu) \;:=\; \sqrt{\tfrac{1}{3}\sum_{i<j}\Delta_{ij}^2(\mu)}\,. \tag{3.36}$$

We say the flow exhibits *near-unification* in a window $\mu \in [\mu_1, \mu_2]$ if $\epsilon_{\mathrm{unif}}(\mu) \le \varepsilon_\star$ for some tolerance $\varepsilon_\star \ll 1$ (e.g. 1%–5%), with all gates of Sec. 3.3 satisfied.

Anchor vs. dynamical convergence. A *unified-anchored* fiber (Class U; Sec. 3.2) sets $\mathcal{N}_1 \simeq \mathcal{N}_2 \simeq \mathcal{N}_3$ at Λ_{spec}, so $\epsilon_{\mathrm{unif}}(\Lambda_{\mathrm{spec}}) \approx 0$ up to small δZ_{IJ}. Conversely, in *split-anchored* fibers (Class S) the running and thresholds may nonetheless drive $\epsilon_{\mathrm{unif}} \to 0$ at an intermediate scale.

2. Analytic estimates for the unification scale

Baseline two-line crossing (without geometry). Neglecting geometric running and kinetic mixing for the moment, the one-loop solution of (3.9) gives the pairwise crossing scale of (i, j):

$$\ln\frac{\Lambda_{ij}}{\mu_0} \;=\; \frac{2\pi}{b_j - b_i}\left[\, \alpha_i^{-1}(\mu_0) \;-\; \alpha_j^{-1}(\mu_0) \,\right] \;+\; \mathcal{O}(\text{thresholds}), \tag{3.37}$$

with $\alpha_I = g_I^2/4\pi$ and b_I the one-loop coefficients from (3.4) (expressed in the same convention). Exact unification requires $\Lambda_{12} \simeq \Lambda_{23} \simeq \Lambda_{13}$.

Including thresholds and geometry (leading). Let $\delta_{ij}^{\rm thr}$ denote the net threshold correction between μ_0 and Λ (sharp matching: Eq. (3.18); smooth matching: integrate occupancies). Let $\delta_{ij}^{\rm geom}(\mu_0 \to \Lambda)$ be the geometric shift from (3.6). Then

$$\boxed{\ln \frac{\Lambda_{ij}}{\mu_0} \;=\; \frac{2\pi}{b_j - b_i}\left[\, \alpha_i^{-1}(\mu_0) - \alpha_j^{-1}(\mu_0) \;+\; \delta_{ij}^{\rm geom} \;+\; \delta_{ij}^{\rm thr} \,\right] \;+\; \mathcal{O}(\text{mixing})\,.} \tag{3.38}$$

A necessary condition for (near-)unification is the *compatibility* of the three Λ_{ij} computed from (3.38).

Hypercharge normalization. If the physical $U(1)_Y$ is embedded with normalization k_Y relative to the canonical fiber basis (so that $g_1^2 = k_Y\, g_Y^2$), one replaces $\alpha_1^{-1} \mapsto \alpha_1^{-1}/k_Y$. In the spectral framework k_Y is fixed by the fiber Killing norms: $k_Y = \mathcal{N}_1/\mathcal{N}_Y$.

3. Practical scanning/fit pipeline

Inputs. Fix a reference scale μ_0 (e.g. an electroweak anchor), measured $\{\alpha_1(\mu_0), \alpha_2(\mu_0), \alpha_3(\mu_0)\}$ in the chosen normalization, and spectral fiber data $(\mathcal{N}_I, \kappa_G)$ at $\Lambda_{\rm spec}$. Specify potential heavy multiplets H with masses $M_i(\mathcal{S})$.

Algorithm.

1. **Initialization**: Set boundary conditions at $\Lambda_{\rm spec}$ via (3.2) (Class U or S anchor).
2. **Evolution**: Run down to μ_0 with the matrix RG (3.8), implementing either sharp matching (Eqs. (3.21), (3.22)) or smooth occupancies (3.17). Optionally include two-loop dynamical pieces in the standard way.
3. **Fit**: Minimize a global objective (with nuisance priors for thresholds)

$$\chi^2 \;=\; \sum_{I=1}^{3} \frac{\left[\alpha_I^{-1}(\mu_0) - \alpha_{I,\rm exp}^{-1}(\mu_0)\right]^2}{\sigma_I^2} \;+\; \lambda_\varepsilon\, \epsilon_{\rm unif}^2(\Lambda_*) \;+\; \lambda_{\rm gates}\, \mathcal{P}_{\rm gates}, \tag{3.39}$$

   where Λ_* is the scale minimizing $\epsilon_{\rm unif}$, $\mathcal{P}_{\rm gates}$ penalizes any violation of the S/G/TQ/D conditions (Sec. 3.3), and λ's are weights.
4. **Diagnostics**: Report $(\Lambda_*, \epsilon_{\rm unif}(\Lambda_*))$, the pairwise Λ_{ij} from (3.38), and the predicted $\sin^2\theta_W(\mu)$ from (3.10).

4. Constraints and admissibility

Perturbativity and Landau poles. For each factor G_I, require

$$\alpha_I(\mu) \;<\; \alpha_{\rm max} \;\ll 4\pi \qquad \text{for all} \quad \mu \in [\mu_{\rm IR}, \Lambda_{\rm UV}], \tag{3.40}$$

with $\alpha_{\max}$ a conservative bound ensuring loop control. For Abelian $U(1)$, avoid a Landau pole below $\Lambda_{\rm UV}$:

$$\alpha_1^{-1}(\Lambda_{\rm UV}) \;=\; \alpha_1^{-1}(\mu_0) \;-\; \frac{b_1^{\rm eff}}{2\pi}\ln\frac{\Lambda_{\rm UV}}{\mu_0} \;+\; \delta_1^{\rm geom} \;+\; \delta_1^{\rm thr} \;>\; 0, \tag{3.41}$$

with $b_1^{\rm eff} < 0$ at one loop (screening).

Positivity/causality gates. At the Λ_* minimizing $\epsilon_{\rm unif}$, check the gates of Sec. 3.3: (G) cone inclusion $c_{T,\pm}^2 \le 1$; (S) SPD Hamiltonian matrix (3.27); (TQ) forward-positivity $\mathbf{C} \succeq 0$; (D) nonnegative absorptive parts. These conditions restrict allowed anisotropies and forbid exotic threshold patterns that would violate analyticity bounds.

Precision electroweak and low-energy. Geometric deformations and thresholds shift oblique parameters (S, T, U) and triple/quartic gauge couplings (Sec. 9). Impose

$$|S|,\ |T|,\ |U| \;\lesssim\; \text{few} \times 10^{-2}, \qquad \delta(g_{1,2})/g_{1,2} \text{ within global-fit tolerances}, \tag{3.42}$$

as priors in (3.39). Vectorlike thresholds must also respect flavor/EDM and collider bounds (to be treated in Part VII).

Baryon violation (model dependent). In the absence of a simple GUT with X, Y bosons, the spectral framework need not generate dimension-six proton-decay operators. If heavy baryon-violating composites exist, their mass M_B and coupling c_B must satisfy

$$\frac{M_B}{\sqrt{|c_B|}} \;\gtrsim\; \Lambda_{\rm B}\,, \tag{3.43}$$

with $\Lambda_{\rm B}$ a phenomenological lower bound. Equation (3.43) is included as a constraint only in spectra that allow such operators.

5. Structural correlates and predictions

Geometric weak mixing. If $\mathcal{N}_1/\mathcal{N}_2$ is fixed at $\Lambda_{\rm spec}$, then (3.11) gives a geometric boundary value for $\sin^2\theta_W$, propagated to μ_0 by (3.10). Thus $(\mathcal{N}_1/\mathcal{N}_2)$ is directly testable via precision electroweak fits (with threshold priors).

Correlated thresholds. If heavy matter comes in *complete* multiplets of a putative embedding (e.g. $SU(5)$-like content, even without a GUT), then $b_I^{(H)}$ shifts satisfy

$$b_1^{(H)} : b_2^{(H)} : b_3^{(H)} \text{ fixed (representation theory)} \quad\Longrightarrow\quad \delta_{ij}^{\rm thr} \text{ correlated, easing } \Lambda_{12} \simeq \Lambda_{23}. \tag{3.44}$$

This is a robust route to near-unification in Class S anchors.

Anisotropy–EFT link. Allowed birefringent deformations (Sec. 1.3) map onto definite linear combinations of $(\mathcal{F}^2)^2$ Wilson coefficients constrained by (3.29). Hence any detectable deviation in gauge-boson propagation is accompanied by a *positive* shift in specific EFT operators—an experimentally falsifiable correlation.

6. Benchmarks

Benchmark U (unified anchor). Take $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3$ at $\Lambda_{\rm spec}$, no extra light matter. Then $\epsilon_{\rm unif}(\Lambda_{\rm spec}) \approx 0$; running down reproduces the measured $(\alpha_1, \alpha_2, \alpha_3)$ if and only if threshold corrections near the electroweak/TeV scale account for the known small misalignments. $\Lambda_{\rm spec}$ must satisfy (3.41) and gates in Sec. 3.3.

Benchmark S (split anchor with correlated thresholds). Choose $\mathcal{N}_I$ split at $\Lambda_{\rm spec}$ but add N_Ψ vectorlike multiplets in complete representations at masses M_Ψ. With (3.23) and (3.44), a window Λ_* with $\epsilon_{\rm unif} \lesssim$ few% is achievable while maintaining perturbativity and precision constraints.

7. Summary

(near-)Unification in the spectral framework is controlled by the interplay of (i) *geometric* anchors $(\mathcal{N}_I, \kappa_G)$ at $\Lambda_{\rm spec}$, (ii) *dynamical* logarithms fixed by field content, and (iii) *threshold* corrections from spectral composites. The quantitative criteria (3.35)–(3.36) and the analytic estimator (3.38) guide efficient searches; gates (S/G/TQ/D) and perturbativity bound the admissible region. In Class U anchors, geometry pre-aligns couplings in the UV; in Class S anchors, correlated thresholds can produce *near*-crossing windows. Precision electroweak data and low-energy constraints sharpen the viable domain, yielding concrete, falsifiable predictions for $\sin^2\theta_W$, running patterns, and possible correlated EFT deformations.

Part II

Anomalies and 't Hooft Matching

Chapter 4

ABJ, Gauge–Gravitational, and Global Anomalies

4.1 Anomaly polynomial and descent in the spectral setting

Aim. We formulate gauge and mixed gauge–gravitational anomalies for chiral spectral matter on the emergent spacetime (M, g_{eff}) with internal fiber isometries G (Ch. 1). The anomalies are encoded by a closed, invariant *characteristic form* (the *anomaly polynomial*) in two higher dimensions; its Chern–Simons transgression yields the *consistent* anomaly via the *descent* equations. We make the BRST content explicit, distinguish consistent vs. covariant anomalies (Bardeen–Zumino shift), and state explicit $d = 4$ formulae including the mixed gauge–gravitational term.

1. Spectral chiral matter and the origin of anomalies

Setup. Let Ψ denote a set of Weyl fermions (or other chiral fields) transforming in complex representations R of G (the isometry group of the fiber $(\mathcal{F}, h)$) and minimally coupled to the gauge connection $\mathcal{A} = \mathcal{A}^I T_I$ (Sec. 1.4) and to the spin connection ω of (M, g_{eff}). The Euclidean generating functional after integrating out Ψ reads

$$e^{-W[\mathcal{A},\omega;\mathcal{S}]} \;=\; \prod_R \det{}_{\text{chiral}}\Big(\not{D}_R[\mathcal{A},\omega;\mathcal{S}]\Big)^{\sigma_R}, \qquad \sigma_R = \pm 1 \;\; \text{(L/R chirality)}, \tag{4.1}$$

where $\not{D}_R$ is the relevant Dirac operator (including spectral medium couplings consistent with Sec. 1.3). Under an infinitesimal gauge transformation $\delta_\epsilon \mathcal{A} = -D\epsilon$ and a local Lorentz transformation $\delta_\theta \omega = -D\theta$, the *measure* may acquire a nontrivial Jacobian. The failure of W to be invariant defines the *anomaly*:

$$\delta W \;=\; \int_{M_{2n}} \Big(\epsilon^I \, \mathcal{A}_I^{\text{cons}} \;+\; \tfrac{1}{2}\theta^{ab} \, \mathcal{A}_{ab}^{\text{cons}} \Big), \tag{4.2}$$

in even spacetime dimension $d = 2n$. We focus on $d = 4$ (so $n = 2$), but present general formulas first.

2. Anomaly polynomial: definition and conventions

Characteristic classes. Let $F = d\mathcal{A} + \mathcal{A}^2$ be the gauge curvature and $R = d\omega + \omega^2$ the Riemann curvature two-form (with wedge products implicit and traces in the appropriate

representations). We adopt the standard normalizations

$$\mathrm{ch}_R(F) \;:=\; \mathrm{Tr}_R \exp\!\Big(\frac{i\,F}{2\pi}\Big), \qquad \widehat{A}(R) \;:=\; \prod_j \frac{\frac{x_j}{2}}{\sinh\frac{x_j}{2}} \;=\; 1 - \frac{1}{24}\,p_1(R) + \cdots\,, \tag{4.3}$$

where $\{\pm i x_j\}$ are the skew-eigenvalues of R and $p_1(R) = -\frac{1}{2\pi^2}\,\mathrm{tr}\,R^2$ in our sign convention. The *anomaly polynomial* in $2n{+}2$ dimensions is the $(2n{+}2)$-form

$$\boxed{\; I_{2n+2} \;=\; \Big[\; \sum_R \sigma_R\, \mathrm{ch}_R(F)\;\widehat{A}(R)\;\Big]_{2n+2} \;} \tag{4.4}$$

(equal to the index density of the chiral Dirac operator). It is closed ($dI_{2n+2} = 0$) and invariant under gauge/Lorentz transformations.

Four dimensions ($n = 2$). Expanding (4.4) to six-form order gives

$$I_6 \;=\; \sum_R \sigma_R \left(\frac{i}{2\pi}\right)^3 \left[\frac{1}{6}\,\mathrm{Tr}_R(F^3) \;+\; \frac{1}{24}\,\mathrm{Tr}_R(F)\;\mathrm{tr}(R^2)\right], \tag{4.5}$$

where Tr_R denotes the trace in R (with generators normalized as $\mathrm{Tr}_R(T^a T^b) = T(R)\,\delta^{ab}$) and tr is over tangent-space indices. *Remarks:* (i) The first term produces ABJ gauge anomalies. (ii) The second term is the mixed gauge–gravitational anomaly. (iii) Pure gravitational anomalies vanish in $d = 4$ for fermions, but appear in $d = 2 \bmod 4$; we include $\mathrm{tr}(R^2)$ to prepare the descent.

3. Descent and the consistent anomaly

Chern–Simons transgression. Because I_{2n+2} is exact on contractible patches, there exists a Chern–Simons $(2n{+}1)$-form Q_{2n+1} such that

$$d\,Q_{2n+1}(\mathcal{A}, \omega) \;=\; I_{2n+2}(F, R). \tag{4.6}$$

Under a gauge/Lorentz variation,

$$\delta Q_{2n+1} \;=\; d\,I^{(1)}_{2n}(\epsilon, \theta; \mathcal{A}, \omega), \tag{4.7}$$

which *defines* the *descent* form $I^{(1)}_{2n}$. The *consistent* anomaly is then

$$\boxed{\; \delta W \;=\; 2\pi i \int_{M_{2n}} I^{(1)}_{2n}(\epsilon, \theta; \mathcal{A}, \omega) \;} \tag{4.8}$$

and obeys the Wess–Zumino consistency conditions because I_{2n+2} is closed and invariant.

Explicit $d = 4$ expressions. Suppressing wedges and overall $(i/2\pi)^3$ for brevity, a convenient choice is

$$Q_5^{\mathrm{gauge}}(\mathcal{A}) = \sum_R \sigma_R\,\frac{1}{6}\,\mathrm{Tr}_R\Big(\mathcal{A}\,d\mathcal{A}\,d\mathcal{A} + \tfrac{3}{2}\,\mathcal{A}^3 d\mathcal{A} + \tfrac{3}{5}\,\mathcal{A}^5\Big), \tag{4.9}$$

$$Q_5^{\mathrm{mix}}(\mathcal{A}, \omega) = \sum_R \sigma_R\,\frac{1}{24}\,\mathrm{Tr}_R(\mathcal{A})\;\mathrm{tr}\Big(\omega\,d\omega + \tfrac{2}{3}\,\omega^3\Big), \tag{4.10}$$

so that $d(Q_5^{\rm gauge}+Q_5^{\rm mix})=I_6$. Varying $\mathcal{A}\mapsto\mathcal{A}-D\epsilon$ and $\omega\mapsto\omega-D\theta$ yields the consistent $I_4^{(1)}$:

$$I_4^{(1)}\Big|_{\rm gauge}=\sum_R\sigma_R\,\frac{1}{6}\,{\rm Tr}_R\Big(\epsilon\,d\mathcal{A}\,d\mathcal{A}+\tfrac{3}{2}\,\epsilon\,\mathcal{A}^2d\mathcal{A}+\tfrac{3}{5}\,\epsilon\,\mathcal{A}^4\Big), \tag{4.11}$$

$$I_4^{(1)}\Big|_{\rm mix}=\sum_R\sigma_R\,\frac{1}{24}\,{\rm Tr}_R(\epsilon)\;{\rm tr}(R^2)\;+\;\sum_R\sigma_R\,\frac{1}{24}\,{\rm Tr}_R(F)\;{\rm tr}(D\theta\,\omega). \tag{4.12}$$

The purely gauge piece (4.11) reduces, in components, to the familiar ABJ divergence $\propto d_R^{abc}\,F^b\tilde{F}^c$ (with $d_R^{abc}={\rm Tr}_R\{T^a,T^b\}T^c$), while (4.12) gives the mixed term $\propto{\rm Tr}_R(\epsilon)\;{\rm tr}(R\wedge R)$.

4. BRST cohomology and Wess–Zumino consistency

BRST form. Promote $\epsilon\to c$ and $\theta\to\chi$ to ghosts; the BRST differential s acts as $s\mathcal{A}=Dc$, $sc=-\frac{1}{2}[c,c]$, $s\omega=D\chi$, $s\chi=-\frac{1}{2}[\chi,\chi]$. The descent becomes a cohomological ladder:

$$d\,I_{2n+1}^{(0)}=I_{2n+2},\qquad s\,I_{2n+1}^{(0)}+d\,I_{2n}^{(1)}=0,\qquad s\,I_{2n}^{(1)}+d\,I_{2n-1}^{(2)}=0,\;\ldots \tag{4.13}$$

and the *anomalous* Zinn–Justin identity reads

$$\mathcal{S}(\Gamma)\;=\;2\pi i\int_{M_{2n}}I_{2n}^{(1)}(c,\chi;\mathcal{A},\omega). \tag{4.14}$$

The Wess–Zumino consistency conditions are equivalent to $s^2=0$ modulo exact forms, i.e. the anomaly is a nontrivial element of the local BRST cohomology $H^1_{\rm loc}(s|d)$.

5. Consistent vs. covariant anomalies (Bardeen–Zumino)

BZ polynomial and current shift. The consistent anomaly (4.8) follows from the variation of a local functional, hence satisfies WZ consistency but yields *consistent* currents $J^\mu_{\rm cons}$ that do not transform covariantly. Adding the Bardeen–Zumino local polynomial $X_{2n}^{\rm BZ}(\mathcal{A},\omega)$ shifts currents

$$J_{\rm cov}\;=\;J_{\rm cons}\;+\;\frac{\delta}{\delta\mathcal{A}}\int X_{2n}^{\rm BZ},\qquad \delta W\Big|_{\rm cov}\;=\;2\pi i\int I_{2n}^{(1)}\Big|_{\rm cov}, \tag{4.15}$$

producing the (manifestly) *covariant* anomaly. In $d=4$, this changes the ABJ coefficient by a factor (e.g. $1/24\pi^2\to1/32\pi^2$ in standard normalizations) while preserving physical content. We will use the *consistent* form for algebraic cancellation conditions and the *covariant* form for transport/response.

6. Explicit $d=4$ component formulae

Non-Abelian ABJ (consistent). For a simple factor with generators T^a,

$$\partial_\mu J^{a\,\mu}_{\rm cons}\;=\;\frac{1}{24\pi^2}\sum_R\sigma_R\,d_R^{abc}\;F^b_{\mu\nu}\,\widetilde{F}^{c\,\mu\nu},\qquad \widetilde{F}^{\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{4.16}$$

in a convention where ${\rm Tr}_R(T^aT^b)=\frac{1}{2}\delta^{ab}$ gives $d_R^{abc}={\rm Tr}_R(T^a\{T^b,T^c\})$.

Mixed gauge–gravitational (consistent). For an Abelian factor (or diagonal generator) with charge matrix Q_R,

$$\partial_\mu J^\mu_{\rm cons}\Big|_{\rm mix} = \frac{1}{192\pi^2}\sum_R \sigma_R\,{\rm Tr}_R(Q_R)\;R_{\mu\nu\rho\sigma}\,\widetilde{R}^{\mu\nu\rho\sigma}\,, \tag{4.17}$$

with $R\widetilde{R} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu}{}^{ab}R_{\alpha\beta\,ab}$. Pure gravitational anomalies vanish in $d = 4$, but the *mixed* term provides a nontrivial constraint (Sec. 4.2).

7. Spectral features and matching

Spectral origin of I_{2n+2}. In the spectral medium the fermion content arises from chiral zero modes of composite operators tied to the fiber $(\mathcal{F}, h)$ (Vol. II/III). The index density—hence I_{2n+2}—is insensitive to smooth deformations of constitutive tensors $(\rho, \eta, \chi^\mu, T)$ as long as the gap protecting chirality does not close. Therefore anomaly coefficients are *topological* and do not run with μ (they match across thresholds up to decoupling of complete vectorlike pairs).

Thresholds and decoupling. Integrating out a heavy *chiral* multiplet H changes I_{2n+2} by its contribution (Appendix 24.26); conversely, vectorlike pairs contribute zero net I_{2n+2}. Hence the anomaly cancellation conditions we derive in Sec. 4.2 are RG-invariant and must hold at any scale.

Background anisotropies. Small CPT-even anisotropies (Sec. 1.3) alter two-point kinematics but *not* the local cohomology class of I_{2n+2}, so the descent and WZ consistency remain unchanged.

8. Roadmap

What follows. In Sec. 4.2 we evaluate (4.5) for the spectral fermion content, deriving the ABJ and mixed gauge–gravitational conditions (including hypercharge normalization and Bardeen counterterms). Section 4.3 treats *global* anomalies (e.g. Witten's $\pi_4(SU(2)) = \mathbb{Z}_2$) in the spectral setting. Section 4.4 presents anomaly inflow and boundary terms, clarifying how higher-dimensional Chern–Simons functionals in the spectral bulk cancel boundary anomalies—providing a geometric origin for consistent vs. covariant forms and for topological transport.

4.2 ABJ and mixed gauge–gravitational anomalies

Aim. We compute and organize *local* (perturbative) anomalies in $d = 4$ for chiral spectral matter, derive algebraic *cancellation conditions* for gauge and mixed gauge–gravitational anomalies, and discuss scheme choices (Bardeen counterterms) that distribute anomalies among currents while preserving physical gauge invariances. Throughout, anomaly coefficients are read from the six-form polynomial I_6 of Sec. 4.1 and expressed in a representation-theoretic basis adapted to the fiber isometries $G = \prod_i G_i \times \prod_a U(1)_a$.

1. Algebraic form of anomaly coefficients

Notation. For each simple factor G_i with generators T_i^a and a chiral Weyl multiplet in representation R:

$$\mathrm{Tr}_R\!\left(T_i^a T_i^b\right) \;=\; T_i(R)\,\delta^{ab}, \qquad \mathrm{Tr}_R\!\left(T_i^a\{T_i^b,T_i^c\}\right) \;=\; A_i(R)\,d_i^{abc}, \tag{4.18}$$

with Dynkin index $T_i(R)$, cubic index $A_i(R)$ (vanishing for groups with $d_i^{abc}=0$ such as $SU(2)$, $SO(N)$), and d_i^{abc} the totally symmetric invariant tensor. For Abelian factors $U(1)_a$, let $q_a(\Psi)$ be the charge of Ψ, and for a product representation $R=\bigotimes_i R_i$ write $\dim R=\prod_i \dim R_i$. The chiral sign is $\sigma_R=+1$ for left-handed and -1 for right-handed Weyl fields.

Anomaly tensors. The consistent anomalies are encoded by the symmetric tensors

$$\mathcal{A}^{(i)}_{\mathrm{ABJ}} := \sum_{\Psi\ \text{chiral}} \sigma_R\, A_i(R), \tag{4.19}$$

$$\mathcal{A}^{(i|a)}_{G^2\text{–}U(1)} := \sum_{\Psi\ \text{chiral}} \sigma_R\, T_i(R)\, q_a(\Psi), \tag{4.20}$$

$$\mathcal{A}^{(abc)}_{U(1)^3} := \sum_{\Psi\ \text{chiral}} \sigma_R\, q_a(\Psi)\, q_b(\Psi)\, q_c(\Psi), \tag{4.21}$$

$$\mathcal{A}^{(a)}_{\mathrm{grav}^2\text{–}U(1)} := \sum_{\Psi\ \text{chiral}} \sigma_R\, q_a(\Psi)\, \dim R. \tag{4.22}$$

In terms of these, the *consistent* divergences of the gauge and Abelian currents read (suppressing adjoint indices and wedge products, with $\widetilde{F}^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$)

$$\partial_\mu J^{a\,\mu}_{(i)\,\mathrm{cons}} \;=\; \frac{1}{24\pi^2}\,\mathcal{A}^{(i)}_{\mathrm{ABJ}}\; d_i^{abc}\, F^{b}_{(i)\,\mu\nu}\,\widetilde{F}^{c\,\mu\nu}_{(i)}, \tag{4.23}$$

$$\partial_\mu J^{\mu}_{(a)\,\mathrm{cons}}\Big|_{G_i^2U(1)_a} \;=\; \frac{1}{8\pi^2}\,\mathcal{A}^{(i|a)}_{G^2\text{–}U(1)}\;\mathrm{tr}\!\left(F_{(i)\,\mu\nu}\,\widetilde{F}^{\mu\nu}_{(i)}\right), \tag{4.24}$$

$$\partial_\mu J^{\mu}_{(a)\,\mathrm{cons}}\Big|_{U(1)^3} \;=\; \frac{1}{24\pi^2}\sum_{b,c}\mathcal{A}^{(abc)}_{U(1)^3}\; F_{(b)\,\mu\nu}\,\widetilde{F}^{\mu\nu}_{(c)}, \tag{4.25}$$

$$\partial_\mu J^{\mu}_{(a)\,\mathrm{cons}}\Big|_{\mathrm{grav}^2U(1)} \;=\; \frac{1}{192\pi^2}\,\mathcal{A}^{(a)}_{\mathrm{grav}^2\text{–}U(1)}\; R_{\mu\nu\rho\sigma}\,\widetilde{R}^{\mu\nu\rho\sigma}. \tag{4.26}$$

Equations (4.23)–(4.26) reproduce the component form of Sec. 4.1 (consistent scheme), with normalizations fixed by (4.5).

2. Anomaly cancellation conditions (local)

Necessary and sufficient (perturbative). A chiral spectral sector in $d = 4$ is free of *local* anomalies iff the following algebraic constraints hold:

$$\boxed{\mathcal{A}^{(i)}_{\mathrm{ABJ}} = \sum_{\Psi} \sigma_R\, A_i(R) = 0 \;\; \text{for each simple factor } G_i} \qquad \text{(non-Abelian ABJ)}, \tag{4.27}$$

$$\boxed{\mathcal{A}^{(i|a)}_{G^2\text{–}U(1)} = \sum_{\Psi} \sigma_R\, T_i(R)\, q_a(\Psi) = 0 \;\; \text{for each } (G_i, U(1)_a)} \qquad (G^2\text{–}U(1)), \tag{4.28}$$

$$\boxed{\mathcal{A}^{(abc)}_{U(1)^3} = \sum_{\Psi} \sigma_R\, q_a q_b q_c = 0 \;\; \text{for all symmetric } (a, b, c)} \qquad (U(1)^3), \tag{4.29}$$

$$\boxed{\mathcal{A}^{(a)}_{\mathrm{grav}^2\text{–}U(1)} = \sum_{\Psi} \sigma_R\, q_a \,\dim R = 0 \;\; \text{for each } U(1)_a} \qquad \text{(mixed grav–}U(1)\text{)}. \tag{4.30}$$

Vectorlike pairs $(R_L \oplus \overline{R}_R)$ obey $\sigma_R A_i(R) + \sigma_{\overline{R}} A_i(\overline{R}) = 0$, etc., and thus do not contribute. Conditions (4.27)–(4.30) are *RG invariant* (Sec. 4.1): integrating out heavy vectorlike states leaves them unchanged; integrating out a heavy *chiral* multiplet violates them unless compensated by other chiral degrees of freedom.

Special cases. For $SU(2)$ and $SO(N)$, $d^{abc} = 0$ so (4.27) is automatic (no perturbative $SU(2)^3$ anomaly). However, $SU(2)$ admits a *global* Witten anomaly (Sec. 4.3) imposing a separate mod-2 condition on the number of doublets.

3. Bardeen counterterms and scheme choices

Consistent vs. covariant currents. The local functional ambiguity $W \to W + \int X_4^{\mathrm{BZ}}$ (Bardeen–Zumino polynomial) shifts currents by local terms. For two Abelian factors $U(1)_A \times U(1)_B$, a canonical Bardeen choice preserving $U(1)_B$ gauge invariance yields

$$\partial_\mu J^\mu_{A,\mathrm{cov}} \;=\; \frac{1}{24\pi^2}\left(\mathcal{A}_{AAA}\, F_A \widetilde{F}_A + 3\,\mathcal{A}_{AAB}\, F_A \widetilde{F}_B\right), \qquad \partial_\mu J^\mu_{B,\mathrm{cov}} \;=\; 0, \tag{4.31}$$

with $\mathcal{A}_{AAB} = \sum \sigma q_A^2 q_B$, etc. In non-Abelian$\times U(1)$ systems, one chooses Bardeen terms so that the *non-Abelian* currents are covariantly conserved, shifting all mixed $G^2U(1)$ anomalies into the Abelian sector; this matches the background-field scheme used in our RG analysis.

Physical content. Scheme choices do not alter the algebraic constraints (4.27)–(4.30) (which must hold), but they affect the partition of consistent vs. covariant pieces among currents and are useful to keep gauge symmetries manifest in the sectors gauged by true dynamical fields.

4. Hypercharge normalization and kinetic mixing

Normalization. Let the physical hypercharge be Y with generator T_Y and normalization k_Y relative to the canonical fiber basis (Sec. 3.1). Rescaling $Y \to \lambda Y$ rescales all Abelian anomaly sums by λ^n $(n = 1, 3)$, hence (4.29) and (4.30) are *scale-invariant*: if they vanish for one normalization, they vanish for all. The *values* of nonvanishing anomaly coefficients (for global symmetries) do depend on normalization and can be matched across dual descriptions.

Kinetic mixing. Off-diagonal kinetic terms among $U(1)$'s do not change the *algebraic* anomaly sums; they only modify current redefinitions. An anomaly-free set remains anomaly-free after any nonsingular linear transformation in the Abelian sector.

5. Worked illustration: SM-like chiral sector

One family (charges in the usual normalization). Take $G = SU(3)_c \times SU(2)_L \times U(1)_Y$ and the chiral set

$$Q_L : (\mathbf{3},\mathbf{2})_{+1/6}, \quad u_R : (\mathbf{3},\mathbf{1})_{+2/3}, \quad d_R : (\mathbf{3},\mathbf{1})_{-1/3}, \quad L_L : (\mathbf{1},\mathbf{2})_{-1/2}, \quad e_R : (\mathbf{1},\mathbf{1})_{-1}.$$

Then

$$SU(3)^3 : \ \sum \sigma\, A_{SU(3)}(R) = A(\mathbf{3}) - A(\overline{\mathbf{3}}) = 0 \quad \text{(vectorlike in color)}, \tag{4.32}$$

$$SU(2)^3 : \ \text{automatic}(d^{abc} = 0),$$

$$SU(3)^2U(1)_Y : \ \sum \sigma\, T_{SU(3)}(R)\, Y = 2 \times \tfrac{1}{2} \times \tfrac{1}{6} \ - \ \tfrac{1}{2} \times \tfrac{2}{3} \ - \ \tfrac{1}{2} \times \Big(-\tfrac{1}{3}\Big) = 0, \tag{4.33}$$

$$SU(2)^2U(1)_Y : \ \sum \sigma\, T_{SU(2)}(R)\, Y = 3 \times \tfrac{1}{2} \times \tfrac{1}{6} \ + \ 1 \times \tfrac{1}{2} \times \Big(-\tfrac{1}{2}\Big) = 0, \tag{4.34}$$

$$U(1)_Y^3 : \ \sum \sigma\, Y^3 = 3\Big[2\Big(\tfrac{1}{6}\Big)^3 - \Big(\tfrac{2}{3}\Big)^3 - \Big(-\tfrac{1}{3}\Big)^3\Big] + \Big[2\Big(-\tfrac{1}{2}\Big)^3 - (-1)^3\Big] = 0, \tag{4.35}$$

$$\text{grav}^2\text{–}U(1)_Y : \ \sum \sigma\, Y \ \dim R = 3\Big[2 \times \tfrac{1}{6} - \tfrac{2}{3} - \Big(-\tfrac{1}{3}\Big)\Big] + \Big[2 \times \Big(-\tfrac{1}{2}\Big) - \Big(-1\Big)\Big] = 0. \tag{4.36}$$

Thus the SM-like chiral family saturates (4.27)–(4.30). In the spectral setting, any deformation of the fiber that preserves this chiral content keeps the anomaly constraints satisfied; vectorlike spectral composites may be added without affecting the sums.

6. Spectral remarks: thresholds and matching

Vectorlike decoupling. Spectral thresholds typically appear as *vectorlike* pairs (Sec. 3.2), hence do not modify anomaly sums. If truly chiral heavy multiplets are present, their removal at low energies must be accompanied by compensating chiral states or by anomaly inflow (Sec. 4.4).

Background anisotropies. CPT-even anisotropies (Sec. 1.3) renormalize kinetic terms and higher-dimension operators, but do not alter (4.27)–(4.30), which depend only on representation theory and charges.

7. Summary

Local anomaly freedom amounts to the algebraic vanishing of the tensors (4.19)–(4.22), i.e. the conditions (4.27)–(4.30). These are independent of RG flow and insensitive to small spectral anisotropies. Bardeen counterterms allow one to keep non-Abelian gauge invariances manifest by shifting mixed anomalies into Abelian currents without affecting the constraints. A SM-like spectral sector satisfies all conditions per generation (Eqs. (4.33)–(4.36)); global (nonperturbative) anomalies, notably the $SU(2)$ Witten anomaly, impose additional mod-2 constraints and are treated next (Sec. 4.3).

4.3 Global/Witten anomalies and consistency

Aim. We analyze *global* (nonperturbative) gauge anomalies in $d = 4$ for chiral spectral matter. Unlike local (ABJ) anomalies, which arise from infinitesimal gauge transformations and are detected by the 6-form polynomial I_6 (Sec. 4.1), global anomalies appear when the fermion path integral acquires a nontrivial *phase* under a *large* gauge transformation with topology in $\pi_4(G)$. The canonical example is Witten's $\mathbb{Z}_2$ anomaly for $SU(2)$ gauge theories. We give: a cohomological/APS-index formulation, a practical representation-theoretic criterion, examples (SM-like content), and consistency/inflow options in the spectral framework.

1. Global anomalies as topological phases

Mapping torus and η-invariant. Let $g \in \mathcal{G}$ represent a large gauge transformation in the homotopy class $[g] \in \pi_4(G)$. Consider the *mapping torus* $X_g := M_4 \times [0,1]/(x,0) \sim (g\cdot x, 1)$ with a bundle interpolating between A and A^g. The chiral determinant changes by a phase

$$\frac{\det\{\}_{\text{chiral}}(\not{D}_{A^g})}{\det\{\}_{\text{chiral}}(\not{D}_A)} = \exp\Big\{ i\pi\, \nu_g \Big\}, \qquad \nu_g \;\equiv\; \text{Index}_2\Big(\not{D}_5\Big) \;\in\; \mathbb{Z}_2, \tag{4.37}$$

where $\text{Index}_2(\not{D}_5)$ is the *mod 2 index* of a 5D Majorana/real Dirac operator on a spin 5-manifold Y_5 bounding X_g (Atiyah–Patodi–Singer). Equivalently, the phase is $\exp\{\, i\pi\, \eta/2 \,\}$ with η the APS η-invariant. A nontrivial $\nu_g = 1$ signals a *global anomaly*: the path integral changes sign and the theory is inconsistent.

Cobordism viewpoint. For fermionic theories in $d = 4$, global gauge anomalies are classified by the torsion subgroup of the spin-bordism group $\Omega^5_{\text{spin}}(BG)$. For $G = SU(2)$, $\Omega^5_{\text{spin}}(BSU(2)) \cong \mathbb{Z}_2$ and the nontrivial element reproduces Witten's anomaly.

2. Witten's $SU(2)$ anomaly: mod-2 criterion

Statement. For an $SU(2)$ gauge theory with left-handed Weyl fermions in (possibly repeated) irreducible representations of isospin $j = \frac{1}{2}, \frac{3}{2}, \ldots$, the theory is globally anomaly-free iff

$$\boxed{\sum_{\text{Weyl } \Psi \text{ in irrep } j} 2\,T_{SU(2)}(j) \;\equiv\; 0 \pmod 2\,.} \tag{4.38}$$

Here $T_{SU(2)}(j)$ is the Dynkin index defined by $\text{Tr}_j(T^a T^b) = T_{SU(2)}(j)\,\delta^{ab}$ with $T_{SU(2)}(\frac{1}{2}) = \frac{1}{2}$. Equivalently,

$$2\,T_{SU(2)}(j) \;=\; \frac{2}{3}\,j(j+1)(2j+1) \;\in\; \mathbb{Z}, \qquad 2\,T_{SU(2)}(j) \;\equiv\; \begin{cases} 1 \pmod 2, & j = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots \\ 0 \pmod 2, & j = \frac{3}{2}, \frac{7}{2}, \ldots \end{cases} \tag{4.39}$$

so a single doublet ($j = \frac{1}{2}$) yields an anomalous sign, while two doublets cancel modulo 2.

Minimal examples.

- One $SU(2)$ doublet: $\sum 2T = 1 \pmod 2 \Rightarrow$ *anomalous*.
- Two doublets or one quartet ($j = \frac{3}{2}$): $\sum 2T = 0 \pmod 2 \Rightarrow$ *consistent*.

SM-like content (one family). Left-handed $SU(2)$ doublets are Q_L (three colors) and L_L (one lepton doublet): total of $3+1=4$ doublets,

$$\sum_{\text{family}} 2T \;=\; 4\times 1 \;\equiv\; 0 \;(\text{mod } 2), \tag{4.40}$$

so the electroweak $SU(2)_L$ sector is free of the Witten anomaly per family.

3. Other gauge groups and discrete/global variants

Groups with $\pi_4(G)=\mathbb{Z}_2$. Any 4D gauge theory with a simple factor locally isomorphic to $SU(2)$ (e.g. $SO(3)\simeq SU(2)/\mathbb{Z}_2$ or product groups containing an $SU(2)$ factor) can exhibit a $\mathbb{Z}_2$ global anomaly governed by the same parity condition (4.38) applied to the representations under that factor. For simple groups with $\pi_4(G)=0$ (e.g. $SU(3)$) there is no *4D* Witten-type global anomaly.[1]

Discrete gauge/global symmetries. If a discrete symmetry Γ is gauged or used as a selection rule, one must also check discrete 't Hooft anomalies classified by $\Omega^5_{\text{spin}}(B\Gamma)$ (torsion groups). In practice these reduce to mod-N constraints on charge assignments; see App. 24.26 for a summary table and descent formulae.

4. Spectral-model implications and matching

RG invariance and thresholds. Global anomaly conditions are *topological*: vectorlike pairs do not affect (4.38); adding/removing a *chiral* multiplet in an $SU(2)$ irrep changes the parity sum and must be compensated by other chiral states *or* by anomaly inflow (Sec. 4.4). Therefore the parity constraint must hold in every effective description across spectral thresholds and symmetry-breaking stages ('t Hooft matching).

Higgsing and quotient groups. If an $SU(2)$ factor is Higgsed or modded by a discrete center (e.g. $SO(3)$ bundles), the global-anomaly test is performed for the *unbroken* $SU(2)$-like factor in the UV and must be matched in the IR (possibly realized via topological terms).

5. Inflow resolution (preview)

Invertible 5D bulk. A nonvanishing mod-2 index ν_g can be canceled by coupling the 4D boundary theory to a 5D invertible TQFT with bulk action $S_{\text{inflow}}[A]$ whose variation on a manifold with boundary reproduces the anomalous phase, yielding the combined partition function

$$\mathcal{Z}_{\text{total}} \;=\; \mathcal{Z}_{\text{boundary}}[A]\;\exp\Big\{\, i\,S_{\text{inflow}}[A]\,\Big\}, \qquad \delta_g \ln \mathcal{Z}_{\text{total}} \;=\; i\pi\Big(\nu_g-\nu_g\Big)=0. \tag{4.41}$$

In Sec. 4.4 we present the explicit Chern–Simons/Wess–Zumino inflow functionals generating both local and global pieces in the spectral setting.

[1] This does not preclude global anomalies in other spacetime dimensions, where the relevant homotopy/bordism groups differ.

6. Practical checklist (spectral sector)

1. **Local anomalies**: verify the algebraic conditions of Sec. 4.2 (all ABJ/mixed sums vanish).
2. $SU(2)$ **factors**: compute $\sum_\Psi 2T_{SU(2)}(j_\Psi)$ modulo 2 (Eq. (4.38)) for every $SU(2)$-like factor (including quotients).
3. **Thresholds/Higgsing**: ensure the parity condition is preserved when integrating out heavy chiral states or breaking the gauge group; otherwise introduce appropriate inflow.
4. **Discrete symmetries**: if present, check discrete anomalies (Appendix 24.26).

7. Summary

Global anomalies probe the topology of gauge transformations beyond infinitesimals. In $d = 4$, the only continuous gauge *global* obstruction of Witten type occurs for $SU(2)$-like factors and is fully captured by the mod-2 index condition (4.38); a single $SU(2)$ doublet is inconsistent, whereas an even number is allowed. These constraints are RG-invariant and must match across thresholds and phases; if violated by the chiral spectrum, they can be remedied by anomaly inflow from a suitable 5D invertible bulk, to be constructed in Sec. 4.4.

4.4 Anomaly inflow and boundary terms

Aim. We complete the anomaly analysis by constructing a *five-dimensional inflow* functional whose variation reproduces the four-dimensional anomalies of Secs. 4.1–4.3. The inflow lives on a 5D bulk $(Y_5, \partial Y_5 = M_4)$ endowed with extensions of the gauge connection $\mathcal{A}$ and spin connection ω. Its exterior derivative equals the anomaly polynomial I_6, while its boundary variation cancels the *consistent* anomaly of the 4D theory. We also derive the boundary transgression (Bardeen–Zumino) terms that relate consistent to covariant currents, and we state the cobordism (global) inflow for the $SU(2)$ Witten anomaly.

1. Bulk–boundary setup and master inflow

Extension and Chern–Simons transgression. Let (M_4, g_{eff}) be the physical spacetime and choose a compact, oriented spin 5-manifold Y_5 with boundary $\partial Y_5 = M_4$. Extend the gauge and spin connections smoothly into Y_5:

$$\mathcal{A} \rightsquigarrow \mathcal{A}_Y, \qquad \omega \rightsquigarrow \omega_Y.$$

By Sec. 4.1, the anomaly polynomial is a closed 6-form

$$I_6(F,R) \;=\; \Big[\, \sum_R \sigma_R \,\mathrm{ch}_R(F)\, \widehat{A}(R) \,\Big]_6, \qquad dI_6 = 0. \tag{4.42}$$

On contractible patches there exists a Chern–Simons 5-form $Q_5(\mathcal{A}, \omega)$ such that

$$d\,Q_5(\mathcal{A},\omega) \;=\; I_6(F,R). \tag{4.43}$$

Master inflow action and its variation. Define the inflow functional

$$\boxed{S_{\rm inflow}[\mathcal{A},\omega] \;=\; 2\pi i \int_{Y_5} Q_5(\mathcal{A}_Y,\omega_Y)\,.} \tag{4.44}$$

Under an infinitesimal gauge/Lorentz transformation, $\delta\mathcal{A} = -D\epsilon$, $\delta\omega = -D\theta$, the descent equation $\delta Q_5 = d\,I_4^{(1)}(\epsilon,\theta;\mathcal{A},\omega)$ (Sec. 4.1) yields

$$\delta S_{\rm inflow} \;=\; 2\pi i \int_{Y_5} d\,I_4^{(1)} \;=\; 2\pi i \int_{M_4} I_4^{(1)}(\epsilon,\theta;\mathcal{A},\omega), \tag{4.45}$$

so that the combined variation of *bulk + boundary* vanishes:

$$\delta\Big(-W_{\rm bdry}[\mathcal{A},\omega] \;+\; S_{\rm inflow}[\mathcal{A},\omega]\Big) \;=\; 0, \tag{4.46}$$

because $\delta W_{\rm bdry} = 2\pi i \int_{M_4} I_4^{(1)}$ by (4.8). Thus $S_{\rm inflow}$ *cancels* the consistent anomaly on the boundary.

2. Explicit 5D forms in $d = 4$

Gauge and mixed terms. A convenient representative of Q_5 is the sum

$$Q_5^{\rm gauge}(\mathcal{A}) = \sum_R \sigma_R\, \frac{1}{6}\Big(\frac{i}{2\pi}\Big)^3 \mathrm{Tr}_R\Big(\mathcal{A}\,d\mathcal{A}\,d\mathcal{A} + \tfrac{3}{2}\,\mathcal{A}^3 d\mathcal{A} + \tfrac{3}{5}\,\mathcal{A}^5\Big), \tag{4.47}$$

$$Q_5^{\rm mix}(\mathcal{A},\omega) = \sum_R \sigma_R\, \frac{1}{24}\Big(\frac{i}{2\pi}\Big)^3 \mathrm{Tr}_R(\mathcal{A})\;\mathrm{tr}\Big(\omega\,d\omega + \tfrac{2}{3}\,\omega^3\Big), \tag{4.48}$$

satisfying $d(Q_5^{\rm gauge}+Q_5^{\rm mix}) = I_6$ (compare (4.9)–(4.10)). The full inflow is

$$S_{\rm inflow} \;=\; 2\pi i \int_{Y_5}\Big(Q_5^{\rm gauge} + Q_5^{\rm mix}\Big). \tag{4.49}$$

Its boundary variation reproduces Eqs. (4.23)–(4.26) in the *consistent* scheme.

Abelian sector and normalizations. For $U(1)$ factors with field strengths F_a, $Q_5^{\rm gauge}$ reduces to $(i/2\pi)^3\sum_\Psi \sigma\,\frac{1}{6}\,q_a^3\,A_a\,F_a^2$ plus cross terms; $Q_5^{\rm mix}$ contains $(i/2\pi)^3\sum_\Psi \sigma\,\frac{1}{24}\,q_a\,A_a\,\mathrm{tr}(\omega d\omega + \frac{2}{3}\omega^3)$, matching the coefficients of (4.25) and (4.26).

3. Boundary transgression and Bardeen–Zumino currents

Relative Chern–Simons and transgression Q_4^T. Let $\bar{\mathcal{A}}$ be a *reference* connection (e.g. a background field in the BFM split). The relative CS form obeys

$$Q_5(\mathcal{A}) - Q_5(\bar{\mathcal{A}}) \;=\; d\,Q_4^T(\mathcal{A},\bar{\mathcal{A}}), \qquad Q_4^T(\mathcal{A},\bar{\mathcal{A}}) \;=\; 3\int_0^1 dt\;\mathrm{Tr}\Big[(\mathcal{A}-\bar{\mathcal{A}})\,F_t^{\,2}\Big], \tag{4.50}$$

with $F_t = d\mathcal{A}_t + \mathcal{A}_t^2$ and $\mathcal{A}_t = \bar{\mathcal{A}} + t(\mathcal{A}-\bar{\mathcal{A}})$. The boundary counterterm

$$X_4^{\rm BZ} \;:=\; 2\pi i\; Q_4^T(\mathcal{A},\bar{\mathcal{A}})\Big|_{M_4} \tag{4.51}$$

shifts consistent to covariant currents:

$$J_{\rm cov} \;=\; J_{\rm cons} \;+\; \frac{\delta}{\delta\mathcal{A}}\int_{M_4} X_4^{\rm BZ}, \qquad \delta W_{\rm cov} \;=\; 2\pi i \int_{M_4} I_4^{(1)}\Big|_{\rm cov}, \tag{4.52}$$

and can be chosen to keep non-Abelian gauge invariances manifest (Sec. 4.2). An analogous transgression applies to the mixed gauge–gravitational sector with $(\mathcal{A},\omega)$ and their references $(\bar{\mathcal{A}},\bar{\omega})$.

4. Global (Witten) anomaly inflow

Cobordism invariant. For $G = SU(2)$, the global anomaly is classified by $\Omega^5_{\text{spin}}(BSU(2)) \cong \mathbb{Z}_2$. There exists a *mod-2* bulk invariant $\mathcal{I}_5[A] \in \{0,1\}$ such that the invertible TQFT

$$\boxed{S^{\text{global}}_{\text{inflow}}[A] \;=\; i\pi \int_{Y_5} \mathcal{I}_5[A] \quad (\text{mod } 2\pi i)} \tag{4.53}$$

produces the anomalous phase $(-1)^{\mathcal{I}_5}$ under large $SU(2)$ transformations, canceling the boundary sign. A representative is the Stiefel–Whitney expression

$$\mathcal{I}_5[A] \;=\; \int_{Y_5} w_2(E) \smile w_3(E) \quad (\text{mod } 2), \tag{4.54}$$

with E the $SU(2)$ bundle. The combined (boundary+bulk) partition function is then invariant under all gauge transformations (Sec. 4.3).

5. Spectral interpretation and domain walls

Spectral bulk as an auxiliary extension. In the quantum–aether setting, Y_5 is an *auxiliary* extension (no new propagating modes are required). Physically, Q_5 can be viewed as encoding a *Wess–Zumino functional* of slow spectral deformations: introduce a homotopy parameter $s \in [0,1]$ and interpolate $\mathcal{A}(x,s)$ between a reference at $s=0$ and the physical connection at $s=1$; then

$$S_{\text{inflow}} \;=\; 2\pi i \int_{M_4 \times [0,1]} Q_5(\mathcal{A}(x,s), \omega(x,s)), \tag{4.55}$$

which is independent of the interpolation up to integers determined by I_6.

Interfaces and defects. If the spectral medium supports interfaces (e.g. where a composite mass changes sign), chiral boundary modes localized on the interface carry the consistent anomaly; the jump of the bulk Q_5 across the interface supplies the compensating inflow, ensuring overall conservation—precisely as in topological insulators.

6. Torsion remark (Nieh–Yan)

When torsion is present. If the emergent connection includes torsion, an additional 4-form $NY = d(e^a \wedge T_a) - R_{ab} \wedge e^a \wedge e^b$ (Nieh–Yan) contributes to *axial* anomalies with a UV-sensitive coefficient. In our torsionless baseline (ω = Levi–Civita of g_{eff}), NY is absent; if torsionful phases are considered in Vol. V, the inflow must be augmented by torsional CS forms so that $dQ_5 = I_6 + \kappa_{\text{NY}}\, d(\cdots)$ while preserving the gates of Sec. 3.3.

7. Summary and practical checklist

- **Construct** $S_{\text{inflow}} = 2\pi i \int_{Y_5} Q_5$ with $dQ_5 = I_6$ (Eqs. (4.43)–(4.49)); its boundary variation cancels the *consistent* anomaly (4.45).
- **Choose** boundary transgression X_4^{BZ} (4.51) to trade consistent for covariant currents while keeping the dynamical non-Abelian symmetries manifest (4.52).

- **Include** the $\mathbb{Z}_2$ bulk phase (4.53) for $SU(2)$ global anomalies (Sec. 4.3); check the parity condition at thresholds ('t Hooft matching).
- **Implement** the homotopy construction (4.55) in numerical codes to evaluate WZ/inflow terms from spectral data.

Conclusion. Anomaly inflow provides the geometric mechanism that reconciles local (ABJ and mixed) and global (Witten) anomalies with gauge invariance of the full system. In the spectral model, it cleanly separates boundary chiral dynamics from a universal, topological bulk functional determined by the anomaly polynomial I_6, while Bardeen–Zumino boundary terms fix the scheme and ensure that the physically gauged symmetries remain manifest.

Chapter 5

WZW and Chern–Simons Terms as Emergent Functionals

5.1 Deriving WZW from spectral phase textures

Aim. We derive the Wess–Zumino–Witten (WZW) functional for Nambu–Goldstone (NG) fields that arise as *spectral phase textures* of the quantum–aether medium. The derivation: (i) identifies the NG target G/H generated by spontaneous breaking of a global symmetry G (acting on the internal fiber $\mathcal{F}$) down to H; (ii) constructs the *ungauged* WZ 5-form on the NG manifold and proves its coefficient is quantized; (iii) *gauges* the functional by minimal coupling to the emergent gauge connection $\mathcal{A}$ (Sec. 1), using descent to reproduce the ABJ and mixed anomalies of Secs. 4, 4.2; and (iv) matches low-energy amplitudes (e.g. $\pi^0 \to \gamma\gamma$) with a *spectral index* fixed by UV anomaly data.

1. Spectral phase textures and the NG field

Order parameter and coset coordinates. Let $\Phi(x)$ denote a complex spectral order parameter (Vol. I) transforming in a linear representation of a compact global group G acting on the internal fiber $(\mathcal{F}, h)$. Assume a homogeneous ground state $\langle\Phi\rangle \neq 0$ spontaneously breaks $G \to H$. The long-wavelength phase textures are NG fields taking values in the coset

$$U(x) \;\in\; G/H, \qquad U(x) = e^{i\,\pi^a(x) X_a / f}, \quad X_a \in \mathfrak{g}/\mathfrak{h}, \tag{5.1}$$

with f a spectral stiffness scale. We use a CCWZ frame $U^{-1}\partial_\mu U = i(e^a_\mu X_a + \omega^i_\mu H_i)$ to split coset vielbein e^a_μ and compensator $\omega^i_\mu \in \mathfrak{h}$.

Kinetic term. The leading two-derivative Lagrangian follows from the fiber metric pullback:

$$\mathcal{L}_{\text{kin}} \;=\; \tfrac{f^2}{4}\, g^{\mu\nu}_{\text{eff}}\, \mathrm{Tr}(e_\mu e_\nu) = \tfrac{f^2}{4}\, g^{\mu\nu}_{\text{eff}}\, \mathrm{Tr}\Big[(U^{-1}\partial_\mu U)_\perp\, (U^{-1}\partial_\nu U)_\perp\Big]\,, \tag{5.2}$$

where $\perp$ denotes the coset projection (Sec. 1.4 for conventions).

2. The Wess–Zumino 5-form on G/H

Closed 5-form and quantization. On the NG manifold, define the left-invariant one-form $\theta := U^{-1}dU$. The *ungauged* WZ 5-form

$$\Omega_5(U) \;:=\; \frac{i^3}{240\pi^2}\,\mathrm{Tr}\big(\theta^5\big), \qquad d\Omega_5 = 0, \tag{5.3}$$

is closed, G-invariant, and represents a cohomology class in $H^5(G/H,\mathbb{Z})$. When $\pi_5(G/H) \neq 0$ (e.g. $G = SU(N_f)$, $H = SU(N_f)$ modulo a discrete center for $N_f \geq 3$) the integral of Ω_5 over a closed 5-cycle is an integer. The *WZ action* is defined on a 5D extension B^5 with $\partial B^5 = M_4$:

$$S_{\mathrm{WZ}}[U] \;=\; k\int_{B^5}\Omega_5(U), \tag{5.4}$$

and is *well-defined* modulo $2\pi\mathbb{Z}$ provided

$$k \;\in\; \mathbb{Z}. \tag{5.5}$$

Indeed, if B^5 and $B^{5\prime}$ are two fillings of M_4, their difference is a closed 5-manifold Σ^5 and $S_{\mathrm{WZ}}[U]\big|_{B^5} - S_{\mathrm{WZ}}[U]\big|_{B^{5\prime}} = k\int_{\Sigma^5}\Omega_5 \in 2\pi k\,\mathbb{Z}$.

Spectral interpretation of k. In the quantum–aether model, k equals a UV *anomaly index* determined by the chiral spectral matter coupled to the flavor currents:

$$k \;=\; \sum_{\Psi\ \text{chiral}} \sigma_\Psi\,\kappa_\Psi, \qquad \kappa_\Psi \;\propto\; \text{cubic index in the flavor representation.} \tag{5.6}$$

For QCD-like sectors this reproduces $k = N_c$ (number of color degrees of freedom). In general, k is the coefficient that appears in the ABJ flavor anomaly polynomial (Sec. 4.2), ensuring 't Hooft matching.

3. Gauging and the WZW functional

Background vs. dynamical gauge fields. Let $G_L \times G_R$ act chirally on fermions and on Φ, with background (or dynamical) gauge connections A_L, A_R. Introduce

$$\mathcal{A} \;=\; A_L \oplus A_R, \qquad \mathcal{F} \;=\; d\mathcal{A} + \mathcal{A}^2, \qquad \mathcal{U} \;=\; \begin{pmatrix} U & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{5.7}$$

The *gauged* WZW action is obtained by minimal coupling and adding local 4D counterterms such that its variation reproduces the *consistent* anomaly computed from the 6-form

$$I_6^{\mathrm{flavor}} \;=\; \frac{1}{24\pi^2}\,k\,\Big[\mathrm{Tr}(F_L^3) - \mathrm{Tr}(F_R^3)\Big], \tag{5.8}$$

cf. Eq. (4.5). A compact form uses transgression between $\mathcal{A}$ and $\mathcal{A}^U := \mathcal{U}^{-1}\mathcal{A}\mathcal{U} + \mathcal{U}^{-1}d\mathcal{U}$:

$$S_{\mathrm{WZW}}[U;A_L,A_R] \;=\; k\int_{B^5}\Big[\Omega_5(\mathcal{A}^U) - \Omega_5(\mathcal{A})\Big] \;+\; \int_{M_4}\Gamma_4^{\mathrm{BZ}}[U;A_L,A_R], \tag{5.9}$$

where $\Omega_5(\mathcal{A}) = \frac{i^3}{240\pi^2}\mathrm{Tr}(\mathcal{A}\,d\mathcal{A}\,d\mathcal{A} + \frac{3}{2}\mathcal{A}^3 d\mathcal{A} + \frac{3}{5}\mathcal{A}^5)$ and Γ_4^{BZ} is a local Bardeen–Zumino functional that fixes the consistent/covariant scheme (Sec. 4.2).

Descent check (variation). Under a gauge variation $\delta A_{L,R} = -D\epsilon_{L,R}$ and $\delta U = i(\epsilon_L U - U\epsilon_R)$,

$$\delta S_{\rm WZW} \;=\; 2\pi i \int_{M_4} I_4^{(1)}(\epsilon_L, \epsilon_R; A_L, A_R), \tag{5.10}$$

with $I_4^{(1)}$ obtained by descent from $I_6^{\rm flavor}$, so that $\delta(S_{\rm ferm} + S_{\rm WZW}) = 0$ once the fermions generating the anomaly are integrated out. Thus $S_{\rm WZW}$ *encodes* anomaly matching in the NG sector.

4. Abelian factors and electromagnetic embedding

Projection to $U(1)_{\rm em}$. For physical electromagnetism $A_{\rm em}$ embedded via $Q = T_L^3 + Y$, the $U(1)$ projection of the gauged WZW term yields

$$\mathcal{L}_{\pi^0\gamma\gamma} \;=\; \frac{k\,\alpha_{\rm em}}{4\pi f}\,\pi^0\,F_{\mu\nu}\,\widetilde{F}^{\mu\nu} \;+\; \cdots, \tag{5.11}$$

with f the NG decay constant from (5.2). In the QCD-like case $k = N_c$, reproducing the classic $\pi^0 \to \gamma\gamma$ rate. In the spectral model, k equals the *spectral* anomaly index (5.6); thus (5.11) becomes a direct low-energy probe of the UV spectral content.

5. Relation to the spectral bulk and inflow

Bulk construction. Equation (5.9) is the boundary transgression of the *same* 5D inflow functional used in Sec. 4.4. Writing $Y_5 = B^5$ with $\partial Y_5 = M_4$, one has

$$S_{\rm inflow}^{\rm flavor}[\mathcal{A}^U] - S_{\rm inflow}^{\rm flavor}[\mathcal{A}] \;=\; 2\pi i\,k \int_{B^5}\Big(\Omega_5(\mathcal{A}^U) - \Omega_5(\mathcal{A})\Big) \;=\; S_{\rm WZ}[U] \pmod{2\pi}. \tag{5.12}$$

Hence the WZW functional is the *boundary avatar* of spectral anomaly inflow.

6. Worked example: $G = SU(N_f)_L \times SU(N_f)_R$, $H = SU(N_f)_V$

Ungauged form. For $N_f \geq 3$, the NG field $U(x) \in SU(N_f)$ and

$$S_{\rm WZ}[U] \;=\; -\frac{k}{240\pi^2}\int_{B^5} {\rm Tr}\Big(U^{-1}dU\Big)^5. \tag{5.13}$$

Expanding to $\mathcal{O}(\pi^5)$ gives the unique parity-odd 5-derivative operator on M_4 (after eliminating the bulk).

Gauged form and Bardeen split. Coupling to $A_{L,R}$ yields the standard gauged WZW functional (a polynomial in U, $A_{L,R}$ and their curvatures) with a Bardeen choice that preserves vector-current conservation, shifting all anomaly to the axial current (Sec. 4.2). The resulting vertices include $\pi^0\gamma\gamma$, $\pi\omega\rho$, etc., with overall coefficient k.

7. Consistency gates and quantization

S/G/TQ/D gates. *(S)* The WZW term is first-order in time derivatives but topological; it does not alter the positivity of the quadratic Hamiltonian from (5.2). *(G)* It is metric-independent up to total derivatives and thus preserves hyperbolicity of the two-derivative dynamics. *(TQ)* It is required by anomaly matching; omitting it violates

unitarity/analyticity constraints in amplitudes like $\pi^0 \to \gamma\gamma$. *(D)* Dissipative effects enter only via gauged sectors (imaginary parts fixed by the optical theorem); the WZW term contributes purely to the dispersive (real) part.

Quantization and discrete identifications. If (G/H) has a nontrivial center or discrete identifications, k must satisfy additional congruences ensuring single-valuedness under non-contractible loops in field space. These are automatically met when k equals the anomaly index of the microscopic spectral matter.

8. Summary

The WZW functional in the spectral framework is the unique, quantized, parity-odd term for NG fields built from spectral phase textures. Its coefficient k equals a UV anomaly index fixed by the chiral spectral content; the *gauged* WZW functional reproduces the consistent flavor anomalies via descent and thus enforces 't Hooft matching. Physically, it governs topological processes and anomalous decays (e.g. $\pi^0 \to \gamma\gamma$) and arises as a boundary transgression of the 5D spectral inflow constructed in Sec. 4.4. The next section treats Chern–Simons functionals in odd dimensions and their parity structure, which play an analogous role for topological gauge responses on defects and in lower-dimensional spectral phases.

5.2 Chern–Simons in odd dimensions and parity structure

Aim. We develop Chern–Simons (CS) functionals in odd dimensions as *emergent* topological responses of the spectral medium, emphasizing: (i) their gauge-variation and quantization under large transformations, (ii) their *parity* (P) and time-reversal (T) structure (parity anomaly), (iii) induced levels from integrating out massive spectral fermions, (iv) topologically massive gauge dynamics in 2+1D, and (v) the gravitational CS counterpart and thermal Hall response. We keep normalizations consistent with Secs. 4–5.

1. CS forms in $2m{+}1$ dimensions: definitions

General construction. Let $\mathcal{A}$ be a (matrix-valued) connection one-form for a compact group G, with curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$ and Tr an invariant quadratic form (Sec. 4.1). The CS $(2m{+}1)$-form $Q_{2m+1}(\mathcal{A})$ is defined by transgression:

$$d\,Q_{2m+1}(\mathcal{A}) \;=\; \mathrm{Tr}\,\mathcal{F}^{\,m+1}. \tag{5.14}$$

Explicitly, using the homotopy $\mathcal{A}_t = t\,\mathcal{A}$,

$$Q_{2m+1}(\mathcal{A}) \;=\; (m{+}1)\int_0^1 dt\;\mathrm{Tr}\Big(\mathcal{A}\,\mathcal{F}_t^{\,m}\Big), \qquad \mathcal{F}_t = d\mathcal{A}_t + \mathcal{A}_t^2. \tag{5.15}$$

The first two cases (used below) are

$$Q_3(\mathcal{A}) = \;\mathrm{Tr}\Big(\mathcal{A}\,d\mathcal{A} + \tfrac{2}{3}\,\mathcal{A}^3\Big), \tag{5.16}$$

$$Q_5(\mathcal{A}) = \;\mathrm{Tr}\Big(\mathcal{A}\,d\mathcal{A}\,d\mathcal{A} + \tfrac{3}{2}\,\mathcal{A}^3 d\mathcal{A} + \tfrac{3}{5}\,\mathcal{A}^5\Big), \tag{5.17}$$

matching the conventions of Secs. 4.1, 4.4, 5.1.

Action and normalization in 2+1D. On a closed three-manifold M_3 we write

$$S^{(3)}_{\mathrm{CS}}[\mathcal{A}] \;=\; \frac{k}{4\pi}\int_{M_3} Q_3(\mathcal{A}), \qquad Q_3(\mathcal{A}) = \mathrm{Tr}\Big(\mathcal{A}d\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3\Big), \tag{5.18}$$

with Tr normalized as $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ for $SU(N)$ fundamentals, and as the canonical product for $U(1)$.

2. Gauge variation, large transformations, and level quantization

Infinitesimal variation. Under $\delta_\epsilon\mathcal{A} = -D\epsilon$ one finds $\delta_\epsilon Q_{2m+1} = d\,\mathrm{Tr}\big(\epsilon\,\mathcal{F}^m\big)$, hence on closed manifolds S_{CS} is invariant under *small* gauge transformations.

Large transformations. For a finite gauge transformation $g : M_{2m+1} \to G$, one has

$$S^{(2m+1)}_{\mathrm{CS}}[\mathcal{A}^g] - S^{(2m+1)}_{\mathrm{CS}}[\mathcal{A}] \;=\; 2\pi\, k\; \mathsf{w}_{2m+1}(g), \qquad \mathsf{w}_{2m+1}(g) \;\in\; \mathbb{Z}, \tag{5.19}$$

where $\mathsf{w}_{2m+1}(g)$ is the winding number (e.g. $\mathsf{w}_3(g) = \frac{1}{24\pi^2}\int \mathrm{Tr}(g^{-1}dg)^3$). Therefore the quantum weight $e^{iS_{\mathrm{CS}}}$ is gauge-invariant provided

$$k \;\in\; \mathbb{Z}. \tag{5.20}$$

For $U(1)$ the same condition follows from Dirac quantization of fluxes. On *spin* manifolds, half-integer refinements can occur only in conjunction with fermionic determinants (Sec. 5.2).

3. Induced levels and the parity anomaly

Massive fermions in 2+1D. Integrating out a Dirac fermion of mass m in representation R induces a CS term

$$\Delta k \;=\; \frac{1}{2}\,\mathrm{sgn}(m)\,\mathsf{T}(R), \qquad \mathsf{T}(R) := \frac{\mathrm{Tr}_R(T^aT^b)}{\mathrm{Tr}_{\mathrm{fund}}(T^aT^b)}\,, \tag{5.21}$$

in the normalization (5.18) (for $U(1)$, $\mathsf{T} = q^2$). For an *odd* number of such fermions, any gauge-invariant regulator produces an *effective* level $k_{\mathrm{eff}} = k_{\mathrm{bare}} + \frac{1}{2}\sum \mathrm{sgn}(m)\mathsf{T}(R)$ that is half-integer unless one adds a local counterterm *breaking* P,T: this is the *parity anomaly*.

Spectral medium viewpoint. In the quantum–aether framework, gapped composite fermions on a 2+1D defect (domain wall, vortex sheet) generate Δk via their spectral asymmetry (the η-invariant). Gauge invariance of e^{iS} demands $k_{\mathrm{eff}} \in \mathbb{Z}$; if the microscopic spectrum enforces $k_{\mathrm{eff}} \in \mathbb{Z}+\frac{1}{2}$, anomaly inflow from a 3+1D bulk cancels the mismatch, at the price of P,T breaking localized on the defect (Sec. 4.4).

4. Parity, time reversal, and edge modes

Discrete transformations. In 2+1D, $S^{(3)}_{\mathrm{CS}}$ changes sign under orientation reversal; hence it is odd under P and T (and also under CPT-odd if C is a spectator). Thus a nonzero k implies P/T breaking.

Boundary variation and chiral edges. If M_3 has a boundary $\partial M_3 = \Sigma_2$, the CS variation produces a boundary anomaly. Gauge invariance is restored by adding gapless edge modes on Σ_2 with a 1+1D chiral anomaly that *matches* the bulk variation (Callan–Harvey mechanism). The number and chirality of edge modes encode k.

5. Topologically massive gauge bosons (Maxwell–CS)

Abelian case. Consider

$$\mathcal{L} \;=\; -\frac{1}{4g^2}\,F_{\mu\nu}F^{\mu\nu} \;+\; \frac{k}{4\pi}\,\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{5.22}$$

on (M_3, g_{eff}). Linearizing in Lorenz gauge yields a massive dispersion

$$\omega^2(\mathbf{p}) \;=\; \mathbf{p}^2 \;+\; m_T^2, \qquad m_T \;=\; \frac{k\,g^2}{2\pi}, \tag{5.23}$$

with a *single* propagating polarization (helicity fixed by $\mathrm{sgn}(k)$). Thus the CS term *topologically* gaps the gauge boson without a Higgs field. Non-Abelian generalizations exist perturbatively around the trivial background for $|k|$ large.

Spectral anisotropy. Small CPT-even anisotropies from χ^μ (Sec. 1.3) deform the Maxwell part but *do not* affect the quantized level k; hence the topological mass (5.23) remains protected up to rescalings of g^2.

6. Gravitational Chern–Simons and thermal Hall

Definition. Let ω be the spin connection and $R = d\omega + \omega^2$. The gravitational CS action on a closed three-manifold M_3 is

$$S^{(3)}_{\text{gCS}}[\omega] \;=\; \frac{k_{\text{g}}}{4\pi}\int_{M_3}\mathrm{tr}\Big(\omega\,d\omega + \tfrac{2}{3}\,\omega^3\Big), \qquad d\,\mathrm{tr}\Big(\omega\,d\omega + \tfrac{2}{3}\omega^3\Big) = \mathrm{tr}\,R\wedge R. \tag{5.24}$$

On spin manifolds, $e^{iS_{\text{gCS}}}$ is well-defined when $k_{\text{g}} \in \frac{1}{2}\mathbb{Z}$ (integer quantization on general oriented manifolds).

Thermal Hall and chiral central charge. Coupling to background geometry, the (dimensionless) chiral central charge c_- of edge modes equals k_{g} in the above normalization, and determines the thermal Hall conductance

$$\frac{\kappa_{xy}}{T} \;=\; \frac{\pi^2 k_B^2}{3h}\,c_-\,, \qquad c_- = k_{\text{g}}. \tag{5.25}$$

Integrating out massive Dirac fermions induces $\Delta k_{\text{g}} = \frac{1}{2}\,\mathrm{sgn}(m)$ per real two-component fermion, mirroring (5.21). As in the gauge case, half-integer k_{g} requires inflow from a 3+1D bulk to maintain invariance.

7. Mixed Abelian CS, BF couplings, and multi-$U(1)$ structure

Mutual CS/BF. For $U(1)_a$ gauge fields $A^{(a)}$, general topological responses include

$$S^{(3)}_{\text{top}} \;=\; \frac{1}{4\pi}\int_{M_3}\Big(K_{ab}\, A^{(a)} dA^{(b)} \;+\; 2\,\Lambda_{ab}\, B^{(a)} dA^{(b)}\Big), \tag{5.26}$$

with integer symmetric K (levels) and integer Λ for BF-type mutual couplings (B another $U(1)$ gauge field). The K-matrix encodes anyonic statistics and Hall responses; in the spectral setting, K is fixed by microscopic anomaly indices of gapped composites living on 2+1D defects. Small Maxwell/anisotropic terms do not spoil the integrality of K.

8. Higher odd dimensions and relation to inflow

Five dimensions. The Q_5 in (5.17) underlies the anomaly inflow used in Secs. 4.4, 5.1. On a closed Y_5,

$$S^{(5)}_{\text{CS}}[\mathcal{A}] \;=\; 2\pi i\, k \int_{Y_5} \frac{1}{(2\pi i)^3}\, Q_5(\mathcal{A}) \quad\Rightarrow\quad dS^{(5)}_{\text{CS}} \;\propto\; \int_{Y_6} \operatorname{Tr} \mathcal{F}^3, \tag{5.27}$$

and on manifolds with boundary it supplies the precise boundary anomaly by descent (Sec. 4.1). Thus CS in odd $= 2m{+}1$ dimensions is the bulk avatar of $2m$-dimensional anomalies.

9. Practical rules and spectral consistency

- **Quantization**: ensure $k \in \mathbb{Z}$ (and $k_{\text{g}} \in \frac{1}{2}\mathbb{Z}$ on spin manifolds) for gauge/gravitational CS so that e^{iS} is well-defined on all bundles.
- **Induced levels**: gapped spectral fermions shift (k, k_{g}) by $\frac{1}{2}\text{sgn}(m)$ times their quadratic indices; enforce integrality via local counterterms *or* anomaly inflow.
- **Parity**: any nonzero k breaks P, T; edge states realize the associated anomalies.
- **Dynamics**: in 2+1D, Maxwell+CS gives a topological mass $m_T = \frac{kg^2}{2\pi}$ with a single helicity; anisotropies renormalize g^2 but not k.
- **Matching**: on defects/interfaces of the spectral medium, the CS levels are fixed by UV anomaly data and equal the boundary inflow coefficients.

10. Summary

Chern–Simons functionals Q_{2m+1} provide the universal odd-dimensional topological responses of the spectral gauge/gravity sectors. Their levels are *quantized* by large-gauge invariance, their P/T-odd nature encodes the parity anomaly, and their induced values are computable from gapped spectral matter via index/eta invariants. In 2+1D they yield topologically massive gauge dynamics and quantized (thermal) Hall responses; in 4+1D they implement anomaly inflow for 3+1D boundaries. Throughout, small constitutive anisotropies of the medium do not affect integrality or anomaly matching, only the non-topological kinetic sector.

5.3 Bardeen counterterms and scheme dependence

Aim. We formalize how *local* polynomial counterterms—the Bardeen (BZ) counterterms—rearrange anomalies among currents without changing physical content. We (i) relate *consistent* and *covariant* anomalies by explicit transgression functionals, (ii) give constructive formulae for Abelian $U(1)$ mixtures and for non-Abelian$\times U(1)$ sectors, (iii) state the BRST-cohomological meaning of scheme changes, (iv) connect to triangle-diagram momentum routing, and (v) fix a *spectral* scheme adapted to our background-field RG and to gauged WZW/inflow matching.

1. Consistent vs. covariant: definitions and relation

Consistent anomaly. Let $W[\mathcal{A},\omega]$ be the quantum effective action after integrating out chiral spectral matter. Under gauge/Lorentz variations $(\delta\mathcal{A},\delta\omega)=(-D\epsilon,-D\theta)$,

$$\delta W \;=\; 2\pi i\int_{M_4} I_4^{(1)}(\epsilon,\theta;\mathcal{A},\omega), \tag{5.28}$$

with $I_4^{(1)}$ the descent of the anomaly polynomial I_6 (Sec. 4.1). The associated *consistent* currents $J_{\rm cons}$ satisfy the Wess–Zumino (WZ) conditions but do not transform covariantly.

Covariant anomaly and BZ shift. Add a local polynomial $X_4^{\rm BZ}(\mathcal{A},\omega)$:

$$W_{\rm cov} \;:=\; W \;+\; \int_{M_4} X_4^{\rm BZ}(\mathcal{A},\omega). \tag{5.29}$$

Define covariant currents by $J_{\rm cov}:=\delta W_{\rm cov}/\delta\mathcal{A}$. Since $X_4^{\rm BZ}$ is local,

$$\delta W_{\rm cov} \;=\; 2\pi i\int_{M_4} I_4^{(1)}\Big|_{\rm cov}, \qquad J_{\rm cov} \;=\; J_{\rm cons} \;+\; \frac{\delta}{\delta\mathcal{A}}\int X_4^{\rm BZ}. \tag{5.30}$$

Thus *scheme choices* correspond to different BZ polynomials, preserving the cohomology class of I_6 while redistributing anomaly among currents. Inflow (Sec. 4.4) fixes a canonical $X_4^{\rm BZ}$ as a boundary transgression.

2. General transgression form of BZ counterterms

Relative Chern–Simons. Let $\bar{\mathcal{A}}$ be a reference (background) connection. The relative Chern–Simons identity gives

$$Q_5(\mathcal{A})-Q_5(\bar{\mathcal{A}}) \;=\; d\,Q_4^T(\mathcal{A},\bar{\mathcal{A}}), \qquad Q_4^T(\mathcal{A},\bar{\mathcal{A}}) \;=\; 3\int_0^1 dt\ \mathrm{Tr}\Big[(\mathcal{A}-\bar{\mathcal{A}})\,F_t^{\,2}\Big], \tag{5.31}$$

with $\mathcal{A}_t=\bar{\mathcal{A}}+t(\mathcal{A}-\bar{\mathcal{A}})$, $F_t=d\mathcal{A}_t+\mathcal{A}_t^2$. A convenient and geometric choice is

$$\boxed{\;X_4^{\rm BZ}(\mathcal{A};\bar{\mathcal{A}}) \;=\; 2\pi i\ k\ Q_4^T(\mathcal{A},\bar{\mathcal{A}})\,,\;} \tag{5.32}$$

with k the anomaly index entering I_6. This realizes the consistent→covariant shift and is precisely the boundary counterterm converting inflow into a manifestly gauge-invariant bulk+boundary system (Sec. 4.4).

3. Abelian $U(1)_A \times U(1)_B$: explicit algebra

Consistent anomaly data. Let charges q_A, q_B define the sums

$$\mathcal{A}_{AAA} = \sum_\Psi \sigma\, q_A^3, \quad \mathcal{A}_{AAB} = \sum_\Psi \sigma\, q_A^2 q_B, \quad \mathcal{A}_{ABB} = \sum_\Psi \sigma\, q_A q_B^2, \quad \mathcal{A}_{BBB} = \sum_\Psi \sigma\, q_B^3, \tag{5.33}$$
$$\mathcal{A}_{\text{grav-A}} = \sum_\Psi \sigma\, q_A, \qquad \mathcal{A}_{\text{grav-B}} = \sum_\Psi \sigma\, q_B.$$

The *consistent* Ward identities (cf. Sec. 4.2) read, in differential form notation ($F = dA$),

$$d \star J_A^{\text{cons}} = \frac{1}{24\pi^2}\Big(\mathcal{A}_{AAA}\, F_A F_A + 3\,\mathcal{A}_{AAB}\, F_A F_B\Big) \;+\; \frac{1}{192\pi^2}\,\mathcal{A}_{\text{grav-A}}\,\mathrm{tr}(RR), \tag{5.34}$$
$$d \star J_B^{\text{cons}} = \frac{1}{24\pi^2}\Big(\mathcal{A}_{BBB}\, F_B F_B + 3\,\mathcal{A}_{ABB}\, F_A F_B\Big) \;+\; \frac{1}{192\pi^2}\,\mathcal{A}_{\text{grav-B}}\,\mathrm{tr}(RR). \tag{5.35}$$

BZ polynomial and covariant split. Consider the BZ functional (suppressing wedge $\wedge$)

$$X_4^{\text{BZ}}[A,B] \;=\; \frac{1}{24\pi^2}\,(c_1\, A\,A\,F_A \;+\; c_2\, A\,A\,F_B \;+\; c_3\, A\,B\,F_A \;+\; c_4\, A\,B\,F_B)\,, \tag{5.36}$$

with real constants c_i. Its variation gives local shifts of the Ward identities. One can *preserve* $U(1)_B$ gauge invariance ($d \star J_B^{\text{cov}} = 0$) by choosing

$$c_4 \;=\; -\mathcal{A}_{BBB}, \qquad c_3 \;=\; -3\,\mathcal{A}_{ABB}, \tag{5.37}$$

after which the entire Abelian anomaly is carried by J_A^{cov}:

$$d\star J_A^{\text{cov}} \;=\; \frac{1}{24\pi^2}\Big(\mathcal{A}_{AAA}\, F_A F_A + 3\,\mathcal{A}_{AAB}\, F_A F_B\Big) \;+\; \frac{1}{192\pi^2}\,\mathcal{A}_{\text{grav-A}}\,\mathrm{tr}(RR), \qquad d\star J_B^{\text{cov}} = 0. \tag{5.38}$$

The remaining (c_1, c_2) are scheme redundancies that shift contact terms only. Equation (5.38) is the Abelian analogue of the "vector-preserving" Bardeen choice in the V/A basis.

Triangle routing map. In momentum space the one-loop AVV/AAA triangle admits a routing parameter a; different a correspond to adding local polynomials $\propto \epsilon^{\mu\nu\rho\sigma} A_\mu A_\nu \partial_\rho A_\sigma$, i.e. to different $(c_1, \ldots, c_4)$. Choosing (5.37) is equivalent to the standard Bardeen routing that enforces the $U(1)_B$ Ward identity.

4. Non-Abelian$\times U(1)$: keeping G manifest

Goal. For a simple factor G with connection $\mathcal{A}$ and an Abelian $U(1)$ gauge field A, we want non-Abelian *covariant* conservation, shifting all $G^2U(1)$ anomaly into the Abelian current.

BZ functional. A minimal choice is

$$X_4^{\text{BZ}}[\mathcal{A},A] \;=\; \frac{1}{8\pi^2}\,\mathcal{A}_{G^2\text{-}U(1)}\,\mathrm{Tr}\Big(A\,\mathcal{A}\,d\mathcal{A} + \tfrac{1}{2}\,A\,\mathcal{A}^3\Big), \tag{5.39}$$

with $\mathcal{A}_{G^2\text{-}U(1)}$ the tensor of Eq. (4.20). Then

$$D_\mu J^{\mu\,a}_{G,\mathrm{cov}} \;=\; 0, \qquad \partial_\mu J^{\mu}_{U(1),\mathrm{cov}}\Big|_{G^2U(1)} \;=\; \frac{1}{8\pi^2}\,\mathcal{A}_{G^2\text{-}U(1)}\;\mathrm{tr}\,(F_G\widetilde{F}_G), \tag{5.40}$$

while purely non-Abelian ABJ pieces remain encoded in the flavor/WZW sector if G is global. This is precisely the scheme used in Secs. 4.2 and 5.1 to keep dynamical G-gauge invariance manifest.

5. BRST cohomology: what changes and what does not

Cohomological class is fixed. Let s be the BRST differential. The consistent anomaly is a representative $\mathcal{A}_{\mathrm{cons}} \in H^1_{\mathrm{loc}}(s|d)$. Adding X_4^{BZ} changes the Ward identity by $sX_4^{\mathrm{BZ}} + d(\cdots)$, i.e.

$$\mathcal{A}_{\mathrm{cov}} \;=\; \mathcal{A}_{\mathrm{cons}} \;+\; s\Big(\int X_4^{\mathrm{BZ}}\Big)\,, \qquad [\mathcal{A}_{\mathrm{cov}}] = [\mathcal{A}_{\mathrm{cons}}] \in H^1_{\mathrm{loc}}(s|d). \tag{5.41}$$

Hence scheme dependence is the freedom to add s-exact (plus d-exact) terms; it cannot remove a nontrivial cohomology class and therefore cannot rescue an anomalous gauge symmetry.

6. Physical invariants vs. contact terms

Unchanged observables. S-matrix elements, anomaly-induced decay rates (e.g. $\pi^0 \to \gamma\gamma$), and transport coefficients determined by *integrated* inflow are *scheme independent*. What changes under X_4^{BZ} are local *contact terms* and the distribution of the anomaly among currents (consistent vs. covariant).

Matching to WZW and inflow. The gauged WZW functional (Sec. 5.1) plus the inflow action (Sec. 4.4) fix a preferred scheme: choose X_4^{BZ} so that the dynamical gauge sector obeys *exact* gauge Ward identities, while the entire anomaly is saturated by NG/inflow degrees of freedom. This is our default *spectral scheme*.

7. Spectral scheme: background-field RG and thresholds

Background-field method (BFM). Split $\mathcal{A} = \bar{\mathcal{A}} + a$ (background $\bar{\mathcal{A}}$, quantum a). We adopt $X_4^{\mathrm{BZ}}(\mathcal{A};\bar{\mathcal{A}})$ of Eq. (5.32) so that *background* Ward identities are exact and RG running respects background gauge invariance (Sec. 3).

Threshold decoupling. When a heavy *vectorlike* multiplet is integrated out, anomaly sums remain zero and X_4^{BZ} is unchanged up to d-exact pieces. For heavy *chiral* multiplets, anomaly inflow/WZW degrees of freedom must appear or persist so that the background Ward identities remain exact after the RG step.

8. Worked V/A example (one Dirac fermion)

Setup. Couple a Dirac fermion to V_μ and A_μ via $\bar\psi\gamma^\mu(V_\mu + \gamma_5 A_\mu)\psi$. The consistent anomalies (one-loop) distribute between J_V and J_A. Adding

$$X_4^{\mathrm{BZ}}[V,A] \;=\; \frac{1}{24\pi^2}\int \epsilon^{\mu\nu\rho\sigma}\Big(6\,V_\mu A_\nu \partial_\rho V_\sigma \;+\; 3\,A_\mu A_\nu \partial_\rho A_\sigma\Big)\,d^4x \tag{5.42}$$

enforces $\partial_\mu J_V^{\rm cov} = 0$ and shifts the full anomaly to $J_A^{\rm cov}$:

$$\partial_\mu J_A^{\rm cov} \;=\; \frac{1}{16\pi^2}\Big(F_V\widetilde{F}_V + \tfrac{1}{3}F_A\widetilde{F}_A\Big), \qquad \partial_\mu J_V^{\rm cov} = 0. \tag{5.43}$$

This is the classic Bardeen choice and matches the gauged WZW normalization (Sec. 5.1).

9. Practical checklist

1. Identify which gauge symmetries are *dynamical*. Choose $X_4^{\rm BZ}$ so their Ward identities are exact (covariant conservation).
2. Use the transgression form (5.32) in BFM to maintain background gauge invariance along the RG flow.
3. In Abelian mixes, fix the nonanomalous $U(1)$ by (5.37); leave residual (c_1, c_2) free or fit to contact-term conventions.
4. Match WZW and inflow: the same $X_4^{\rm BZ}$ must render the boundary variation equal to the consistent descent while keeping the dynamical sector manifestly invariant.
5. Remember: BZ counterterms cannot fix a truly anomalous gauge symmetry; they only repackage the same cohomology class.

10. Summary

Bardeen counterterms are local, polynomial functionals that implement the passage from *consistent* to *covariant* Ward identities. In practice they (i) preserve the gauge invariance of dynamical sectors, (ii) shift anomalies into designated (typically Abelian or axial) currents, (iii) align the low-energy WZW functional with UV anomaly data, and (iv) dovetail with inflow as boundary transgressions. Their freedom is exactly the addition of s- and d-exact pieces in BRST cohomology and affects only contact terms, never the anomaly class or physical observables.

5.4 Matching to low-energy EFTs

Aim. We provide a concrete and general *matching* framework that connects the spectral topological functionals (WZW, Chern–Simons, and Bardeen counterterms) to standard low-energy Effective Field Theories (EFTs). The central principle is that the *generating functional with sources* must reproduce the same local and global anomalies in both the microscopic (spectral) and the infrared EFT descriptions. This fixes the normalization of parity-odd couplings and yields sharp predictions for anomalous amplitudes.

1. Matching principle via source generating functionals

Background sources. Let $J = \{A_\mu^a,\ A_\mu^{(U(1))},\ \omega_\mu^{ab},\ \dots\}$ denote background gauge and geometric sources coupled to conserved currents in the UV spectral theory. The full (Euclidean) generating functional is

$$W_{\rm UV}[J] \;=\; W_{\rm reg}[J] \;+\; W_{\rm anom}[J], \qquad \delta W_{\rm anom}[J] \;=\; 2\pi i\int_{M_4} I_4^{(1)}(\epsilon,\theta;J), \tag{5.44}$$

with $I_4^{(1)}$ the descent of the 6-form I_6 (Sec. 4.1). In the IR, one writes an EFT for light fields φ (e.g. Nambu–Goldstone modes $U(x)$, photons, weak bosons, ...):

$$W_{\rm IR}[J] \;=\; -\ln\!\int\![\mathcal{D}\varphi]\;\exp\Big\{-S_{\rm EFT}[\varphi;J]\Big\} \;=\; W^{\rm EFT}_{\rm reg}[J] \;+\; W^{\rm EFT}_{\rm anom}[J]. \tag{5.45}$$

Matching requirement:

$$\boxed{W^{\rm EFT}_{\rm anom}[J] \;=\; W_{\rm anom}[J] \quad\Longleftrightarrow\quad \delta W^{\rm EFT}_{\rm anom}[J] \;=\; \delta W_{\rm anom}[J]\,.} \tag{5.46}$$

In practice, $W^{\rm EFT}_{\rm anom}$ is provided by the gauged WZW functional on M_4 plus Bardeen counterterms, and by CS inflow from Y_5 when appropriate (Secs. 4.4, 5.1, 5.2, 5.3).

Consequences. Equation (5.46) *fixes* all parity-odd, anomaly-saturating couplings in the IR (up to local scheme choices that do not change observables). Anomaly coefficients do *not* run; threshold changes appear only through the decoupling of vectorlike states (no effect) or through compensating WZW/inflow sectors if chiral multiplets are integrated out.

2. Chiral perturbation theory (even/odd intrinsic parity)

Kinematics and power counting. For $G_{\rm flavor} = SU(N_f)_L \times SU(N_f)_R \to H = SU(N_f)_V$ with $N_f \geq 2$, the NG field $U(x) \in SU(N_f)$, and CCWZ one-forms $\theta = U^{-1}dU$, u_μ, the parity-even Lagrangian at $\mathcal{O}(p^2)$ is

$$\mathcal{L}^{(2)} \;=\; \frac{f^2}{4}\,\mathrm{Tr}\big(u_\mu u^\mu\big) \;+\; \frac{f^2}{4}\,\mathrm{Tr}\big(\chi U^\dagger + U\chi^\dagger\big), \tag{5.47}$$

while the unique parity-odd term at $\mathcal{O}(p^4)$ is the *Wess–Zumino–Witten* functional

$$S_{\rm WZW}[U;A_L,A_R] \;=\; k\int_{B^5}\Omega_5(\mathcal{A}^U) - k\int_{B^5}\Omega_5(\mathcal{A}) \;+\; \int_{M_4}\Gamma_4^{\rm BZ}[U;A_L,A_R]. \tag{5.48}$$

Matching (5.46) implies

$$\boxed{k \;=\; k_{\rm spec} \;=\; \sum_{\Psi\ \text{chiral}} \sigma_\Psi\,\kappa_\Psi\,,} \tag{5.49}$$

with $k_{\rm spec}$ the spectral anomaly index (Sec. 5.1).

Benchmark amplitude $\pi^0 \to \gamma\gamma$. Projecting $A_{L,R}$ onto electromagnetism, $A_{\rm em} = s_W B + c_W W^3$, one obtains

$$\mathcal{L}_{\pi^0\gamma\gamma} \;=\; \frac{k\,\alpha_{\rm em}}{4\pi f}\;\pi^0\,F_{\mu\nu}\widetilde{F}^{\mu\nu}\,, \qquad \mathcal{A}(\pi^0\to\gamma\gamma) \;=\; \frac{k\,\alpha_{\rm em}}{\pi f}, \tag{5.50}$$

independent of even-parity low-energy constants (LECs). In QCD-like sectors $k = N_c$ exactly reproduces the observed rate; in the spectral model (5.50) measures $k_{\rm spec}$ directly.

Odd-intrinsic-parity vertices. Expanding (5.48) yields fixed coefficients for processes such as $\omega \to 3\pi$, $\gamma\pi \to \pi\pi$, and anomalous $\pi\rho\omega$ couplings; these are *not* free LECs but are determined by k and the gauge embedding (Sec. 5.1).

3. Electroweak EFT (EWχL and SMEFT) embeddings

Higgs-doublet (SMEFT) basis and CP-odd operators. In a linear realization with Higgs doublet H, the lowest-dimension CP-odd gauge operators are

$$\mathcal{O}_{\widetilde{G}} = f^{ABC} G^{A}_{\mu\nu} G^{B\,\nu\rho}\, \widetilde{G}^{C\mu}_{\rho}, \quad \mathcal{O}_{\widetilde{W}} = \epsilon^{abc} W^{a}_{\mu\nu} W^{b\,\nu\rho}\, \widetilde{W}^{c\mu}_{\rho}, \tag{5.51}$$

$$\mathcal{O}_{H\widetilde{W}B} = (H^\dagger \tau^a H)\, W^{a}_{\mu\nu}\, \widetilde{B}^{\mu\nu}, \qquad \mathcal{O}_{H\widetilde{W}} = (H^\dagger H)\, W^{a}_{\mu\nu}\, \widetilde{W}^{a\,\mu\nu}, \quad \text{etc.} \tag{5.52}$$

The *gauge* topological densities $\mathrm{Tr}\, F\widetilde{F}$ are total derivatives and do not affect perturbative S-matrix elements, but the mixed operator $\mathcal{O}_{H\widetilde{W}B}$ produces physical $Z\gamma$ and $\gamma\gamma$ form factors. Matching the *gauged* WZW functional onto SMEFT at scales $\mu \ll \Lambda_{\rm spec}$ yields

$$\boxed{\frac{c_{H\widetilde{W}B}}{\Lambda^2} = \frac{k_{\rm spec}}{16\pi^2 f^2}\, \mathcal{C}_{\rm embed}(s_W, c_W) + \mathcal{O}\!\left(\frac{p^2}{\Lambda^2_{\rm spec}}\right)}, \tag{5.53}$$

where $\mathcal{C}_{\rm embed}$ encodes the group-theory projection of the flavor anomaly onto electroweak hypercharge and is fixed once the $SU(2)_L \times U(1)_Y$ embedding is specified (Sec. **??**). Equation (5.53) predicts correlated shifts in CP-odd hVV form factors and rare decays (*scheme-independent* combinations).

Electroweak chiral Lagrangian (EWχL). If the Higgs is treated as a singlet and EWSB is nonlinearly realized, the parity-odd sector at NLO is entirely controlled by the gauged WZW functional built from the $SU(2)_L \times U(1)_Y / U(1)_{\rm em}$ coset. Thus the coefficients in the odd-intrinsic-parity basis of EWχL are *fixed* by $k_{\rm spec}$ and the embedding, with no free parameters beyond even-parity LECs.

4. Matching in 2+1D: Hall and thermal responses

K-matrix and conductivities. For multi-$U(1)$ CS theories in 2+1D,

$$S^{(3)}_{\rm top} = \frac{1}{4\pi} \int \Big(K_{ab}\, A^{(a)} dA^{(b)} + 2\, \Lambda_{ab}\, B^{(a)} dA^{(b)} \Big), \tag{5.54}$$

matching to a microscopic spectral defect with anomaly vector t_a gives the electric Hall conductivity

$$\boxed{\sigma_{xy} = \frac{e^2}{h}\, t^\top K^{-1} t\,, \quad K \in \mathbb{Z}^{n\times n} \text{ fixed by UV indices.}} \tag{5.55}$$

Gravitational CS (Sec. 5.2) matches the thermal Hall coefficient:

$$\frac{\kappa_{xy}}{T} = \frac{\pi^2 k_B^2}{3h}\, c_-\,, \qquad c_- = k_{\rm g} = (\text{UV spectral } \eta\text{-index}). \tag{5.56}$$

Equations (5.55)–(5.56) are robust against small Maxwell/anisotropy deformations.

5. Procedure: practical matching algorithm

Step A: Fix sources and scheme. Declare the set of background sources and choose the *spectral Bardeen scheme* in which dynamical gauge symmetries are manifest (Sec. 5.3). This uniquely specifies $W_{\rm anom}[J]$.

Step B: Build the IR EFT. Write the most general EFT consistent with symmetries and power counting: even-parity operators with free LECs; odd-parity sector given by WZW (and CS in odd dimensions).

Step C: Enforce anomaly equality. Impose (5.46): the variation of the EFT functional under gauge/Lorentz transformations equals that of the UV spectral theory. This *fixes* the normalization of WZW/CS terms (hence k, K, k_{g}).

Step D: Map to operator bases. Expand the gauged WZW functional in physical fields and project onto the desired basis (ChPT, EWχL, SMEFT). Read off Wilson coefficients (e.g. $c_{H\widetilde{W}B}$) as in (5.53).

Step E: Validate with soft theorems and low-energy limits. Check Adler zeroes, anomaly pole structures (AVV triangle), and soft-photon/pion theorems. Odd-parity amplitudes at $\mathcal{O}(p^4)$ are *nonrenormalized* in their overall normalization by higher loops (Adler–Bardeen).

6. Examples and correlated predictions

Anomalous decays and form factors.

$$\Gamma(\pi^0 \to \gamma\gamma) \;\propto\; \left(\frac{k\,\alpha_{\mathrm{em}}}{\pi f}\right)^2, \qquad F_{Z\gamma}(q^2) \;\supset\; \frac{k_{\mathrm{spec}}}{16\pi^2 f^2}\,\mathcal{C}_{\mathrm{embed}} \;+\; \cdots \tag{5.57}$$

The same k_{spec} controls $\gamma\pi \to \pi\pi$ and $\omega \to 3\pi$ vertices; fitting one process predicts the others.

Electroweak precision and rare processes. Odd-parity operators induced by WZW matching predict correlated shifts in $h \to Z\gamma$, $Z \to \gamma\gamma^*$, and in triple gauge vertices of the CP-odd class. These must satisfy the positivity/causality gates of Sec. 3.3 (odd-sector contributions appear in dispersive sum rules consistent with analyticity).

7. Remarks on thresholds and decoupling

Vectorlike thresholds. Integrating out vectorlike pairs leaves anomaly sums unchanged and does not modify k_{spec}, hence all odd-parity predictions remain intact; only even-parity LECs receive threshold shifts.

Chiral thresholds and inflow. If a chiral multiplet is removed, its anomaly must be carried by WZW/inflow degrees of freedom that stay in the EFT. Practically, k_{spec} is piecewise constant across scales and changes only when matched to explicit inflow terms.

8. Summary

Anomaly matching (5.46) elevates the WZW/CS functionals from model-building options to *obligatory* components of low-energy EFTs. Their coefficients (k, K, k_{g}) are quantized and fixed by UV spectral indices, independent of RG running and insensitive to small anisotropies. In four dimensions this determines all odd-intrinsic-parity couplings of NG

bosons and their electroweak projections; in three dimensions it fixes electric and thermal Hall responses. The resulting correlations across amplitudes constitute sharp, falsifiable predictions of the spectral framework.

Chapter 6

't Hooft Anomaly Matching in the Spectral Model

6.1 Statement and construction of matching

Aim. We state and prove (in the EFT sense) the *'t Hooft anomaly matching principle* adapted to the spectral (quantum–aether) framework. We identify the relevant global symmetry group, define the ultraviolet (UV) anomaly data (local and global), and give a *constructive* infrared (IR) recipe that produces an effective theory—comprised of massless composites, Nambu–Goldstone (NG) modes with a gauged WZW functional, and possibly a topological sector coupled by inflow—whose anomaly equals the UV one. The result applies in $d = 4$ to both continuous and discrete global symmetries and includes mixed gauge–gravitational pieces.

1. Symmetry data and sources

Global vs. gauged. Let $\mathsf{G}_{\text{gauge}}$ be the (emergent) gauge group realized by fiber isometries (Ch. 1) and G_{gl} the exact global symmetry used for *'t Hooft probes*. By definition, all local anomalies of $\mathsf{G}_{\text{gauge}}$ vanish (Sec. 4); possible *global* (Witten) anomalies of gauged factors are canceled by inflow (Sec. 4.4). The matching principle pertains to G_{gl} and to any faithful subgroup thereof that remains unbroken in the UV.

Background sources. Couple G_{gl} to non-dynamical gauge fields $\mathcal{B}$ and the spacetime to a spin connection ω. Denote by

$$W[J] \equiv W[\mathcal{B}, \omega] \;=\; -\ln Z[\mathcal{B}, \omega] \tag{6.1}$$

the Euclidean generating functional. UV chiral spectral matter (Vols. II/III) induces a nontrivial variation

$$\delta W_{\text{UV}} \;=\; 2\pi i \int_{M_4} I_4^{(1)}(\epsilon, \theta; \mathcal{B}, \omega), \qquad dI_4^{(1)} = 0, \quad dQ_5 = I_6, \tag{6.2}$$

with I_6 the anomaly polynomial of Sec. 4.1 restricted to the *global* sources $\mathcal{B}$, and (ϵ, θ) the parameters of G_{gl} and local Lorentz transformations.

2. Statement: spectral 't Hooft matching

Local (perturbative) matching. Let I_6^{UV} be the six-form computed from the UV chiral content. Any IR description valid below a scale Λ_{IR} must satisfy

$$\boxed{I_6^{\mathrm{UV}}(\mathcal{B}, R) \;=\; I_6^{\mathrm{IR}}(\mathcal{B}, R)} \tag{6.3}$$

as differential forms (i.e. with identical coefficients of $\mathrm{Tr}F^3$, $F^aF^bF^c$, and $\mathrm{Tr}F\,\mathrm{tr}R^2$), where I_6^{IR} includes *all* light contributions: massless chiral composites, the *gauged* WZW functional of NG fields, and any boundary transgressions required by inflow.

Global (nonperturbative) matching. Let $\Omega^5_{\mathrm{spin}}(B\mathsf{G}_{\mathrm{gl}})$ classify global anomalies. The torsion class of the UV theory

$$[\mathfrak{a}_{\mathrm{UV}}] \;\in\; \Omega^5_{\mathrm{spin}}(B\mathsf{G}_{\mathrm{gl}}) \tag{6.4}$$

must equal that of the IR theory:

$$\boxed{[\mathfrak{a}_{\mathrm{UV}}] \;=\; [\mathfrak{a}_{\mathrm{IR}}] \;\in\; \Omega^5_{\mathrm{spin}}(B\mathsf{G}_{\mathrm{gl}})\,,} \tag{6.5}$$

where $[\mathfrak{a}_{\mathrm{IR}}]$ includes the contribution of a possible invertible 5D bulk (inflow) attached to the IR boundary. For instance, for an $SU(2)$ flavor factor this reduces to the mod-2 Witten constraint (Sec. 4.3).

Functional (descent) form. Equivalently, there exists a *local* 4D Bardeen polynomial X_4^{BZ} such that

$$\boxed{W_{\mathrm{UV}}[J] \;-\; W_{\mathrm{IR}}[J] \;=\; \int_{M_4} X_4^{\mathrm{BZ}}[J] \;+\; 2\pi i \int_{Y_5} \Big(Q_5^{\mathrm{UV}} - Q_5^{\mathrm{IR}}\Big),} \tag{6.6}$$

with $dQ_5^{\mathrm{UV}} = I_6^{\mathrm{UV}}$ and $dQ_5^{\mathrm{IR}} = I_6^{\mathrm{IR}}$. Taking a variation yields (6.3); the Y_5 term ensures equality of global classes (6.5).

3. Construction: IR building blocks

(A) Massless composites. Enumerate gauge-invariant chiral composites $\psi_\alpha^{\mathrm{IR}}$ that remain massless. Their contribution to I_6^{IR} is computed by the same group-theory tensors as in Sec. 4.2.

(B) NG sector and gauged WZW. If $\mathsf{G}_{\mathrm{gl}} \to \mathsf{H}$ is spontaneously broken, include $U(x) \in \mathsf{G}_{\mathrm{gl}}/\mathsf{H}$ with the *gauged* WZW functional (Sec. 5.1):

$$S_{\mathrm{WZW}}[U;\mathcal{B}] \;=\; k_{\mathrm{spec}}\Big[\int_{B^5}\Big(\Omega_5(\mathcal{B}^U) - \Omega_5(\mathcal{B})\Big)\Big] \;+\; \int_{M_4} \Gamma_4^{\mathrm{BZ}}[U;\mathcal{B}], \tag{6.7}$$

with k_{spec} the UV spectral anomaly index. This saturates the *difference* between UV and IR fermionic anomalies in broken directions.

(C) Invertible TQFT (inflow). If no massless dof can carry a global torsion class (e.g. Witten's $\mathbb{Z}_2$), couple the boundary IR theory to a 5D invertible bulk with action

$$S_{\text{inflow}}[J] \;=\; 2\pi i \int_{Y_5} Q_5[J] \quad \text{(local anomalies)}\,, \qquad S^{\text{tors}}_{\text{inflow}}[J] \;=\; i\pi \int_{Y_5} \mathcal{I}_5[J] \quad \text{(global torsion)}, \tag{6.8}$$

as in Sec. 4.4. The combined system matches both (6.3) and (6.5).

4. Spectral thresholds and invariance

Vectorlike decoupling. Integrating out a heavy *vectorlike* pair does not change I_6 nor the torsion class. Thus (6.3)–(6.5) are RG-invariant.

Chiral thresholds. Integrating out a heavy *chiral* multiplet changes I_6 by its contribution; the IR must compensate by adding the corresponding WZW/inflow terms that persist below the threshold. Hence k_{spec} is piecewise constant and jumps only when the EFT explicitly acquires/removes anomaly-saturating topological functionals.

5. Worked templates

Confinement without massless fermions. If a confining phase leaves no chiral composites, the entire anomaly is reproduced by a NG WZW term (if G_{gl} is broken) and/or by an invertible bulk. This is the QCD-like template with $k_{\text{spec}} = N_c$.

Higgs phase with unbroken subgroup. If $\mathsf{G}_{\text{gl}} \to \mathsf{H}$, then I_6^{UV} restricted to H must equal the sum of I_6 from massless H-multiplets plus the H-gauged projection of S_{WZW}.

6. Algorithmic construction (practical matching)

1. **Identify** G_{gl} and compute I_6^{UV} and (if present) the torsion class $[\mathfrak{a}_{\text{UV}}]$.
2. **Choose** the IR phase (confinement/breaking/topological) consistent with the spectral gates (S/G/TQ/D).
3. **Assemble** I_6^{IR} from: (i) massless composites; (ii) gauged WZW with coefficient k_{spec}; (iii) inflow terms.
4. **Fix scheme** with a Bardeen polynomial so that dynamical gauge symmetries remain manifest (Sec. 5.3).
5. **Verify** (6.3) coefficient-wise and (6.5) in bordism; test 3-point Ward identities and soft limits.

7. Remarks on discrete and mixed symmetries

Discrete $\mathbb{Z}_N$ and higher-form symmetries. For discrete Γ and higher-form symmetries, replace I_6 by the appropriate cohomology/cobordism invariants and repeat the steps above; inflow is again the unifying mechanism. Mixed anomalies with $\mathsf{G}_{\text{gauge}}$ are *forbidden* (must vanish), but mixed *global* anomalies with gravity are matched exactly as in (6.3).

8. Summary

The spectral version of 't Hooft matching states that *all* anomaly data of the UV global symmetry—local polynomial I_6 and global bordism class—are RG invariants that must be reproduced in any IR phase by a combination of massless composites, NG WZW functionals, and inflow topological terms. Equations (6.3)–(6.6) provide both the *constraints* and a *construction* that will be executed in Secs. 6.2–6.4 for the spectral fermion content and its phenomenology.

6.2 Spectral fermion content and currents

Aim. We specify the *microscopic* and *effective* descriptions of fermionic degrees of freedom in the spectral framework and derive the associated conserved currents that enter anomaly matching. The construction proceeds in three layers: (i) chiral zero modes bound to spectral textures (microscopic origin of fermions), (ii) the covariant source formalism that defines currents and Ward identities, and (iii) benchmark SM-like assignments and composite currents (baryon, lepton, flavor). Throughout we work on the emergent spacetime (M, g_{eff}) with internal fiber $(\mathcal{F}, h)$ and gauge connection $\mathcal{A}$ as developed in Ch. 1.

1. Microscopic origin: chiral zero modes on spectral textures

Spectral Dirac operator. Let Ψ be a multi-component spinor valued in a complex vector bundle $\mathcal{E} \to M$ associated to the fiber representation $\mathcal{R}$ of the gauge/isometry group G. Couple Ψ minimally to the G-connection and to a composite Yukawa background built from the order parameters $\mathcal{S} = \{\Phi, \rho, \eta, \chi^\mu, T\}$:

$$\mathcal{L}_\Psi = \bar{\Psi}\, i\not{D}\, \Psi - \bar{\Psi}\, \mathsf{M}[\Phi, \chi]\Psi, \qquad \not{D} := \gamma^\mu\big(\nabla_\mu + \mathcal{A}_\mu\big), \tag{6.9}$$

with ∇_μ the Levi–Civita spin connection for g_{eff} and M a Hermitian matrix in flavor/fiber space. We assume a *texture* (topological defect or domain) of (Φ, χ) such that $\mathsf{M}(x)$ varies slowly and crosses zero on a codimension-k locus Σ.

Index and chiral zero modes. The Atiyah–Singer index (bulk) plus Atiyah–Patodi–Singer (APS) boundary/defect contributions give the net number of chiral zero modes localized on Σ:

$$\operatorname{Index} \not{D} = \int_M \widehat{A}(R)\, \operatorname{ch}_{\mathcal{R}}(F) \;+\; \tfrac{1}{2}\Big(\eta_{\partial M} + \dim\ker \not{D}_{\partial M}\Big) \;+\; \mathfrak{I}_{\text{tex}}[\Phi, \chi], \tag{6.10}$$

where $\mathfrak{I}_{\text{tex}}$ is a *texture index* determined by the homotopy class of the map $(\Phi, \chi) : S^d \to \mathcal{T}$ (the target $\mathcal{T}$ of normalized order parameters); explicit forms reduce to winding/Hopf numbers in Sec. 13. Equation (6.10) counts $n_L - n_R$ of normalizable zero modes of definite chirality on Σ, thereby generating *chiral* fermions at low energy.

Generations as topological multiplicity (preview). If the texture supports an integer index $Q_{\text{topo}} \in \mathbb{Z}$ (e.g. Hopf degree), the number of replicated chiral families equals $|Q_{\text{topo}}|$:

$$N_{\text{gen}} = |Q_{\text{topo}}|, \qquad \operatorname{sign}(Q_{\text{topo}}) = \text{overall chirality}. \tag{6.11}$$

This will be made precise in Sec. 13; here we use it to parametrize the fermion content.

2. Source formalism and current definitions

Background sources. Introduce non-dynamical sources J for global symmetries:

$$J = \{\, \mathcal{B}_\mu^A T_A \text{ (global gauge/flavor)}, \quad \omega_\mu^{ab} \text{ (spin)}, \quad b_{\mu\nu} \text{ (2-form sources for higher-form symmetries, if an} \tag{6.12}$$

The Euclidean generating functional (with a fixed Bardeen scheme, Sec. 5.3) is

$$W[J] \equiv -\ln Z[J], \qquad J_A^\mu(x) = \frac{\delta W}{\delta \mathcal{B}_\mu^A(x)}, \quad T^{\mu\nu}(x) = \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g_{\mu\nu}(x)}, \quad J_{(b)}^{\mu\nu}(x) = \frac{\delta W}{\delta b_{\mu\nu}(x)}. \tag{6.13}$$

Consistent vs. covariant currents. As in Secs. 4–5, we distinguish

$$J_{\rm cov}^\mu = J_{\rm cons}^\mu + \frac{\delta}{\delta \mathcal{B}_\mu}\int X_4^{\rm BZ}, \qquad \partial_\mu J_{\rm cov}^\mu \;=\; \text{covariant anomaly density.} \tag{6.14}$$

The *spectral scheme* keeps dynamical gauge Ward identities exact while assigning global anomalies to NG/inflow sectors.

Noether (microscopic) form. At the level of fields,

$$J_A^\mu \;=\; \bar\Psi\,\gamma^\mu T_A \Psi + J_{A,\rm comp}^\mu[\mathcal{S}, U]\,, \qquad J_{A,\rm comp}^\mu = \text{(composite pieces from NG/WZW and medium respon} \tag{6.15}$$

The composite contribution ensures that the full current reproduces the anomaly via WZW/inflow even when fermions confine/higgs.

3. Decompositions: vector/axial, chiral, and higher-form

Vector/axial in a flavor group. For $G_{\rm flavor} = G_L \times G_R$, define

$$J_V^\mu = J_L^\mu + J_R^\mu, \qquad J_A^\mu = J_L^\mu - J_R^\mu, \qquad J_{L/R}^\mu = \bar\Psi \gamma^\mu P_{L/R}\, T\, \Psi + \cdots, \tag{6.16}$$

with $P_{L/R} = (1 \mp \gamma_5)/2$ and the ellipsis denoting NG/WZW pieces. In the Bardeen vector-preserving scheme, $\partial_\mu J_{V,\rm cov}^\mu = 0$ while $\partial_\mu J_{A,\rm cov}^\mu$ reproduces the consistent ABJ density.

Baryon/lepton currents. For SM-like assignments,

$$J_B^\mu = \frac{1}{3}\sum_{\text{quarks}} \bar\Psi\gamma^\mu\Psi \;+\; J_{B,\rm comp}^\mu, \qquad J_L^\mu = \sum_{\text{leptons}} \bar\Psi\gamma^\mu\Psi \;+\; J_{L,\rm comp}^\mu. \tag{6.17}$$

The anomaly-free combination under the SM gauge group is $B-L$ at the *global* level; gauging $B-L$ requires the presence of right-handed neutrinos (see the benchmark below).

Higher-form currents (if present). If the strong sector admits a conserved $U(1)$ one-form symmetry (e.g. center symmetry in a confining phase), couple a two-form source b. The generalized current $J_{(b)}^{\mu\nu}$ obeys

$$\partial_\mu J_{(b)}^{\mu\nu} = \frac{1}{4\pi}\,\kappa^{ab}\,\epsilon^{\nu\rho\sigma\lambda} F_{\rho\sigma}^a F_{\lambda\mu}^b u^\mu + \cdots, \tag{6.18}$$

encoding mixed higher-form anomalies (to be used in Parts IV, VII).

4. Benchmark SM-like spectral content

Field content per generation. Let $\mathsf{G}_{\rm gauge} = SU(3)_c \times SU(2)_L \times U(1)_Y$ and consider one chiral family (LH Weyl fields):

$$Q_L : (\mathbf{3},\mathbf{2})_{+1/6}, \quad u_R : (\mathbf{3},\mathbf{1})_{+2/3}, \quad d_R : (\mathbf{3},\mathbf{1})_{-1/3}, \quad L_L : (\mathbf{1},\mathbf{2})_{-1/2}, \quad e_R : (\mathbf{1},\mathbf{1})_{-1}\,, \tag{6.19}$$

optionally $\nu_R : (\mathbf{1},\mathbf{1})_0$ if $U(1)_{B-L}$ is to be *gauged*. The microscopic origin is a set of $n_{\rm gen}$ chiral zero modes counted by (6.10)–(6.11).

Gauge currents. With generators T_c^a, t^i and hypercharge Y:

$$J^{\mu,a}_{(3)} = \sum_{\text{colored}} \bar\Psi \gamma^\mu T_c^a \Psi\,, \qquad J^{\mu,i}_{(2)} = \sum_{\text{doublets}} \bar\Psi \gamma^\mu t^i \Psi\,, \qquad J^\mu_Y = \sum_{\text{all}} Y\, \bar\Psi \gamma^\mu \Psi\,, \tag{6.20}$$

supplemented by composite pieces so that $D_\mu J^{\mu,a}_{(3),\rm cov} = D_\mu J^{\mu,i}_{(2),\rm cov} = 0$ in our scheme, while $\partial_\mu J^{\rm cov}_Y$ carries mixed $G^2U(1)$ anomalies as in Sec. 4.2.

Global currents. The canonical flavor group in the chiral-symmetric limit is $U(3)^5$ (one factor per multiplet in (6.19)); Yukawa couplings break it to subgroups. The anomaly-relevant Abelian combinations are

$$B = \tfrac{1}{3}(Q_L + u_R + d_R), \qquad L = L_L + e_R(+\nu_R), \qquad B-L, \tag{6.21}$$

with $B-L$ anomaly-free with respect to $\mathsf{G}_{\rm gauge}$ in the presence of ν_R (global anomaly-free without ν_R, but *not* gaugeable).

Anomaly tensors (per generation). Define (consistent) sums as in Eqs. (4.19)–(4.22). For the set (6.19) one has

$$\mathcal{A}_{SU(3)^3} = 0, \quad \mathcal{A}_{SU(2)^3} = 0, \quad \mathcal{A}_{SU(3)^2U(1)_Y} = 0, \quad \mathcal{A}_{SU(2)^2U(1)_Y} = 0, \quad \mathcal{A}_{U(1)_Y^3} = 0, \quad \mathcal{A}_{\rm grav^2U(1)_Y} = 0, \tag{6.22}$$

and the mod-2 Witten condition for $SU(2)_L$ is satisfied since there are 4 doublets per family (Sec. 4.3). If $U(1)_{B-L}$ is gauged, the presence of ν_R ensures all $G^2(B-L)$ and $(B-L)^3$ anomalies vanish.

5. Composite currents and matching when fermions are gapped

Confinement/Higgsing. If some microscopic fermions confine or become massive via spectral Higgsing, their anomalies must be reproduced by NG fields and/or inflow. Decompose the full current as

$$J^\mu_A \;=\; J^\mu_{A,\text{light fermions}} \;+\; J^\mu_{A,\rm WZW} \;+\; J^\mu_{A,\rm inflow}, \tag{6.23}$$

where $J^\mu_{A,\rm WZW} = \delta S_{\rm WZW}/\delta \mathcal{B}^A_\mu$ and $J^\mu_{A,\rm inflow}$ is a boundary functional derivative of the 5D CS action (Sec. 4.4). Then $d \star J_A$ reproduces exactly the UV anomaly polynomial restricted to the IR sources, realizing (6.3).

Baryon number from topology (strong sector). In confining sectors, the baryon current can acquire a topological term (Skyrme/WZW):

$$J^{\mu}_{B,\mathrm{top}} \;=\; \frac{k}{24\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big(U^{-1}\partial_\nu U\,U^{-1}\partial_\rho U\,U^{-1}\partial_\sigma U\Big), \tag{6.24}$$

whose charge equals the winding number of $U(x)$ (baryon number), with k fixed by the UV anomaly index k_{spec} (Sec. 5.1).

6. Spectral constraints and gates

S/G/TQ/D gates for fermion sectors. *(S)* Positive-definite kinetic terms and absence of ghosts for fermions and NG modes. *(G)* Hyperbolicity of Dirac and NG equations in (M, g_{eff}); small χ^μ-induced anisotropies must preserve the light-cone ordering (Sec. 1.3). *(TQ)* Exact anomaly matching via WZW/inflow; Adler–Bardeen nonrenormalization protects the normalization of odd-parity couplings. *(D)* Dissipative channels (finite η) may renormalize even-parity LECs but *cannot* modify anomaly coefficients.

Thresholds. Vectorlike thresholds do not change anomaly sums nor k_{spec}; chiral thresholds do, and the jump is carried by persistent WZW/inflow terms (Sec. **??**).

7. Practical checklist for model instances

1. **Choose textures:** specify (Φ, χ) and compute the texture index $\mathfrak{I}_{\mathrm{tex}}$; deduce N_{gen} via (6.11).
2. **Assign reps:** embed zero modes into G-representations as in (6.19) (or a chosen extension).
3. **Compute I_6^{UV}:** evaluate anomaly tensors with the chiral content; verify (6.22).
4. **Build currents:** write J^μ_A via (6.15), add $J^\mu_{A,\mathrm{WZW}}$ and (if needed) $J^\mu_{A,\mathrm{inflow}}$.
5. **Fix scheme:** adopt the spectral Bardeen choice so that dynamical $\mathsf{G}_{\mathrm{gauge}}$ currents are covariantly conserved.
6. **Validate:** check Ward identities, soft theorems, and anomaly-induced amplitudes (e.g. $\pi^0 \to \gamma\gamma$) at low energy.

8. Summary

Fermions in the spectral model arise as chiral zero modes localized on topological textures of the medium; their multiplicity (generations) is topological. Currents are defined unambiguously by functional differentiation with respect to background sources and decompose into microscopic, NG/WZW, and inflow pieces. In SM-like benchmarks the gauge anomalies vanish per family and the global $SU(2)$ Witten constraint is satisfied. Upon confinement or Higgsing, composite currents—fixed by the same anomaly data—guarantee exact 't Hooft matching in the IR.

6.3 Implications for flavor structure and generations

Aim. We develop a *geometric–topological* account of flavor in the spectral model. Generations arise as *topological multiplicities* of chiral zero modes; hierarchies and mixings follow from *overlap integrals* of localized mode profiles on the internal fiber $\mathcal{F}$ and along codimension-one spectral textures in M. CP violation is attributed to *Berry phases* (holonomies) of families in parameter space. We state explicit formulas for Yukawa textures, CKM/PMNS mixing matrices, and neutrino seesaw structures, and we list anomaly/matching constraints on family symmetries.

1. Generations as topological multiplicities

Family index. Let $\mathscr{T}$ denote the target of normalized order parameters (e.g. phase/director variables of Φ, χ). A smooth configuration (Φ, χ) with asymptotics defines a class $[\varphi] \in \pi_d(\mathscr{T})$ on linking spheres S^d around the locus Σ where the composite mass $\mathsf{M}(x)$ changes sign. The net number of normalizable chiral zero modes (per representation $\mathcal{R}$) equals the *texture index*

$$N_{\text{gen}} \;=\; \Big|\mathfrak{I}_{\text{tex}}[\Phi,\chi]\Big| \in \mathbb{Z}, \qquad \operatorname{sign}\mathfrak{I}_{\text{tex}} = \text{chirality}. \tag{6.25}$$

Typical realizations are degree (winding) maps for $\pi_3(S^3) = \mathbb{Z}$ or Hopf degree for $\pi_3(S^2) = \mathbb{Z}$, yielding three families when $|\mathfrak{I}_{\text{tex}}| = 3$. This *family index* is RG-invariant and must be preserved through phase transitions by 't Hooft matching (Sec. 6.1).

Zero-mode bundle. Zero modes span a rank-N_{gen} complex vector bundle $\mathcal{E}_{\text{fam}} \to \Sigma$ with an induced Berry connection $\mathcal{A}^{\text{fam}}$ from adiabatic transport in texture moduli. Its curvature $\mathcal{F}^{\text{fam}}$ controls geometric phases relevant for CP violation (Sec. 6.3).

2. Mode localization and Yukawa overlaps

Profiles. Left- and right-chiral zero modes localize near distinct centers in $(M \times \mathcal{F})$:

$$\Psi_L^{(i)}(x,y) \;\simeq\; \varphi_L^{(i)}(x)\,\phi_L^{(i)}(y)\,u_L(x), \qquad \Psi_R^{(j)}(x,y) \;\simeq\; \varphi_R^{(j)}(x)\,\phi_R^{(j)}(y)\,u_R(x), \tag{6.26}$$

where $u_{L/R}$ are 4D spinors and $\phi_{L/R}^{(i)}$ are normalized internal/fiber wavefunctions, sharply peaked around $y = y_{L/R}^{(i)}$ with widths $\sigma_{L/R}^{(i)}$ determined by the spectral stiffness and the slope of $\mathsf{M}(x)$ across the texture.

Yukawa matrices as overlaps. Let H denote the (composite) Higgs resonance with internal profile $h(y)$ and vacuum amplitude v. The *Dirac* Yukawas are

$$(Y_f)_{ij} \;=\; \lambda_f \int_{\mathcal{F}} d\mu_h(y)\; \big(\phi_L^{(i)}(y)\big)^{\dagger}\, h(y)\, \phi_R^{(j)}(y), \qquad m_f = \frac{v}{\sqrt{2}}\, Y_f, \quad f = u,d,e,\nu_D. \tag{6.27}$$

For Gaussian localization on $\mathcal{F}$ equipped with metric h_{AB},

$$\phi_{L/R}^{(i)}(y) \propto \exp\Big[-\tfrac{1}{2}\,(y-y_{L/R}^{(i)})^A\,\Sigma_{AB}^{L/R}\,(y-y_{L/R}^{(i)})^B\Big], \qquad h(y) \approx \text{const}, \tag{6.28}$$

one obtains the *factorized* texture

$$(Y_f)_{ij} \;\simeq\; Y_{0,f}\;\exp\Big[-\tfrac{1}{2}\,\Delta y_{ij}^A\;\Lambda_{AB}\;\Delta y_{ij}^B\Big], \qquad \Delta y_{ij} := y_L^{(i)} - y_R^{(j)}\,, \tag{6.29}$$

with a positive matrix Λ_{AB} fixed by widths and h_{AB}. Exponential sensitivity in geodesic separation $d_{ij}^2 := \Delta y_{ij} \Lambda \, \Delta y_{ij}$ naturally generates mass hierarchies.

Proposition (hierarchies from geometry). *If* $(Y_f)_{ij}$ has the form (6.29) with generic, non-degenerate centers, then (i) the singular values (masses) are exponentially ordered by the smallest d_{ij} per row/column; (ii) left and right diagonalizing rotations inherit *angle* dependence from the triangles spanned by $\{y_L^{(i)}\}$ and $\{y_R^{(j)}\}$ in $\mathcal{F}$. Hence CKM/PMNS mixing angles are controlled by *relative geometry* of localization centers.

3. CKM/PMNS from spectral geometry

CKM structure. Let $Y_u = U_{uL}^\dagger \, D_u \, U_{uR}$, $Y_d = U_{dL}^\dagger \, D_d \, U_{dR}$ with $D_{u,d}$ diagonal and positive. The CKM matrix is $V_{\rm CKM} = U_{uL} U_{dL}^\dagger$. In the overlap model (6.29), for nearly collinear triplets $\{y_L^{(i)}\}$ and mild misalignment between $\{y_R^{(j)}\}$ across sectors,

$$|V_{us}| \;\sim\; \exp\!\Big[-\tfrac{1}{4}\Big(d_{12}^2(u) - d_{11}^2(u) + d_{12}^2(d) - d_{11}^2(d) \Big)\Big], \tag{6.30}$$

and similarly for V_{cb}, V_{ub}. Thus small CKM angles arise from *near alignment* of left-handed localization centers across up/down sectors.

PMNS largeness. Leptonic angles are large if the left-handed lepton centers $\{y_{L,\ell}^{(i)}\}$ are clustered (degenerate geometry) while right-handed charged leptons are split; neutrino mixings additionally depend on the Majorana sector (next subsection), favoring near-democratic left profiles.

4. CP violation from Berry holonomy

Family Berry connection. Let $\boldsymbol{\lambda}$ denote slow parameters of the texture (positions, phases in Φ). The family bundle $\mathcal{E}_{\rm fam} \to \mathcal{M}_\lambda$ carries a Berry connection $(\mathcal{A}^{\rm fam})_{ij} = i\langle \phi_i | \partial_{\boldsymbol{\lambda}} \phi_j \rangle \cdot d\boldsymbol{\lambda}$. For a closed loop $\gamma \subset \mathcal{M}_\lambda$, the holonomy $\mathcal{U}_\gamma = \mathcal{P} \exp \oint_\gamma \mathcal{A}^{\rm fam}$ contributes a physical phase to the product of Yukawas.

Jarlskog invariant as flux. In a weakly deformed regime, the CKM Jarlskog invariant admits the representation

$$J_{\rm CKM} \;\simeq\; \frac{\mathrm{Area}_\triangle(\mathcal{F}_{\rm eff})}{\prod_{i<j}(m_{u,i}^2 - m_{u,j}^2)\,\prod_{k<\ell}(m_{d,k}^2 - m_{d,\ell}^2)}\;\Phi_{\rm Berry}, \qquad \Phi_{\rm Berry} := \int_\Sigma \mathrm{tr}\, \mathcal{F}^{\rm fam}, \tag{6.31}$$

where Σ is a 2-chain in parameter space bounded by the adiabatic path defining the triangle of left-handed centers. Thus $\delta_{\rm CP}$ is a *geometric* phase, stable under local rephasings and insensitive to short-distance details (topological protection).

5. Neutrinos: spectral seesaw and mixing

Majorana sector from textures. Introduce sterile modes $N_R^{(a)}$ localized on a distinct texture with centers $y_N^{(a)}$ and a Majorana mass kernel $\mathcal{M}_R(y, y')$ supported near $y \simeq y'$.

The overlaps give

$$(M_R)_{ab} \;=\; \Lambda_R \int_{\mathcal{F}} d\mu_h \, \left(\phi_N^{(a)}\right)^* \mathcal{K}_R(y)\, \phi_N^{(b)}, \qquad (m_D)_{ia} \;=\; \lambda_\nu \int_{\mathcal{F}} d\mu_h \, \left(\phi_{L,\ell}^{(i)}\right)^* h(y)\, \phi_N^{(a)}. \tag{6.32}$$

The light neutrino mass matrix follows the Type-I seesaw:

$$M_\nu \;=\; -\, m_D\, M_R^{-1}\, m_D^{\top}. \tag{6.33}$$

Large PMNS angles emerge when m_D has near-democratic left overlap (clustered $y_{L,\ell}^{(i)}$) while M_R is mildly hierarchical. Majorana phases trace back to Berry phases of the sterile bundle.

6. Selection rules, texture zeros, and family symmetries

Geometric selection rules. If $\mathcal{F}$ admits isometries $U(1)^r$ with charges $q_\alpha[\phi^{(i)}]$, then the overlap (6.27) vanishes unless charge is conserved:

$$q_\alpha[\phi_L^{(i)}] + q_\alpha[h] - q_\alpha[\phi_R^{(j)}] = 0 \quad \forall\ \alpha. \tag{6.34}$$

This realizes *texture zeros* and rank deficiencies without fine tuning. Small symmetry breaking in the medium lifts zeros with controlled powers, reproducing Froggatt–Nielsen-like suppressions.

Anomaly conditions for family gaugeings. If a family symmetry $U(1)_F$ is gauged, anomaly cancellation imposes

$$\sum_{\Psi} \sigma\, q_F(\Psi)\, T_2^G(\Psi) = 0, \quad \sum_{\Psi} \sigma\, q_F(\Psi)^3 = 0, \quad \sum_{\Psi} \sigma\, q_F(\Psi) = 0, \tag{6.35}$$

for each gauged factor G and gravity. Otherwise $U(1)_F$ is global and its anomaly is matched by a WZW term in the NG sector (Sec. 5).

7. Worked geometric template

Minimal fiber template. Take $\mathcal{F}$ with local coordinates $(\theta, \vec{\zeta})$ where $\theta \in S^1$ (phase) and $\vec{\zeta} \in \mathbb{R}^2$ (two stiffness eigen-directions). Place three LH centers at nearly collinear $\vec{\zeta}_L^{(i)}$ and RH centers $\vec{\zeta}_R^{(j)}$ spread in a skew direction; choose h uniform. Then

$$Y_{ij} \propto \exp\!\Big[-\tfrac{1}{2}\big(\vec{\zeta}_L^{(i)} - \vec{\zeta}_R^{(j)}\big)^{\top} \Lambda \big(\vec{\zeta}_L^{(i)} - \vec{\zeta}_R^{(j)}\big) \Big]\, e^{\,i(\theta_L^{(i)} - \theta_R^{(j)} + \theta_h)}, \tag{6.36}$$

which yields (i) hierarchical masses, (ii) small CKM angles (near alignment of $\vec{\zeta}_L$ between up/down), and (iii) an irreducible CKM phase from a nontrivial loop in θ-space.

8. RG stability and spectral gates

Stability under running. The exponential geometry in (6.29) is renormalized multiplicatively by wavefunction factors; the *ratios* of distances d_{ij}^2 and the Berry holonomies are invariant under soft deformations. Anomaly-related quantities (k in WZW, inflow coefficients) do *not* run.

S/G/TQ/D gates. (S) Positive-definite kinetic terms for families and NG sectors; (G) hyperbolic propagation with small χ^μ anisotropies; (TQ) anomaly matching enforced by WZW/inflow; (D) dissipative renormalizations cannot alter quantized Berry/WZW indices.

9. Practical checklist

1. **Fix topology:** choose (Φ, χ) with $\mathfrak{I}_{\text{tex}} = 3$; compute zero-mode bundle $\mathcal{E}_{\text{fam}}$ and $\mathcal{F}^{\text{fam}}$.
2. **Localize centers:** determine $y_{L/R}^{(i)}$ and widths from the background $\mathsf{M}(x)$ and fiber stiffness; evaluate Y_f via (6.27).
3. **Diagonalize:** obtain V_{CKM}, U_{PMNS}; extract J_{CKM} and Majorana phases.
4. **Impose selection rules:** apply (6.34); identify texture zeros and their controlled liftings.
5. **Validate anomalies:** if a family $U(1)_F$ is gauged, enforce (6.35); otherwise include WZW for broken global factors.

10. Summary

In the spectral model, *why three families?*—because the family number equals a topological index of the underlying texture. *Why hierarchies and small CKM angles?*—because Yukawas are exponentially sensitive to geometric separations of localized modes on $\mathcal{F}$, with near alignment for left quarks. *Why large leptonic mixing?*—because lepton left modes are clustered and the neutrino sector employs a spectral seesaw. *Where does CP violation come from?*—from Berry holonomies of the family bundle. All of these features are compatible with 't Hooft anomaly matching, with odd-parity normalizations fixed by UV spectral indices and protected along the RG flow.

6.4 Constraints on emergent symmetry breaking

Aim. We formulate *nonnegotiable* constraints on spontaneous/emergent symmetry breaking patterns in the spectral (quantum–aether) framework. These constraints combine: (i) 't Hooft anomaly matching (local and global), (ii) homotopy/cohomology data of the vacuum manifold (WZW quantization, inflow), (iii) S/G/TQ/D gates (stability, hyperbolicity, unitarity/positivity, dissipation) of Vol. I and Secs. 3.3, 5.1–5.3, and (iv) RG and decoupling rules of Ch. 3. We state general theorems with proof sketches and provide practical tests for candidate breaking patterns.

1. Kinematical and topological prerequisites

Goldstone and coset data. A breaking pattern $\mathsf{G}_{\text{gl}} \to \mathsf{H}$ (global sector) defines a coset $\mathcal{M} = \mathsf{G}_{\text{gl}}/\mathsf{H}$. Let $\dim \mathcal{M} = N_{\text{NG}}$ denote the number of type-A NG bosons (possible type-B pairs require nonzero density of noncommuting charges; absent at zero density). The NG kinetic term is fixed by the pullback of the fiber metric (Sec. 5.1).

WZW admissibility. A necessary and sufficient condition for a nontrivial WZ functional is

$$H^5(\mathcal{M},\mathbb{Z}) \neq 0 \quad \Longleftrightarrow \quad \exists\, \Omega_5 \text{ closed, integer-quantized}\,, \tag{6.37}$$

with coefficient $k \in \mathbb{Z}$ fixed by UV anomaly indices (Sec. 5.1). If (6.37) holds, any IR phase realizing $\mathsf{G}_{\rm gl} \to \mathsf{H}$ *must* include $S_{\rm WZW}$ with that k.

Global anomalies and torsion. Let the global anomaly class be $[\mathfrak{a}] \in \Omega^5_{\rm spin}(B\mathsf{G}_{\rm gl})$ (Sec. 4.3). If $[\mathfrak{a}] \neq 0$, a trivially gapped symmetric phase is impossible; either massless modes, SSB with WZW, or a 5D invertible bulk (inflow) must appear.

2. No-go theorem for symmetric trivial gapped phases

Theorem (spectral anomaly obstruction). *Consider a UV spectral theory with global symmetry* $\mathsf{G}_{\rm gl}$ *and anomaly data* $(I_6,[\mathfrak{a}]) \neq (0,0)$. *Then no IR phase at zero temperature can be simultaneously:*

1. *fully symmetric* (no SSB of $\mathsf{G}_{\rm gl}$),
2. *trivially gapped* (no massless excitations, no topological order/inflow),
3. *local, unitary, Lorentz-compatible* (S/G/TQ/D satisfied).

Proof (sketch). Assume such a phase exists. Its generating functional $W_{\rm IR}[J]$ is local and gauge/Lorentz invariant for sources J, hence $\delta W_{\rm IR} = 0$ under all background transformations. 't Hooft matching (Sec. 6.1) requires $\delta W_{\rm UV} = \delta W_{\rm IR}$; but $\delta W_{\rm UV} = 2\pi i \int I_4^{(1)} \neq 0$ if $I_6 \neq 0$, or a nontrivial torsion phase if $[\mathfrak{a}] \neq 0$. Contradiction unless the IR includes anomaly carriers: massless fields, WZW/inflow, or topological order. □

Corollary (allowed resolutions). An anomalous UV may flow only to IR phases of the form:

$$\text{IR} \;=\; (\text{gapless}) \;\oplus\; (\text{SSB with WZW}) \;\oplus\; (\text{invertible TQFT inflow})\,, \tag{6.38}$$

with the sum saturating both I_6 and $[\mathfrak{a}]$.

3. Constraints on $\mathsf{G}_{\rm gauge}$ and mixed patterns

Gauge-sector admissibility. Local gauge anomalies of $\mathsf{G}_{\rm gauge}$ must vanish:

$$I_6\big|_{\mathsf{G}_{\rm gauge}} \;=\; 0, \qquad [\mathfrak{a}]\big|_{\mathsf{G}_{\rm gauge}} \;=\; 0, \tag{6.39}$$

with the $SU(2)$ Witten condition satisfied mod 2 (Sec. 4.3). Consequently, any proposed Higgs pattern $\mathsf{G}_{\rm gauge} \to \mathsf{H}_{\rm gauge}$ must preserve these cancellations across thresholds (Ch. 3); if a chiral multiplet is decoupled, its contribution must be replaced by persistent WZW/inflow terms *in the global sector only.*

Mixed global–gauge constraints. For a global $U(1)_X$ mixing with gauge factors, the IR must reproduce the covariant Ward identities (Sec. 5.3):

$$\partial_\mu J^\mu_{X,\rm cov} \;=\; \frac{1}{24\pi^2}\, C_{abc}\, F^a F^b F^c \;+\; \frac{1}{192\pi^2}\, C_{\rm grav}\, \mathrm{tr}(RR)\,, \tag{6.40}$$

either via massless fermions or via the gauged WZW functional. Gauging $U(1)_X$ is *forbidden* unless the RHS vanishes identically (e.g. $B-L$ requires ν_R).

4. Fiber/texture compatibility and homotopy obstructions

Texture index conservation. Let $\mathfrak{I}_{\text{tex}}$ be the family (zero-mode) index from Sec. 6.2. Any continuous deformation (RG flow, Higgsing) that preserves S/G/TQ/D gates satisfies

$$\Delta\mathfrak{I}_{\text{tex}} \;=\; 0 \quad \text{mod (emergence/annihilation in pairs on defect junctions)}, \tag{6.41}$$

so the number of generations cannot change unless accompanied by creation/annihilation events constrained by bordism (domain-wall inflow).

Allowed vacua and $\pi_n(\mathcal{M})$. If $\pi_1(\mathcal{M}) \neq 0$ or $\pi_2(\mathcal{M}) \neq 0$, stable strings/walls exist and must carry anomaly inflow consistent with I_6 restricted to the unbroken subgroup (Sec. 4.4). A breaking pattern that *forces* anomalous defects without available edge modes is disallowed.

5. Positivity/causality gates across phases

Forward-limit bounds. In any broken phase, the even-parity EFT coefficients obey forward positivity (Sec. 3.3): schematically,

$$\frac{\partial^{2n}}{\partial s^{2n}}\,\mathcal{A}_{\text{fwd}}(s)\Big|_{s=0} \;\geq\; 0 \quad\Rightarrow\quad \alpha_n(\text{Higgs/NG}) \;\geq\; 0, \tag{6.42}$$

for appropriate linear combinations α_n of Wilson coefficients ($(\partial\phi)^4$, $(D_\mu F^{\mu\nu})^2$, etc.). Violation signals inadmissible breaking (ghosts/superluminality).

Odd sector nonrenormalization. Coefficients of WZW/CS terms are quantized (Sec. 5.1, 5.2) and *do not* run (Adler–Bardeen). Any breaking pattern changing their normalization without an accompanying threshold jump is disallowed.

6. Complementarity, confinement vs. Higgs

Complementarity (spectral version). If two IR descriptions (Higgs-like vs. confining) preserve the same unbroken global symmetry and reproduce identical anomaly data $(I_6, [\mathfrak{a}])$, then they are *complementary* descriptions of the same long-distance phase. Practically, the baryon current in the confining side (Skyrmion number) maps to a topological charge of the Higgs/NG manifold with the same WZW coefficient k_{spec} (Sec. 5.1).

7. Electroweak and family case studies

EW breaking. For $\mathsf{G}_{\text{gauge}} = SU(2)_L \times U(1)_Y \to U(1)_{\text{em}}$:

1. Witten mod-2 constraint: number of $SU(2)_L$ doublets is even per UV theory; satisfied per SM-like generation set (Sec. 6.2).
2. Mixed anomalies vanish (Sec. 4.2), hence no obstruction from (6.39).
3. Parity-odd IR couplings (e.g. $\pi^0\gamma\gamma$ in QCD-like sector, CP-odd hVV form factors) are fixed by k_{spec} (Sec. 5.4); any EW-breaking pattern changing these normalizations without a threshold jump is excluded.

Gauging $B-L$. Admissible *iff* right-handed neutrinos are present so that $I_6|_{G^2(B-L)} = (B-L)^3 = \mathrm{grav}^2(B-L) = 0$; otherwise, $B-L$ can be only global and its anomalies must be carried by WZW terms.

Family $U(1)_F$. Gauging $U(1)_F$ requires Eq. (6.35); if violated, $U(1)_F$ may be broken spontaneously with a corresponding WZW term in the flavor NG sector.

8. Quantitative consistency inequalities

Spectral Higgs mass matrix. Let $V_{\mathrm{eff}}(\Phi)$ be the spectral Higgs potential with Hessian $\mathcal{H}_{ab} = \partial_a \partial_b V_{\mathrm{eff}}$ at the vacuum. Stability requires

$$\mathcal{H} \;>\; 0 \quad \text{on} \quad \mathrm{Ker}\, Q^{\top}, \tag{6.43}$$

where Q are the broken generators' charges acting on Φ (Goldstone directions saturate zero modes). Violations imply tachyons or runaway directions.

Gauge-boson mass positivity. From the covariant derivative $D_\mu \Phi = (\partial_\mu - i\mathcal{A}_\mu)\Phi$, the broken generators $T^{\hat{a}}$ yield a mass matrix

$$(M_V^2)_{\hat{a}\hat{b}} = \frac{1}{2}\, g^2 \,\langle \Phi^\dagger \{T^{\hat{a}}, T^{\hat{b}}\} \Phi \rangle \;>\; 0, \tag{6.44}$$

with rank equal to the number of broken gauge directions. A negative eigenvalue is incompatible with the (S) gate.

9. Practical checklist for a candidate breaking pattern

1. **Anomaly ledger:** compute $(I_6, [\mathfrak{a}])$ for G_{gl} with the UV spectral content; record mixed terms with $\mathsf{G}_{\mathrm{gauge}}$.
2. **Coset topology:** evaluate $H^5(\mathcal{M}, \mathbb{Z})$ and $\pi_{1,2}(\mathcal{M})$; if $H^5 \neq 0$, include S_{WZW} with $k = k_{\mathrm{spec}}$.
3. **Defect inflow:** if $\pi_{1,2} \neq 0$, determine required edge theories on strings/walls from the restriction of I_6.
4. **Gauge admissibility:** enforce (6.39); fix Bardeen scheme so dynamical gauge Ward identities are exact (Sec. 5.3).
5. **Positivity:** check forward-limit bounds (6.42); ensure no superluminal cones relative to g_{eff}.
6. **Thresholds:** when integrating out chirals, add persistent WZW/inflow to keep (6.3)–(6.5).
7. **Hessian/masses:** verify (6.43)–(6.44).

10. Summary

Emergent symmetry breaking in the spectral model is tightly constrained by anomaly data and consistency gates. A symmetric, trivially gapped IR is *forbidden* whenever the UV carries nonzero $(I_6, [\mathfrak{a}])$. Admissible IR phases must realize anomalies through some combination of massless fields, WZW functionals fixed by k_{spec}, and anomaly inflow. Coset topology dictates the presence of WZW and of anomalous defect modes; positivity/hyperbolicity restrict even-parity EFT couplings. Gauge sectors must remain anomaly-free across thresholds, with global mixed anomalies reproduced in the IR. These rules provide a rigorous filter for viable breaking patterns preceding the detailed phenomenology of Parts III–VII.

Part III

Electroweak Sector as a Structural Phase

Chapter 7

From Fiber Isometries to $SU(2)_L \times U(1)_Y$

7.1 Embedding of weak isometries and hypercharge

Aim. We give a *constructive* and *metric* embedding of the electroweak group

$$\iota : \ SU(2)_L \times U(1)_Y \hookrightarrow \mathrm{Iso}(\mathcal{F}, h),$$

derive the induced principal-bundle connection on M, fix the generator normalizations from fiber geometry, establish the hypercharge charge lattice from integral cohomology, and state the precise current map needed for later phenomenology. This section refines the structural statements of Ch. 1 by proving that horizontality with respect to the $SU(2) \times U(1)$ orbits *forces* the standard electroweak minimal coupling at leading order.

1. Killing system and moment maps

Killing fields. Assume a smooth, effective left isometric action $\Phi : (g, \theta; y) \mapsto (g, \theta) \cdot y$ with generators

$$\xi_a \in \Gamma(T\mathcal{F}),\ a = 1, 2, 3, \qquad \zeta \in \Gamma(T\mathcal{F}), \qquad [\xi_a, \xi_b] = \epsilon_{abc}\xi_c, \quad [\zeta, \xi_a] = 0, \quad \mathcal{L}_{\xi_a} h = \mathcal{L}_\zeta h = 0. \tag{7.1}$$

Moment maps (Killing potentials). Let ω be a torsion-free metric connection on $(\mathcal{F}, h)$ (Levi–Civita). In local complex coordinates (when available) or in general via Hodge decomposition, define *Killing potentials* (moment maps) $\mu_a, \mu_Y : \mathcal{F} \to \mathbb{R}$ by

$$d\mu_a = \iota_{\xi_a} \vartheta, \qquad d\mu_Y = \iota_\zeta \vartheta, \tag{7.2}$$

where ϑ is any primitive of the fiber symplectic two-form used in the response functional (Vol. I). While μ's are defined up to constants, their gradients select vertical directions and will determine the orthogonal projector needed for horizontality.

2. Ehresmann connection from fiber horizontality

Vertical/horizontal split. At a point $y \in \mathcal{F}$, define the *vertical* subspace

$$\mathcal{V}_y := \mathrm{span}\{\xi_a(y), \zeta(y)\} \subset T_y\mathcal{F}, \qquad \mathcal{H}_y := \mathcal{V}_y^{\perp_h} \subset T_y\mathcal{F}. \tag{7.3}$$

This orthogonal decomposition yields an Ehresmann connection one-form $\mathbb{A}$ on the principal bundle of group orbits in $\mathcal{F}$:

$$\mathbb{A}(v) \in \mathfrak{su}(2) \oplus \mathfrak{u}(1), \qquad \mathbb{A}(\xi_a) = T_a, \quad \mathbb{A}(\zeta) = Y, \quad \ker \mathbb{A} = \mathcal{H}. \tag{7.4}$$

Pullback to spacetime. The internal map $y : M \to \mathcal{F}$ induces a spacetime gauge potential by pullback:

$$\mathcal{A}_\mu(x) := y^* mathbbA\,(\partial_\mu) = g\, W^a_\mu(x)\, T_a + g'\, B_\mu(x)\, Y, \tag{7.5}$$

with real one-forms W^a, B on M and couplings (g, g') defined below. The *horizontality condition*

$$\mathcal{D}_\mu y^A \;:=\; \partial_\mu y^A - \xi^A_a(y)\, g\, W^a_\mu - \zeta^A(y)\, g'\, B_\mu \;\in\; \mathcal{H}_{y(x)} \tag{7.6}$$

minimizes the internal gradient energy and enforces gauge covariance as in Sec. 1.2.

3. Metric normalization and gauge couplings

Invariant quadratic form. Define the G-invariant inner products on vertical vectors by

$$\kappa_{ab}(y) := h(\xi_a, \xi_b)\Big|_y, \qquad \kappa_{YY}(y) := h(\zeta, \zeta)\Big|_y, \qquad \kappa_{aY}(y) := h(\xi_a, \zeta)\Big|_y = 0, \tag{7.7}$$

the last equality holding because $SU(2)$ and $U(1)$ orbits are orthogonal for an isometric *product* action; small cross-terms are allowed but can be absorbed by a basis rotation. Averaging over group orbits defines constants

$$\overline{\kappa}_{ab} = \frac{1}{\mathrm{Vol}(\mathcal{O})} \int_{\mathcal{O}} \kappa_{ab}\, d\mathrm{vol}_h = \kappa_2\, \delta_{ab}, \qquad \overline{\kappa}_{YY} = \kappa_1 > 0. \tag{7.8}$$

Canonical normalization and g, g'. Write the Yang–Mills term induced by fast fiber fluctuations (Sec. 1.4) as

$$\mathcal{L}^{\mathrm{EW}}_{\mathrm{YM}} \;=\; -\frac{Z_2}{4}\, W^a_{\mu\nu} W^{a\,\mu\nu} \;-\; \frac{Z_1}{4}\, B_{\mu\nu} B^{\mu\nu}, \qquad Z_2 = \kappa_G\, \kappa_2, \quad Z_1 = \kappa_G\, \kappa_1, \tag{7.9}$$

with $\kappa_G > 0$ the spectral gauge stiffness. Canonical normalization fixes

$$g^2 = \frac{1}{Z_2} = \frac{1}{\kappa_G\, \kappa_2}, \qquad g'^2 = \frac{1}{Z_1} = \frac{1}{\kappa_G\, \kappa_1}. \tag{7.10}$$

Hence the ratio

$$\tan^2 \theta_W \;=\; \frac{g'^2}{g^2} \;=\; \frac{\kappa_2}{\kappa_1} \tag{7.11}$$

is a *geometric* invariant of the $SU(2)$ vs. $U(1)$ vertical metric (the full electroweak mixing will be derived in Sec. 7.4).

4. Hypercharge lattice and Dirac quantization

Line bundle and first Chern class. Let $\mathcal{L}_Y \to \mathcal{F}/SU(2)$ be the complex line bundle associated with the $U(1)$ action generated by ζ. The curvature two-form of the $U(1)$ connection pulled back to M is $F_Y = dB$. Dirac quantization yields

$$\frac{1}{2\pi}\int_{\Sigma_2} F_Y \;\in\; \mathbb{Z} \qquad \forall\, \Sigma_2 \in H_2(M,\mathbb{Z}) \text{ (or in any } U(1) \text{ orbit surface).} \tag{7.12}$$

Therefore, eigenvalues of the hypercharge generator Y in any associated bundle representation lie on a lattice

$$\mathrm{Spec}\big(Y\big) \;\subset\; y_0\,\mathbb{Z}, \qquad y_0 > 0, \tag{7.13}$$

with y_0 the minimal unit fixed by the orbit length and normalization. Choosing $y_0 = \frac{1}{6}$ realizes the SM-like spectrum.

Electric charge operator. Define the unbroken generator

$$Q := T^3 + Y, \tag{7.14}$$

so that after symmetry breaking the photon couples to Q (Sec. 8). Charge integrality follows from (7.13) and the $SU(2)$ weight lattice.

5. Minimal coupling and covariant currents

Associated bundles and representations. For any field φ in a representation R of $SU(2)\times U(1)$, the spectral covariant derivative is

$$D_\mu\varphi = \partial_\mu\varphi - ig\,W_\mu^a R(T_a)\varphi - ig'\,B_\mu R(Y)\varphi, \tag{7.15}$$

and the Noether currents read

$$J^{\mu,a} = \sum_\varphi \frac{\partial\mathcal{L}}{\partial(D_\mu\varphi)}\,R(T_a)\varphi + \text{h.c.}, \qquad J_Y^\mu = \sum_\varphi \frac{\partial\mathcal{L}}{\partial(D_\mu\varphi)}\,R(Y)\varphi + \text{h.c.}, \tag{7.16}$$

supplemented by composite (WZW/inflow) contributions to ensure exact Ward identities in the spectral Bardeen scheme (Sec. 5.3).

6. Embedding constraints and anomaly cancellations

Gauge-anomaly constraints. For chiral fermion content $\{\Psi\}$, admissibility of the embedding requires (per family)

$$\mathcal{A}_{SU(2)^2U(1)} = \sum_\Psi \sigma\, y(\Psi)\, T_2^{SU(2)}(\Psi) = 0, \qquad \mathcal{A}_{U(1)^3} = \sum_\Psi \sigma\, y(\Psi)^3 = 0, \qquad \mathcal{A}_{\mathrm{grav}^2U(1)} = \sum_\Psi \sigma\, y(\Psi) = 0, \tag{7.17}$$

and the absence of the $SU(2)$ Witten anomaly (even number of doublets). The SM-like assignment

$$Q_L : (\mathbf{2})_{+1/6}, \quad u_R : (\mathbf{1})_{+2/3}, \quad d_R : (\mathbf{1})_{-1/3}, \quad L_L : (\mathbf{2})_{-1/2}, \quad e_R : (\mathbf{1})_{-1} \quad (\text{optionally } \nu_R : (\mathbf{1})_0) \tag{7.18}$$

satisfies (7.17) and lies on the lattice (7.13) with $y_0 = \frac{1}{6}$.

7. Examples of fiber geometries realizing the embedding

Example A: $S^3 \times S^1$. Take $\mathcal{F} = S^3 \times S^1$ with round metrics of radii (R_3, R_1). Let $SU(2)$ act by left translations on $S^3 \simeq SU(2)$, and $U(1)$ rotate S^1. Then

$$\kappa_2 = R_3^2 \,\mathrm{tr}_{\mathrm{fund}}(T_a T_a) = \tfrac{3}{4} R_3^2, \qquad \kappa_1 = R_1^2 \,\mathrm{tr}(Y^2) = \tfrac{1}{2} R_1^2, \tag{7.19}$$

and (7.10)–(7.11) give $\tan^2\theta_W = (3/2)\,(R_3^2/R_1^2)$ up to the common stiffness κ_G. The $U(1)$ lattice spacing is $y_0 \propto 1/R_1$ from (7.12).

Example B: Homogeneous coset G_f/H_f with commuting $U(1)$. If $\mathcal{F} = G_f/H_f$ admits a subgroup $SU(2) \times U(1) \subset G_f$ with $[SU(2), U(1)] = 0$, the induced Killing system satisfies (7.1). Small left-invariant metric deformations renormalize (κ_1, κ_2) but preserve the charge lattice and group structure.

8. Consistency gates (EW embedding)

S/G/TQ/D. *(S) Stability:* $\kappa_{1,2}, \kappa_G > 0$ ensure correct kinetic signs. *(G) Hyperbolicity:* the principal symbol of W_μ^a, B_μ propagation is governed by g_{eff}; small χ^μ-induced anisotropies (Vol. I) must obey subluminal cones. *(TQ) Unitarity/positivity:* forward-limit bounds constrain higher-derivative corrections to (7.9). *(D) Dissipation:* medium couplings may renormalize $Z_{1,2}$ but cannot alter the quantized lattice (7.13) nor the anomaly constraints (7.17).

9. Summary

An isometric embedding $SU(2)_L \times U(1)_Y \subset \mathrm{Iso}(\mathcal{F}, h)$ equips M with a pulled-back connection $\mathcal{A}_\mu = g W_\mu^a T_a + g' B_\mu Y$ obtained from fiber horizontality. The electroweak couplings (g, g') are set by vertical metric coefficients (κ_2, κ_1), and the hypercharge spectrum lies on an integral lattice fixed by the first Chern class of the $U(1)$ action. Admissible chiral embeddings satisfy the usual anomaly cancellations. These geometric facts provide the foundation for chirality and assignments (Sec. 7.2), collective gauge excitations (Sec. 7.3), and the Weinberg angle/photon structure (Sec. 7.4).

7.2 Chirality and left–right assignments

Aim. We specify how electroweak *chirality* is realized in the spectral framework and fix the *left–right* representation assignments under the embedded isometry $SU(2)_L \times U(1)_Y \subset \mathrm{Iso}(\mathcal{F}, h)$ constructed in Sec. 7.1. The construction proceeds in three layers: (i) microscopic origin of chirality from spectral textures (zero-mode index), (ii) representation-theoretic embedding of matter multiplets and their covariant couplings, (iii) gauge-invariant Yukawa structures and mass generation once the Higgs resonance is included (Ch. 8). All results are consistent with anomaly constraints (Chs. 4, 5).

1. Spectral origin of chirality

Chiral projectors and zero modes. Let Ψ be a multi-component spinor in an associated bundle to the $SU(2)_L \times U(1)_Y$ embedding. The Dirac operator on (M, g_{eff})

coupled to the electroweak connection $\mathcal{A}_\mu = g\,W_\mu^a T_a + g' B_\mu Y$ reads

$$\not{D}_{\rm EW} \;:=\; \gamma^\mu\big(\nabla_\mu + ig\,W_\mu^a T_a + ig' B_\mu Y\big)\,, \qquad P_{L/R} := \frac{1 \mp \gamma_5}{2}. \tag{7.20}$$

In the presence of a slowly varying spectral mass kernel $\mathsf{M}(x)$ generated by (Φ, χ) (Vols. I–III), the index theorem (Sec. 6.2) counts the net number of normalizable chiral zero modes:

$$n_L - n_R \;=\; \operatorname{Index} \not{D}_{\rm EW}\big|_{\mathcal{R}} \;=\; \int_M \widehat{A}(R)\,\operatorname{ch}_{\mathcal{R}}(F_{\rm EW}) \;+\; \mathfrak{I}_{\rm tex}[\Phi, \chi]\,, \tag{7.21}$$

where $\mathcal{R}$ is the internal representation and $\mathfrak{I}_{\rm tex}$ is the texture index. Thus *chirality* in the IR is a *topological* property protected against smooth deformations that respect the S/G/TQ/D gates.

Consequence. Zero modes appear with definite chirality and transform under $SU(2)_L$ as doublets or singlets according to their fiber localization (Sec. 6.2). This enforces the left–right pattern below.

2. Representation content and covariant couplings

Left vs. right multiplets. Let ψ_L denote chiral zero modes transforming in *nontrivial* $SU(2)_L$ irreps, and ψ_R those in *trivial* (singlet) irreps. For SM-like matter per generation:

$$\text{Quarks:} \qquad Q_L = (u_L, d_L)^\top : (\mathbf{2})_{+1/6}, \quad u_R : (\mathbf{1})_{+2/3}, \quad d_R : (\mathbf{1})_{-1/3}, \tag{7.22}$$

$$\text{Leptons:} \qquad L_L = (\nu_L, e_L)^\top : (\mathbf{2})_{-1/2}, \quad e_R : (\mathbf{1})_{-1}, \quad (\nu_R : (\mathbf{1})_0 \text{ optional}). \tag{7.23}$$

Hypercharges lie on the lattice fixed by the $U(1)_Y$ orbit (Sec. 7.1); the minimal spacing choice $y_0 = \frac{1}{6}$ realizes (7.22)–(7.23).

Covariant derivatives. With $T_a = \sigma_a/2$ in the fundamental,

$$D_\mu Q_L = \Big(\partial_\mu - ig\,W_\mu^a \tfrac{\sigma_a}{2} - ig' B_\mu \tfrac{1}{6}\Big)\,Q_L, \tag{7.24}$$

$$D_\mu u_R = \Big(\partial_\mu - ig' B_\mu \tfrac{2}{3}\Big)\,u_R, \qquad D_\mu d_R = \Big(\partial_\mu + ig' B_\mu \tfrac{1}{3}\Big)\,d_R, \tag{7.25}$$

$$D_\mu L_L = \Big(\partial_\mu - ig\,W_\mu^a \tfrac{\sigma_a}{2} + ig' B_\mu \tfrac{1}{2}\Big)\,L_L, \tag{7.26}$$

$$D_\mu e_R = (\partial_\mu + ig' B_\mu)\,e_R, \qquad D_\mu \nu_R = \partial_\mu \nu_R \quad \text{(if present)}. \tag{7.27}$$

These follow from the geometric embedding and fix the *minimal* electroweak couplings.

Charged/neutral currents (pre-mixing). Define $W_\mu^\pm := (W_\mu^1 \mp i W_\mu^2)/\sqrt{2}$ and keep (W_μ^3, B_μ) as neutral pre-rotated fields. The interaction Lagrangian is

$$\mathcal{L}_{\rm int} = \frac{g}{\sqrt{2}}\Big(\bar{u}_L \gamma^\mu d_L\, W_\mu^+ + \bar{\nu}_L \gamma^\mu e_L\, W_\mu^+ + \text{h.c.}\Big) + g\,J_3^\mu\,W_\mu^3 + g'\,J_Y^\mu\,B_\mu, \tag{7.28}$$

$$J_3^\mu = \sum_{\text{doublets}} \bar{\psi}_L \gamma^\mu \tfrac{\sigma_3}{2} \psi_L, \qquad J_Y^\mu = \sum_{\text{all}} y\,\bar{\psi}\gamma^\mu\psi. \tag{7.29}$$

After electroweak symmetry breaking the physical neutral interactions follow from the (B, W^3) rotation (Sec. 7.4).

3. Gauge-invariant Yukawa structures and chirality

Higgs representation and hypercharge. The Higgs (structural resonance) is taken as a complex $SU(2)_L$ doublet with $Y_H = +\frac{1}{2}$ (Ch. 8):

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} : \ (\mathbf{2})_{+1/2}, \qquad \widetilde{H} := i\sigma_2 H^* : \ (\mathbf{2})_{-1/2}. \tag{7.30}$$

This choice is *forced* by gauge invariance of Yukawas and by electric charge assignments $Q = T^3 + Y$.

Yukawa couplings. Gauge-invariant and renormalizable Yukawa interactions are

$$\mathcal{L}_Y \ = \ -\bar{Q}_L\, Y_u\, \widetilde{H}\, u_R \ - \ \bar{Q}_L\, Y_d\, H\, d_R \ - \ \bar{L}_L\, Y_e\, H\, e_R \ - \ \bar{L}_L\, Y_\nu\, \widetilde{H}\, \nu_R \ + \ \text{h.c.}, \tag{7.31}$$

where Y_f are complex matrices in generation space determined by overlap integrals on the fiber (Sec. 6.3). If ν_R is absent, a dimension-five Weinberg operator is induced after integrating out the spectral seesaw:

$$\mathcal{L}_\nu^{(5)} \ = \ -\frac{1}{2\Lambda_\nu} \left(\bar{L}_L\, \widetilde{H}\right) C \left(\widetilde{H}^{\top} L_L^{\top}\right) \ + \ \text{h.c.}, \tag{7.32}$$

with Λ_ν controlled by the heavy sterile sector (Sec. 6.3).

Electroweak symmetry breaking (EWSB). When the Higgs acquires a vacuum amplitude $\langle H \rangle = (0,\ v/\sqrt{2})^{\top}$ (Ch. 8), fermion masses arise as

$$m_u = \frac{v}{\sqrt{2}}\, Y_u, \qquad m_d = \frac{v}{\sqrt{2}}\, Y_d, \qquad m_e = \frac{v}{\sqrt{2}}\, Y_e, \qquad M_\nu = \begin{cases} \frac{v^2}{2}\, Y_\nu M_R^{-1} Y_\nu^{\top} & \text{(Type-I seesaw)} \\ \frac{v^2}{2\Lambda_\nu} & \text{(Weinberg)} \end{cases} \tag{7.33}$$

with M_R the sterile Majorana matrix if ν_R exist. Left–right mass pairing preserves the chiral electroweak assignments and rotates to the CKM/PMNS bases (Sec. 6.3).

4. Anomaly and global consistency checks

Gauge anomalies. Per generation, the assignments (7.22)–(7.23) satisfy

$$\mathcal{A}_{SU(2)^2 U(1)_Y} = 0, \qquad \mathcal{A}_{U(1)_Y^3} = 0, \qquad \mathcal{A}_{\text{grav}^2 U(1)_Y} = 0, \tag{7.34}$$

and avoid the $SU(2)$ Witten anomaly (even number of doublets). If $U(1)_{B-L}$ is gauged, ν_R fields are required to cancel $G^2(B-L)$ and $(B-L)^3$ anomalies.

Vectorlike thresholds. Vectorlike pairs (e.g. heavy $(\mathbf{2})_y \oplus (\mathbf{2})_{-y}$) may appear in intermediate regimes and decouple without affecting anomalies; their effect is confined to threshold corrections of even-parity couplings (Ch. 3).

5. Spectral remarks on chirality and medium anisotropy

Chirality protection. χ^μ-induced anisotropies (Vol. I) cannot mix P_L and P_R at leading order because $\{\gamma_5, \not{D}_{\text{EW}}\} = 0$ in the chiral limit; allowed operators respecting S/G/TQ/D appear at higher derivative order and are constrained by positivity (Sec. 3.3).

Custodial structure (preview). With H in $(\mathbf{2})_{1/2}$, the Higgs sector admits an approximate custodial $SU(2)_V$ symmetry when $Y_u = Y_d$ and hypercharge interactions are neglected; this controls the ρ parameter and will be analyzed in Ch. 8 and Sec. 8.2.

6. Worked one-generation template (pre-EWSB)

Lagrangian. Collecting the kinetic and interaction terms,

$$\mathcal{L}_{\text{kin}} = \bar{Q}_L i\not{D} Q_L + \bar{u}_R i\not{D} u_R + \bar{d}_R i\not{D} d_R + \bar{L}_L i\not{D} L_L + \bar{e}_R i\not{D} e_R \; (+\; \bar{\nu}_R i\not{\partial} \nu_R), \tag{7.35}$$

$$\mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}} \left(\bar{u}_L \gamma^\mu d_L W^+_\mu + \bar{\nu}_L \gamma^\mu e_L W^+_\mu + \text{h.c.} \right) + g J^\mu_3 W^3_\mu + g' J^\mu_Y B_\mu, \tag{7.36}$$

$$\mathcal{L}_Y = -\bar{Q}_L Y_u \widetilde{H} u_R - \bar{Q}_L Y_d H d_R - \bar{L}_L Y_e H e_R \; (-\bar{L}_L Y_\nu \widetilde{H} \nu_R) \; + \; \text{h.c.} \tag{7.37}$$

This is the unique renormalizable realization consistent with the geometric embedding of Sec. 7.1.

7. Consistency gates (EW chirality sector)

S/G/TQ/D gates. *(S)* Positive fermion kinetic terms; Yukawa matrices bounded to avoid tachyonic instabilities in the scalar potential (Ch. 8). *(G)* Hyperbolicity controlled by g_{eff}; allowed χ^μ-deformations must preserve causal cones. *(TQ)* Unitarity/positivity constraints on higher-dimensional operators; anomaly cancellations (7.34) enforced. *(D)* Dissipative effects renormalize wavefunctions but cannot change chiral indices nor anomaly coefficients.

8. Summary

Chirality in the spectral model is topologically generated by texture indices and implemented at low energies by placing *left-handed* matter in $SU(2)_L$ doublets and *right-handed* matter in singlets with hypercharges on the quantized $U(1)_Y$ lattice. Minimal electroweak couplings and Yukawa structures follow uniquely from the geometric embedding, while anomaly constraints are satisfied per generation. These assignments prepare the ground for Sec. 7.3, where electroweak gauge bosons are derived as collective χ^μ excitations and their kinematics are connected to the medium response.

7.3 Gauge bosons as collective χ^μ excitations

Aim. We derive the *kinematics* of the electroweak gauge fields as *collective excitations* of the spectral medium driven by the director χ^μ and the internal map $y : M \to \mathcal{F}$. Microscopically, vertical deformations of y along fiber isometries are compensated by a pure-gauge pullback (Sec. 7.1); dynamical gauge bosons arise when the medium's *retarded response* to Noether currents is nonzero. We construct the quadratic effective action from current correlators (Kubo formalism), identify the constitutive tensor for gauge fields, analyze dispersion in the presence of χ^μ, and state positivity/causality gates for admissible anisotropies.

1. From vertical deformations to collective fields

Pure-gauge vs. dynamical pieces. Decompose the electroweak connection (for $I, J \in \{1, 2, 3, Y\}$)

$$\mathcal{A}^I_\mu \;=\; \mathcal{A}^I_{\mu,\mathrm{pg}}[y] \;+\; a^I_\mu, \qquad \mathcal{A}^I_{\mu,\mathrm{pg}}[y] \;:=\; \left(y^*\mathbb{A}\right)^I{}_\mu, \tag{7.38}$$

where $\mathbb{A}$ is the Ehresmann connection on the fiber-orbit bundle (Sec. 7.1). The pure-gauge piece has vanishing curvature if y varies adiabatically along group orbits; the *dynamical* field a^I_μ captures collective, medium-sourced excitations. Gauge transformations act as $a_\mu \mapsto g\, a_\mu g^{-1} - (\partial_\mu g) g^{-1}$, leaving the decomposition covariant.

Noether currents and response. Let J^μ_I denote the conserved (covariant) Noether currents of the fiber isometries (Sec. 1.1):

$$J^\mu_I \;=\; \kappa_y\, h_{AB}\, \xi^A_I(y)\, \mathcal{D}^\mu y^B \;+\; J^\mu_{I,\mathrm{matter}}, \tag{7.39}$$

with $J^\mu_{I,\mathrm{matter}}$ including chiral zero modes and composite sectors. Small a_μ sources these currents via $\delta\mathcal{L} = +J^\mu_I a^I_\mu$, so the medium's retarded correlator determines the dynamics.

2. Quadratic effective action from Kubo

Current correlator and transversality. Define the retarded polarization tensor in Fourier space ($k^\mu = (\omega, \mathbf{k})$)

$$\Pi^{\mu\nu}_{IJ}(k) \;:=\; i\int d^4x\, e^{ik\cdot x}\, \theta(x^0)\, \langle [\, J^\mu_I(x),\, J^\nu_J(0)\,] \rangle_{\mathcal{S}}\ . \tag{7.40}$$

Gauge covariance of J (Ward identities) implies *transversality*

$$k_\mu\, \Pi^{\mu\nu}_{IJ}(k) \;=\; 0 \;=\; \Pi^{\mu\nu}_{IJ}(k)\, k_\nu. \tag{7.41}$$

Local (hydrodynamic) expansion. At wavelengths large compared to micro-scales (set by the spectral correlation lengths), Π admits a derivative expansion constrained by (7.41):

$$\Pi^{\mu\nu}_{IJ}(k) \;=\; \mathcal{Z}_{IJ}\left(k^2 g^{\mu\nu}_{\mathrm{eff}} - k^\mu k^\nu\right) \;+\; \Delta\Pi^{\mu\nu}_{IJ}(k;\chi) \;+\; \mathcal{O}(k^4), \tag{7.42}$$

where $\mathcal{Z}_{IJ} > 0$ are constant (isotropic) stiffnesses and $\Delta\Pi$ collects *anisotropic* but gauge-invariant structures built from u^μ (aether rest frame), the projected director $\hat{\chi}^\mu$ (orthogonal to u^μ), and k^μ.

Quadratic gauge-field action. Integrating out fast degrees yields, up to field redefinitions,

$$\Gamma^{(2)}[a] \;=\; \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\; a^I_\mu(-k)\, \Pi^{\mu\nu}_{IJ}(k)\, a^J_\nu(k) \;\longrightarrow\; -\frac{1}{4}\int d^4x\; \mathcal{K}^{IJ}{}_{\mu\alpha\nu\beta}\, F^{I\,\mu\nu} F^{J\,\alpha\beta}, \tag{7.43}$$

with constitutive tensor $\mathcal{K}$ determined by the low-k limit of Π. In the isotropic limit, $\mathcal{K}^{IJ}{}_{\mu\alpha\nu\beta} = \mathcal{Z}_{IJ}\,(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu})/2$, reproducing the Yang–Mills form with

$$\mathcal{Z}_{ab} = \frac{\delta_{ab}}{g^2}, \qquad \mathcal{Z}_{YY} = \frac{1}{g'^2}, \qquad \mathcal{Z}_{aY} = 0 \quad \text{(orthogonal orbits)}. \tag{7.44}$$

3. Constitutive tensor in an anisotropic medium

Tensor bases. Let u^μ be the unit aether velocity (timelike, $u^2 = -1$) and define spatial projector $P^{\mu\nu} := g_{\rm eff}^{\mu\nu} + u^\mu u^\nu$. Decompose the director as $\hat\chi^\mu := P^\mu{}_\nu \chi^\nu / \sqrt{\chi_\perp^2}$ with $\chi_\perp^2 := \chi_\alpha P^{\alpha\beta} \chi_\beta$. Construct the independent *CPT-even* rank-4 tensors for two-forms:

$$\big[\mathbb{T}_1\big]_{\mu\alpha\nu\beta} := \tfrac{1}{2}\big(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu}\big), \tag{7.45}$$

$$\big[\mathbb{T}_2\big]_{\mu\alpha\nu\beta} := \tfrac{1}{2}\big(P_{\mu\nu}P_{\alpha\beta} - P_{\mu\beta}P_{\alpha\nu}\big) - \mathbb{T}_1, \tag{7.46}$$

$$\big[\mathbb{T}_3\big]_{\mu\alpha\nu\beta} := \tfrac{1}{2}\big(\hat\chi_\mu\hat\chi_\nu P_{\alpha\beta} - \hat\chi_\mu\hat\chi_\beta P_{\alpha\nu} - (\mu \leftrightarrow \alpha)\big), \tag{7.47}$$

and (if allowed by discrete symmetries) the *CPT-odd* Carroll–Field–Jackiw-type structure

$$\big[\mathbb{T}_{\rm CFJ}\big]_{\mu\alpha\nu\beta} := \epsilon_{\mu\alpha\nu\beta}\; u\cdot\hat\chi\ . \tag{7.48}$$

The spectral gates (Sec. 3.3) require the CFJ coefficient to vanish or be highly suppressed to avoid birefringent instabilities; hereafter we set it to zero.

General constitutive law. To quadratic order,

$$\mathcal{K}^{IJ}{}_{\mu\alpha\nu\beta} \;=\; Z_1^{IJ}\,\big[\mathbb{T}_1\big]_{\mu\alpha\nu\beta} \;+\; Z_2^{IJ}\,\big[\mathbb{T}_2\big]_{\mu\alpha\nu\beta} \;+\; Z_3^{IJ}\,\big[\mathbb{T}_3\big]_{\mu\alpha\nu\beta}, \tag{7.49}$$

with real symmetric matrices Z_r^{IJ} in gauge space. Isotropy corresponds to $Z_2^{IJ} = Z_3^{IJ} = 0$, $Z_1^{ab} = \delta^{ab}/g^2$, $Z_1^{YY} = 1/g'^2$, $Z_1^{aY} = 0$. Small anisotropies are encoded by $\delta Z_r^{IJ} \ll Z_1^{IJ}$.

4. Maxwell–Yang–Mills equations and dispersion

Constitutive fields. Define the generalized displacement two-form (per gauge index)

$$H^{I\,\mu\nu} := 2\,\mathcal{K}^{IJ}{}_{\mu\alpha\nu\beta}\,F^{J\,\alpha\beta}, \tag{7.50}$$

so that the field equations read

$$D_\mu H^{I\,\mu\nu} \;=\; J^{I,\nu}_{\rm ext}, \qquad D_{[\mu}F^I_{\nu\rho]} = 0. \tag{7.51}$$

In the unbroken EW phase and in the linearized regime, $D_\mu \to \partial_\mu$.

Plane waves and mode speeds. Consider transverse plane waves $a_\mu^I \propto \epsilon_\mu\, e^{ik\cdot x}$ in the aether rest frame $u^\mu = (1, \mathbf{0})$. Let $\mathbf{n} := \mathbf{k}/|\mathbf{k}|$ and decompose $\hat{\boldsymbol{\chi}}$ into components parallel/perpendicular to $\mathbf{n}$. The dispersion relation for physical polarizations $\epsilon\cdot k = 0$ is

$$\omega^2 \;=\; c_{IJ}^2(\mathbf{n}, \hat{\boldsymbol{\chi}})\;|\mathbf{k}|^2\ , \qquad c_{IJ}^2 \;=\; \Big(Z_1^{IJ} + \alpha^{IJ}\,(\hat{\boldsymbol{\chi}}\cdot\mathbf{n})^2 + \beta^{IJ}\,[1 - (\hat{\boldsymbol{\chi}}\cdot\mathbf{n})^2]\Big)^{-1}, \tag{7.52}$$

where α^{IJ}, β^{IJ} are linear combinations of $Z_{2,3}^{IJ}$ respecting positivity (see below). In the isotropic limit $Z_{2,3} \to 0$ one recovers luminal propagation $\omega^2 = |\mathbf{k}|^2$ in units where $g_{\rm eff}$ sets the light cone.

Kinetic mixing and electroweak angle (pre-EWSB). If the fiber orbits are not exactly orthogonal, $Z_1^{3Y} \neq 0$ induces a *kinetic* mixing between W^3 and B. Diagonalizing the quadratic form defines a *renormalized* mixing angle $\tilde{\theta}_W$ by

$$\tan\!\left(2\tilde{\theta}_W\right) \;=\; \frac{2Z_1^{3Y}}{Z_1^{YY} - Z_1^{33}} \;+\; \mathcal{O}(\delta Z_{2,3}), \tag{7.53}$$

which reduces to $\tan\theta_W = g'/g$ (Sec. 7.1) when $Z_1^{3Y} = 0$. Small anisotropies $\delta Z_{2,3}$ lead to direction-dependent corrections suppressed by $\delta Z/Z_1$.

5. Kubo relations and microscopic expressions

Isotropic stiffnesses. Project Π onto the transverse subspace using $\mathbb{P}^{\mu\nu}(k) = g^{\mu\nu} - k^\mu k^\nu/k^2$. Then

$$\frac{1}{g^2}\,\delta_{ab} \;=\; \frac{1}{6}\left[\frac{\partial}{\partial k^2}\,\mathbb{P}_{\mu\nu}(k)\,\Pi_{ab}^{\mu\nu}(k)\right]_{k=0}, \qquad \frac{1}{g'^2} \;=\; \frac{1}{6}\left[\frac{\partial}{\partial k^2}\,\mathbb{P}_{\mu\nu}(k)\,\Pi_{YY}^{\mu\nu}(k)\right]_{k=0}. \tag{7.54}$$

These relations tie g, g' directly to the UV spectral microphysics encoded in current correlators of the medium.

Anisotropic coefficients. Coefficients of $\mathbb{T}_{2,3}$ follow from directional derivatives at $k \to 0$:

$$Z_{2,3}^{IJ} \;\sim\; \frac{\partial}{\partial(k\cdot\hat{\chi})^2}\Big[\mathbb{P}_{\mu\nu}(k)\,\Pi_{IJ}^{\mu\nu}(k)\Big]_{k=0}, \qquad Z_3^{IJ} \;\propto\; \langle J\cdot J\rangle_{\hat{\chi}\hat{\chi}} - \tfrac{1}{2}\langle J\cdot J\rangle_P, \tag{7.55}$$

where angular brackets denote medium expectation values in a background with fixed $\hat{\chi}$.

6. Positivity, hyperbolicity, and sum rules

Stability (no ghosts). For any real two-form $X_{\mu\nu}^I$ in physical (transverse) directions,

$$\sum_{I,J} X_{\mu\nu}^I\,\mathcal{K}^{IJ\,\mu\alpha\nu\beta}\,X_{\alpha\beta}^J \;>\; 0 \quad\Rightarrow\quad Z_1^{IJ} \text{ positive-definite, and } |\delta Z_{2,3}^{IJ}| \;\ll\; \mathrm{eigmin}(Z_1). \tag{7.56}$$

Causality (subluminal cones). The principal symbol for the wave operator is

$$\mathcal{P}^{IJ}{}_{\mu\nu}(k) \;=\; \mathcal{K}^{IJ}{}_{\mu\alpha\nu\beta}\,k^\alpha k^\beta, \tag{7.57}$$

and hyperbolicity requires that, for each polarization, the characteristic surfaces lie *inside* or on the g_{eff} light cone. This constrains $(\alpha^{IJ}, \beta^{IJ})$ in (7.52) to nonnegative ranges.

Optical theorem and forward bounds. The retarded correlator obeys a Källén–Lehmann representation with positive spectral densities (Ch. VI). Forward-limit dispersion relations imply

$$\partial_{k^2}\Big[\mathbb{P}_{\mu\nu}\Pi_{IJ}^{\mu\nu}\Big]_{k=0} \;>\; 0, \qquad \partial_{(k\hat{\chi})^2}\Big[\mathbb{P}_{\mu\nu}\Pi_{IJ}^{\mu\nu}\Big]_{k=0} \;\geq\; 0, \tag{7.58}$$

consistent with (7.56).

7. Remarks on discrete symmetries and forbidden terms

CPT-odd CFJ term. A term $\propto \epsilon^{\mu\nu\alpha\beta}\, n_\mu A_\nu F_{\alpha\beta}$ (Carroll–Field–Jackiw) would produce vacuum birefringence and violates CPT/Paritiy. In the spectral medium it can only appear if the background violates these symmetries macroscopically; the (TQ) gate then forces its coefficient to be negligible. We set it to zero in the baseline EW phase.

Topological odd terms on defects. Parity-odd Chern–Simons terms *do* arise on 3D defects (Sec. 5.2) via anomaly inflow, but they do not modify bulk 4D propagation.

8. Summary and outlook to EWSB

Gauge bosons are *collective* excitations of the spectral medium: the pure-gauge pullback $\mathcal{A}_{\rm pg}$ enforces horizontality, while dynamical fields a_μ acquire their Yang–Mills kinetics from the retarded current response of the medium. Their quadratic action is governed by a constitutive tensor $\mathcal{K}$ that reduces to isotropic Yang–Mills at leading order and admits controlled, gauge-invariant anisotropies aligned with the director χ^μ. Positivity and hyperbolicity constrain these anisotropies and tie g, g' to Kubo integrals of Noether correlators. In Ch. 8 we will include the Higgs resonance, showing how longitudinal modes emerge and how the (A_μ, Z_μ) mixture inherits (mild) χ^μ-dependent kinematic deformations consistent with the spectral gates.

7.4 Weinberg angle from geometry

Aim. We derive the Weinberg angle θ_W *from first principles of the fiber geometry*, relate it to the vertical metric coefficients of the internal isometries (Sec. 7.1), include the Higgs-induced neutral gauge-boson mass matrix, and present the general algorithm that incorporates noncanonical kinetic terms and mild χ^μ–induced anisotropies (Sec. 7.3). We conclude with renormalization-group (RG) evolution and threshold matching rules linking the high-scale geometric prediction to low-energy observables.

1. Canonical definition and Higgs mass matrix

Canonical (isotropic) kinetic terms. In the orthogonal-orbit limit (no kinetic mixing, isotropy), the neutral electroweak fields B_μ and W^3_μ have canonical Yang–Mills quadratic terms

$$\mathcal{L}^{\rm kin}_{\rm neutral} \;=\; -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^3_{\mu\nu}W^{3\,\mu\nu}, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \;\; W^3_{\mu\nu} = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu. \tag{7.59}$$

The Higgs doublet $H : (\mathbf{2})_{+1/2}$ with $\langle H\rangle = (0,\, v/\sqrt{2})^\top$ produces the neutral mass matrix

$$\mathcal{L}^{\rm mass}_{\rm neutral} \;=\; \frac{v^2}{8}\begin{pmatrix} B_\mu & W^3_\mu \end{pmatrix}\begin{pmatrix} g'^2 & -gg' \\ -gg' & g^2 \end{pmatrix}\begin{pmatrix} B^\mu \\ W^{3\,\mu} \end{pmatrix}. \tag{7.60}$$

Diagonalization by an orthogonal rotation in the (B, W^3) plane,

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \;=\; \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix}, \tag{7.61}$$

yields a massless photon A_μ and

$$m_W^2 = \frac{g^2 v^2}{4}, \qquad m_Z^2 = \frac{(g^2+g'^2)v^2}{4}, \qquad \sin^2\theta_W = \frac{g'^2}{g^2+g'^2}, \quad \cos^2\theta_W = \frac{g^2}{g^2+g'^2}, \tag{7.62}$$

and the electric charge coupling

$$e \;=\; g\,\sin\theta_W \;=\; g'\,\cos\theta_W, \qquad Q \;=\; T^3 + Y. \tag{7.63}$$

Equation (7.62) implies the custodial tree-level relation $\rho \equiv m_W^2/(m_Z^2\cos^2\theta_W) = 1$ (Sec. 8.2).

2. Geometric origin: vertical metric $\Rightarrow \theta_W$

Isometry-induced couplings. From Sec. 7.1, the vertical (Killing) metric on the fiber orbits defines

$$g^2 = \frac{1}{\kappa_G\,\kappa_2}, \qquad g'^2 = \frac{1}{\kappa_G\,\kappa_1}, \qquad \kappa_2 := \overline{h(\xi_a,\xi_a)}\,, \quad \kappa_1 := \overline{h(\zeta,\zeta)}\,, \tag{7.64}$$

with $\kappa_G > 0$ the spectral gauge stiffness and the bar denoting the group-orbit average. Therefore,

$$\tan^2\theta_W \;=\; \frac{g'^2}{g^2} \;=\; \frac{\kappa_2}{\kappa_1}\,, \qquad \sin^2\theta_W \;=\; \frac{1}{1+\kappa_1/\kappa_2}\,. \tag{7.65}$$

Thus the Weinberg angle is a purely geometric ratio of vertical metric coefficients on $\mathcal{F}$. For the model $\mathcal{F} = S^3(R_3)\times S^1(R_1)$ (Example A in Sec. 7.1), $\kappa_2 \propto R_3^2$ and $\kappa_1 \propto R_1^2$, so $\tan^2\theta_W \propto R_3^2/R_1^2$.

3. Noncanonical kinetic terms and mild anisotropy

General quadratic neutral sector. Allow a positive-definite kinetic matrix Z and (small) CPT-even anisotropies from χ^μ (Sec. 7.3):

$$\mathcal{L}^{\rm kin}_{\rm neutral} \;=\; -\frac{1}{4}\,(F_{\mu\nu})^\top Z\,(F^{\mu\nu}), \qquad Z \;=\; \begin{pmatrix} Z_{YY} & Z_{Y3} \\ Z_{Y3} & Z_{33}\end{pmatrix} \;+\; \delta Z(\chi^\mu), \tag{7.66}$$

and the Higgs mass matrix M as in (7.60) written in the (B, W^3) basis. The *physical* θ_W is obtained by the two-step canonicalization:

$$\text{(i) Canonicalize:} \quad \mathsf{A} \;=\; S\,\mathsf{A}_{\rm can}, \quad S^\top Z\,S \;=\; \mathbb{1}_2 \quad\Rightarrow\quad M_{\rm can} \;=\; S^\top M\,S, \tag{7.67}$$

$$\text{(ii) Diagonalize:} \quad O^\top(\theta_W^{\rm phys})\,M_{\rm can}\,O(\theta_W^{\rm phys}) \;=\; \mathrm{diag}(0, m_Z^2). \tag{7.68}$$

Here $\mathsf{A}^\top = (B_\mu,\, W_\mu^3)$ and O is an $SO(2)$ rotation. Gauge invariance guarantees $\det M = 0$ so that the photon eigenvalue remains exactly zero *independently of* Z and $\delta Z(\chi^\mu)$.

Small kinetic mixing (first-order shift). If $|Z_{Y3}| \ll \sqrt{Z_{YY}Z_{33}}$ and $\delta Z(\chi^\mu)$ is negligible in the isotropic limit, one may write $S = Z^{-1/2}$ and expand to first order in

$$\varepsilon \;:=\; \frac{Z_{Y3}}{\sqrt{Z_{YY}Z_{33}}}\,. \tag{7.69}$$

Then the physical mixing angle is shifted from the canonical value $\tan\theta_W = g'/g$ by

$$\theta_W^{\rm phys} \;=\; \theta_W \;+\; \delta\theta_W, \qquad \delta\theta_W \;=\; \frac{\varepsilon}{2}\,\frac{g^2-g'^2}{g^2+g'^2} \;+\; \mathcal{O}(\varepsilon^2)\,, \tag{7.70}$$

while $e = g\sin\theta_W^{\rm phys} = g'\cos\theta_W^{\rm phys}$ continues to hold after canonicalization. Directional χ^μ-induced anisotropies only enter at $\mathcal{O}(\delta Z)$ and are bounded by the positivity/hyperbolicity gates (Sec. 3.3).

4. Charge operator and photon uniqueness

Unbroken generator. Independently of Z and of anisotropies, the *unbroken* $U(1)_{\rm em}$ generator is

$$Q \;=\; T^3 + Y, \tag{7.71}$$

fixed by the Higgs representation $(\mathbf{2})_{1/2}$ and the vacuum alignment. The corresponding gauge boson is the null eigenvector of M and remains *exactly* massless as long as the vacuum preserves Q. In particular, the normalization of electric charge e is set by the canonical kinetic normalization of the photon after the procedure (7.67)–(7.68).

5. RG evolution and geometric matching scale

Running couplings. Let μ be the renormalization scale and $g(\mu), g'(\mu)$ the canonically normalized couplings. Then

$$\sin^2\theta_W(\mu) \;=\; \frac{g'(\mu)^2}{g(\mu)^2+g'(\mu)^2}\,. \tag{7.72}$$

The one-loop RGEs (with SM-like content) read

$$\mu\frac{dg}{d\mu} = \frac{b_2}{16\pi^2}g^3, \qquad \mu\frac{dg'}{d\mu} = \frac{b_Y}{16\pi^2}g'^3, \tag{7.73}$$

with $(b_2, b_Y) = (-19/6,\ 41/6)$ for three chiral families and one Higgs doublet (in the non-GUT normalization of g').[1] Integrating (7.73) relates $\sin^2\theta_W(\mu)$ between any two scales once thresholds are specified.

Geometric matching. At a *geometric* matching scale μ_* (the onset of the effective fiber description),

$$\tan^2\theta_W(\mu_*) \;=\; \frac{\kappa_2(\mu_*)}{\kappa_1(\mu_*)}\,, \tag{7.74}$$

with $\kappa_{1,2}$ computed from the vertical metric of $(\mathcal{F}, h)$. Below μ_* the RG flow (7.73) and threshold/decoupling rules (Ch. 3) transport this boundary condition to the electroweak scale.

Thresholds and universality classes. Vectorlike thresholds shift g^{-2} and g'^{-2} by additive constants without changing anomaly data; chiral thresholds must be accompanied by persistent WZW/inflow pieces (Parts II, V) but do not alter the definition (7.72). See Sec. 3.2 for decoupling schemes.

[1] In the GUT normalization $g_1 = \sqrt{5/3}\,g'$, $b_1 = 41/10$ and $\sin^2\theta_W = \frac{3}{5}\,\frac{g_1^2}{g_1^2+g_2^2}$.

6. Worked template and predictions

Template. Given a concrete fiber, e.g. $\mathcal{F} = S^3(R_3) \times S^1(R_1)$ with weak deformations,

$$\kappa_2 = \alpha\, R_3^2\,(1+\delta_2), \qquad \kappa_1 = \beta\, R_1^2\,(1+\delta_1), \qquad |\delta_{1,2}| \ll 1, \tag{7.75}$$

one predicts

$$\sin^2\theta_W(\mu_*) \;=\; \frac{1}{1+(\beta/\alpha)\,(R_1^2/R_3^2)}\;\left[1+\mathcal{O}(\delta_1-\delta_2)\right], \tag{7.76}$$

to be RG-run down to the weak scale. The small difference $(\delta_1 - \delta_2)$ is controlled by metric anisotropies or mild mixing of orbits.

7. Consistency gates

S/G/TQ/D. *(S)* Z must be positive-definite (no ghosts); *(G)* the principal symbol from the constitutive tensor remains subluminal relative to g_{eff} (Sec. 7.3); *(TQ)* forward-limit positivity for higher-derivative operators holds and odd-parity coefficients (WZW/CS on defects) remain quantized; *(D)* dissipative renormalizations cannot alter the exact null eigenvalue of M nor the definition of Q.

8. Summary

In the spectral model, the Weinberg angle is *geometrically* determined at the emergence scale by the ratio of vertical metric coefficients on the internal fiber, Eq. (7.65). The Higgs mechanism then fixes the neutral mass matrix and the relations m_Z, m_W, e in the canonical limit, while kinetic mixing and mild anisotropy are incorporated by canonicalization followed by mass diagonalization. RG evolution and thresholds connect the geometric boundary condition at μ_* to precision electroweak observables, providing a sharp bridge between fiber geometry and measurable electroweak data.

Chapter 8

Higgs as a Structural Resonance

8.1 Order parameter and potential from spectral stiffness

Aim. We construct the electroweak Higgs as a *structural resonance* of the spectral medium: a soft collective mode that condenses when an $SU(2)_L \times U(1)_Y$ channel of the medium's stiffness crosses a critical value. We define the order parameter as a section of an associated bundle, derive its covariant kinetic term from Kubo response, build the most general gauge-invariant potential compatible with the spectral gates, and state the gauge-independence (Nielsen) identity for the effective potential.

1. Order parameter as an associated-bundle field

Definition. Let $P \to M$ be the $SU(2)_L \times U(1)_Y$ principal bundle induced by the fiber isometries (Sec. 7.1). The Higgs field is a section of the associated complex rank-2 bundle

$$E_H \;:=\; P \times_{(2,\,1/2)} \mathbb{C}^2 \;\longrightarrow\; M, \qquad H(x) \;\in\; \Gamma(E_H), \qquad H \;:\; (\mathbf{2})_{+1/2}. \tag{8.1}$$

With $T_a = \sigma_a/2$ and hypercharge $Y_H = \frac{1}{2}$, the spectral covariant derivative is

$$D_\mu H \;:=\; \Big(\partial_\mu - ig\, W_\mu^a \tfrac{\sigma_a}{2} - ig' B_\mu \tfrac{1}{2}\Big) H. \tag{8.2}$$

Gauge transformations act as $H \mapsto e^{i\alpha^a T_a + i\beta Y_H} H$, while $D_\mu H$ transforms covariantly.

Kinetic term (leading order). The medium's isotropic response fixes a positive *Higgs stiffness* $\kappa_H > 0$ and yields

$$\mathcal{L}_H^{(2)} \;=\; \kappa_H\, g_{\text{eff}}^{\mu\nu}\, (D_\mu H)^\dagger (D_\nu H), \tag{8.3}$$

with g_{eff} the emergent metric (Vol. I). Small, parity-even anisotropies induced by the director χ^μ (Sec. 7.3) enter as

$$\Delta\mathcal{L}_H^{(2)} \;=\; \delta K^{\mu\nu}(\chi;\mathcal{S})\, (D_\mu H)^\dagger (D_\nu H), \qquad \delta K^{\mu\nu} u_\mu u_\nu = 0, \quad \|\delta K\| \ll \kappa_H, \tag{8.4}$$

and are constrained by positivity/hyperbolicity gates (Sec. 3.3). Field renormalization may be used to set $\kappa_H = 1$ at a reference scale μ_*.

2. Spectral origin of κ_H (Kubo formula)

Charged-scalar channel. Let $\mathcal{O}_H$ denote the composite operator that interpolates the Higgs resonance in the microscopic spectral medium (built from $(\Phi, \chi^\mu, \dots)$). The retarded two-point correlator

$$G_R(\omega, \mathbf{k}) \;:=\; -i \int d^4x\, e^{ik\cdot x}\, \theta(t)\, \langle [\, \mathcal{O}_H(x), \mathcal{O}_H^\dagger(0)\,] \rangle_{\mathcal{S}} \tag{8.5}$$

admits a low-momentum expansion

$$G_R(\omega, \mathbf{k}) \;\simeq\; \alpha_0 \;-\; \alpha_2\, (\omega^2 - c_H^2 \mathbf{k}^2) \;+\; \cdots . \tag{8.6}$$

Matching to (8.3) gives the Kubo relations

$$\kappa_H \;=\; \alpha_2, \qquad c_H^2 \;=\; 1 \;+\; \mathcal{O}(\delta K/\kappa_H), \tag{8.7}$$

in units where the g_{eff}-light-cone speed is 1. Thus the Higgs kinetic coefficient is a *spectral stiffness* readable from the slope of the scalar correlator.

3. Gauge-invariant potential from spectral energetics

Invariant building blocks. Gauge invariance and renormalizability allow, up to total derivatives, the local invariants

$$I_1 \;:=\; H^\dagger H, \qquad I_2 \;:=\; \big(H^\dagger \sigma^a H\big)\big(H^\dagger \sigma^a H\big) \;\equiv\; (H^\dagger H)^2 \tag{8.8}$$

(the identity follows from Pauli-matrix algebra). Hence at quartic order the most general $SU(2)_L \times U(1)_Y$-invariant potential is a function of I_1.

Landau functional from the medium. Integrating out fast internal modes in the scalar channel yields (near the origin)

$$V(H; \mathcal{S}) \;=\; m_H^2(\mathcal{S})\, I_1 \;+\; \lambda_H(\mathcal{S})\, I_1^2 \;+\; \sum_{n\geq 3} \frac{\lambda_{2n}(\mathcal{S})}{\Lambda^{2n-4}}\, I_1^n, \tag{8.9}$$

with state-dependent coefficients controlled by the spectral fields $\mathcal{S} = \{\Phi, \rho, \eta, \chi^\mu, T\}$. Stability requires $\lambda_H > 0$ and, if present, $\lambda_{2n} \geq 0$. The sign of $m_H^2(\mathcal{S})$ is governed by a *structural susceptibility* of the medium; an instability in this channel triggers condensation.

Gauge independence (Nielsen identity). Let ξ parametrize the gauge-fixing functional (Ch. 2). The effective potential V_{eff} obeys the Nielsen identity

$$\frac{\partial V_{\text{eff}}}{\partial \xi} \;+\; C(H, \xi)\, \frac{\partial V_{\text{eff}}}{\partial H} \;=\; 0, \tag{8.10}$$

so that its extrema (and in particular the vacuum amplitude v) are *gauge-parameter independent*. This ensures that v and physical masses derived from V_{eff} are unambiguous.

4. Symmetry breaking pattern and vacuum manifold

Minimization. For $m_H^2 < 0$ and $\lambda_H > 0$, the minima form

$$\mathcal{M}_{\rm vac} \;=\; \Big\{ H \in \mathbb{C}^2 \;:\; H^\dagger H = \frac{v^2}{2} \Big\}, \qquad v^2 \;:=\; -\frac{m_H^2}{\lambda_H} \;>\; 0. \tag{8.11}$$

Choosing the standard gauge

$$H(x) \;=\; \frac{1}{\sqrt{2}} \exp\!\Big(\frac{i\,\pi^a(x)\sigma_a}{v} \Big) \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \tag{8.12}$$

identifies three Nambu–Goldstone fields π^a and one radial (Higgs) mode h. The symmetry breaking pattern is

$$SU(2)_L \times U(1)_Y \;\longrightarrow\; U(1)_{\rm em} \quad \text{generated by} \quad Q = T^3 + Y, \tag{8.13}$$

with vacuum manifold diffeomorphic to $SU(2) \simeq S^3$ modulo gauge orbits.

Custodial structure (global). In the limit $g' \to 0$ and neglecting Yukawa splittings, the scalar sector enjoys a global $SU(2)_L \times SU(2)_R$ acting on

$$\Sigma \;:=\; \frac{1}{v} \big(i\sigma_2 H^*,\; H \big) \;\in\; SU(2), \qquad \Sigma \;\mapsto\; L\,\Sigma\,R^\dagger, \tag{8.14}$$

spontaneously broken to the custodial $SU(2)_V$ ($L = R$). This structure will protect $\rho = 1$ at tree level (Sec. 8.2).

5. Spectral expressions for m_H^2 and λ_H

Two- and four-point susceptibilities. Let $W[J]$ be the generating functional with source J coupling linearly to $\mathcal{O}_H$ and define $\phi \equiv \langle \mathcal{O}_H \rangle_J$. Then, at zero momentum,

$$m_H^2 \;=\; \Big[\frac{\partial^2 \Gamma[\phi]}{\partial \phi^\dagger \partial \phi} \Big]_{\phi=0}, \qquad \lambda_H \;=\; \frac{1}{2} \Big[\frac{\partial^4 \Gamma[\phi]}{\partial (\phi^\dagger \phi)^2} \Big]_{\phi=0}, \tag{8.15}$$

where Γ is the Legendre transform of W. Equivalently, m_H^2 is minus the inverse of the zero-momentum two-point function and λ_H the connected four-point vertex in the Higgs channel. Both inherit medium dependence through $\mathcal{S}$ and are renormalization-scale dependent; RG-improved expressions are used in Sec. 8.3.

Signs and gates. The condensation criterion is $m_H^2 < 0$. The (S) gate requires $\lambda_H > 0$ at the scale where the EFT is defined, and the (TQ) gate enforces forward-limit positivity on higher-derivative scalar operators that accompany (8.3).

6. Canonical normalization and reference scales

Field rescaling. Choosing $\kappa_H(\mu_*) = 1$ defines the canonical Higgs field at a *geometric matching scale* μ_* (Sec. 7.4). All parameters (m_H^2, λ_H) are then understood as $\overline{\rm MS}$ (or spectral) couplings at μ_* and must be RG-evolved to phenomenological scales. Thresholds of heavy vectorlike states contribute additively to m_H^2 and multiplicatively to λ_H according to decoupling rules (Ch. 3).

7. Summary (to be used in Secs. 8.2–8.4)

The Higgs is a complex doublet order parameter $H \in \Gamma(E_H)$ whose kinetic term is fixed by a spectral stiffness κ_H given by a Kubo slope of the scalar correlator. The most general renormalizable potential depends only on $I_1 = H^\dagger H$ with coefficients (m_H^2, λ_H) determined by two- and four-point susceptibilities of the medium. For $m_H^2 < 0$, condensation $|H| = v/\sqrt{2}$ breaks $SU(2)_L \times U(1)_Y$ to $U(1)_{\rm em}$ with three NG modes and a radial mode; a custodial $SU(2)_V$ emerges in the symmetric limit. These results furnish the starting point for the mass-generation mechanism and custodial relations (Sec. 8.2), Higgs self-interactions and spectral corrections (Sec. 8.3), and oblique parameter analysis (Sec. 8.4).

8.2 Mass generation: m_W, m_Z and custodial symmetry

Aim. Starting from the order-parameter dynamics of Sec. 8.1, we derive gauge-boson masses and the couplings of the Higgs resonance, expose the custodial symmetry that enforces the tree-level relation $\rho \equiv m_W^2/(m_Z^2 \cos^2\theta_W) = 1$, and identify the controlled sources of custodial breaking in the spectral framework.

1. Mass terms from the covariant derivative

Unitary-gauge parameterization. Let the Higgs doublet be written around the vacuum as

$$H(x) \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \qquad v^2 = -\,\frac{m_H^2}{\lambda_H} \;>0, \tag{8.16}$$

where v is fixed by the potential (8.9). Using the electroweak covariant derivative

$$D_\mu H = \Big(\partial_\mu - i g\, W_\mu^a \tfrac{\sigma_a}{2} - i g' B_\mu \tfrac{1}{2}\Big) H, \tag{8.17}$$

the quadratic part of the kinetic Lagrangian (8.3) produces gauge-boson mass terms.

Charged and neutral combinations. Define

$$W_\mu^\pm := \frac{1}{\sqrt{2}}\,(W_\mu^1 \mp i W_\mu^2), \qquad \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix}, \tag{8.18}$$

with $\tan\theta_W = g'/g$ in the canonical limit (Sec. 7.4). Then

$$\kappa_H\,(D_\mu H)^\dagger D^\mu H \;\supset\; \frac{\kappa_H v^2}{4}\Big[g^2\, W_\mu^+ W^{-\mu} + \frac{(g^2+g'^2)}{2}\, Z_\mu Z^\mu\Big], \tag{8.19}$$

so that, after field canonicalization $\kappa_H \to 1$ at the matching scale,

$$m_W^2 = \frac{g^2 v^2}{4}, \qquad m_Z^2 = \frac{(g^2+g'^2)\,v^2}{4}, \qquad m_A = 0. \tag{8.20}$$

Higgs–vector couplings. Expanding (8.19) to linear and quadratic order in h yields

$$\mathcal{L}_{hVV} = \frac{2m_W^2}{v}\, h\, W_\mu^+ W^{-\mu} + \frac{m_Z^2}{v}\, h\, Z_\mu Z^\mu + \frac{m_W^2}{v^2}\, h^2 W_\mu^+ W^{-\mu} + \frac{m_Z^2}{2v^2}\, h^2 Z_\mu Z^\mu, \tag{8.21}$$

i.e. $g_{hWW} = 2m_W^2/v$ and $g_{hZZ} = 2m_Z^2/v$ at tree level.

2. Nonlinear sigma-model form and eaten modes

Goldstone matrix. Introduce the 2×2 field

$$\Sigma(x) := \frac{1}{v}\left(i\sigma_2 H^*,\ H\right) \in SU(2), \qquad \Sigma = \exp\!\left(\frac{i\,\pi^a \sigma_a}{v}\right), \tag{8.22}$$

which transforms as $\Sigma \mapsto L\,\Sigma\,R^\dagger$ under a global $SU(2)_L \times SU(2)_R$. Gauging $SU(2)_L$ and the $U(1)_Y$ subgroup generated by T^3_R gives

$$D_\mu \Sigma = \partial_\mu \Sigma - ig\,W^a_\mu \tfrac{\sigma_a}{2}\,\Sigma + ig'\,\Sigma\,B_\mu \tfrac{\sigma_3}{2}. \tag{8.23}$$

The term $\frac{v^2}{4}\,\mathrm{Tr}\big[(D_\mu\Sigma)^\dagger D^\mu \Sigma\big]$ reproduces (8.20) and shows that the three fields π^a are eaten to provide the longitudinal polarizations of $W^\pm$ and Z.

3. Custodial symmetry and ρ parameter

Exact statement at tree level. In the limit $g' \to 0$ (and neglecting Yukawa splittings), the scalar Lagrangian

$$\mathcal{L}_\Sigma = \frac{v^2}{4}\,\mathrm{Tr}\big[(D_\mu\Sigma)^\dagger D^\mu \Sigma\big] \tag{8.24}$$

is invariant under global $SU(2)_L \times SU(2)_R$ acting as $\Sigma \mapsto L\Sigma R^\dagger$, spontaneously broken to the *custodial* $SU(2)_V$ ($L = R$). This symmetry enforces equality of charged and neutral mass coefficients in (8.19) and implies

$$\rho \;\equiv\; \frac{m_W^2}{m_Z^2 \cos^2\theta_W} \;=\; 1 \qquad \text{(tree level)}. \tag{8.25}$$

Controlled sources of custodial breaking. In the full theory, deviations $\Delta\rho \equiv \rho - 1$ arise from:

1. **Hypercharge** ($g' \neq 0$): breaks $SU(2)_R \to U(1)_Y$ but preserves (8.25) at tree level.
2. **Yukawa splittings**: $y_t \neq y_b$ (and, more generally, fermion isospin breaking) generate loop-level $\Delta\rho > 0$.
3. **Higher-dimensional operators**: e.g.

$$\frac{c_T}{\Lambda^2}\left(H^\dagger \overleftrightarrow{D}_\mu H\right)^2 \quad \Rightarrow \quad \alpha\, T \;=\; \Delta\rho \;=\; c_T\,\frac{v^2}{\Lambda^2} \;+\; \cdots, \tag{8.26}$$

   with α the fine-structure constant. In the spectral model such operators are induced by integrating out heavy spectral sectors; their coefficients are constrained by the positivity/causality gates (Ch. 18).
4. **Medium anisotropies** (χ^μ): CPT-even deformations of the Higgs kinetic tensor $\delta K^{\mu\nu}$ in (8.4) are *gauge-index blind* at dimension four, so after canonical normalization they do not alter (8.25). Any custodial-violating anisotropy appears first in higher-dimensional, gauge-invariant operators and is therefore suppressed.

4. Gauge fixing, BRST, and Goldstone equivalence (brief)

Covariant R_ξ gauges. Introduce

$$\mathcal{L}_{\rm gf} = -\frac{1}{2\xi}\big(\partial^\mu W^a_\mu - \xi\, m_W\, \pi^a\big)^2 - \frac{1}{2\xi}\big(\partial^\mu Z_\mu - \xi\, m_Z\, \pi^0\big)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2, \tag{8.27}$$

with BRST structure as in Ch. 2. The Goldstone masses are $\sqrt{\xi}\, m_{W,Z}$ and cancel the gauge-parameter dependence in amplitudes.

Equivalence theorem. At energies $E \gg m_{W,Z}$,

$$\mathcal{A}\big(W_L^\pm, Z_L;\ \cdots\big) \;=\; \mathcal{A}\big(\pi^\pm, \pi^0;\ \cdots\big)\,\Big[1 + \mathcal{O}\Big(\frac{m_{W,Z}}{E}\Big)\Big], \tag{8.28}$$

so longitudinal-boson scattering probes the scalar (custodial) sector and its spectral corrections.

5. Spectral ties and parameter counting

Parameter map. At tree level,

$$\{g, g', v\} \;\longrightarrow\; \{m_W, m_Z, e\}\,, \qquad e = g\sin\theta_W = g'\cos\theta_W, \tag{8.29}$$

with g, g' fixed by vertical metric coefficients of the fiber (Sec. 7.1) and θ_W by (7.65). The scale v is determined by the spectral susceptibilities m_H^2, λ_H (Sec. 8.1).

Gates. *(S)* $m_W^2, m_Z^2 > 0$; *(G)* the principal symbols of $W^\pm, Z, A$ waves lie within the $g_{\rm eff}$ cone (anisotropies bounded as in Sec. 7.3); *(TQ)* forward-limit positivity for Higgs/gauge higher-derivative operators; *(D)* dissipative renormalizations cannot generate photon mass nor spoil BRST unitarity.

6. Summary

Gauge-boson masses arise solely from the Higgs kinetic term evaluated on the vacuum, yielding $m_W^2 = \frac{1}{4}g^2v^2$ and $m_Z^2 = \frac{1}{4}(g^2+g'^2)v^2$ with a massless photon. Written in the nonlinear form, the scalar sector exhibits a global $SU(2)_L \times SU(2)_R$ broken to custodial $SU(2)_V$, enforcing $\rho = 1$ at tree level. In the spectral framework, g, g' and θ_W are geometric (fiber-metric) quantities, while v is fixed by scalar susceptibilities; departures from $\rho = 1$ originate only from controlled custodial-breaking sources (hypercharge loops, Yukawa splittings, or higher-dimensional operators), to be quantified in Sec. 8.4.

8.3 Higgs self-couplings and spectral corrections

Aim. We derive the cubic and quartic Higgs self-couplings from the spectral potential, organize beyond-renormalizable corrections induced by heavy spectral sectors, and present the loop-corrected (Coleman–Weinberg) effective potential together with RG relations. Whenever useful we set the Higgs kinetic normalization to unity at the matching scale, $\kappa_H(\mu_*) = 1$ (Sec. 8.1); nontrivial κ_H and anisotropies are included at the end via wavefunction factors.

1. Tree-level self-interactions from the spectral potential

Potential around the minimum. With the gauge-invariant invariant $I_1 = H^\dagger H$, write the local potential up to sextic order

$$V(H) \;=\; m_H^2\, I_1 \;+\; \lambda_H\, I_1^2 \;+\; \frac{c_6}{\Lambda^2}\, I_1^3 \;+\; \mathcal{O}\!\left(\frac{I_1^4}{\Lambda^4}\right), \tag{8.30}$$

where Λ is the spectral gap to heavy modes and c_6 encodes their net contribution. Choose unitary gauge $H^\top = (0,\, (v+h)/\sqrt{2})$ with v fixed by the minimum condition

$$\left.\frac{\partial V}{\partial h}\right|_{h=0} = 0 \quad\Longleftrightarrow\quad m_H^2 + \lambda_H v^2 + \frac{3}{4}\, c_6 \frac{v^4}{\Lambda^2} = 0. \tag{8.31}$$

Physical mass and self-couplings. Define physical parameters by derivatives of V at the minimum,

$$m_h^2 := \left.\frac{\partial^2 V}{\partial h^2}\right|_0, \qquad \lambda_3 := \left.\frac{\partial^3 V}{\partial h^3}\right|_0, \qquad \lambda_4 := \left.\frac{\partial^4 V}{\partial h^4}\right|_0, \tag{8.32}$$

so that $V \supset \frac{1}{2} m_h^2 h^2 + \frac{1}{3!}\lambda_3 h^3 + \frac{1}{4!}\lambda_4 h^4 + \cdots$. Using the chain rule with $I_1 = \frac{1}{2}(v+h)^2$ one finds

$$m_h^2 \;=\; v^2\!\left(2\lambda_H + 3\, c_6\, \frac{v^2}{\Lambda^2}\right), \tag{8.33}$$

$$\lambda_3 \;=\; 6\lambda_H v + 15\, c_6\, \frac{v^3}{\Lambda^2} \;=\; \frac{3m_h^2}{v} \;+\; 6\, c_6\, \frac{v^3}{\Lambda^2}, \tag{8.34}$$

$$\lambda_4 \;=\; 6\lambda_H + 45\, c_6\, \frac{v^2}{\Lambda^2} \;=\; \frac{3m_h^2}{v^2} \;+\; 36\, c_6\, \frac{v^2}{\Lambda^2}. \tag{8.35}$$

In the renormalizable limit ($c_6 \to 0$) this reproduces the SM-like relations $\lambda_3^{(0)} = 3m_h^2/v$ and $\lambda_4^{(0)} = 3m_h^2/v^2$.

Relative shifts from the sextic. It is convenient to quote the fractional deviations

$$\frac{\lambda_3}{3m_h^2/v} = 1 + \delta_3, \qquad \frac{\lambda_4}{3m_h^2/v^2} = 1 + \delta_4, \qquad \delta_3 = \frac{2c_6}{\Lambda^2}\,\frac{v^4}{m_h^2}, \quad \delta_4 = 6\,\delta_3. \tag{8.36}$$

Thus a positive c_6 enhances both self-couplings with a fixed ratio $\delta_4/\delta_3 = 6$ at this order.

2. Derivative operators and wavefunction effects

Kinetic deformations. Spectral matching generally induces the gauge-invariant derivative operator

$$\Delta\mathcal{L}_{\rm der} \;=\; \frac{c_H}{\Lambda^2}\, \partial_\mu (H^\dagger H)\, \partial^\mu (H^\dagger H) \;+\; \cdots . \tag{8.37}$$

In unitary gauge, the h kinetic term becomes

$$\frac{1}{2}\, Z_h\, (\partial h)^2, \qquad Z_h \;=\; 1 \;+\; \frac{c_H v^2}{\Lambda^2} \;+\; \mathcal{O}\!\left(\frac{vh}{\Lambda^2}, \frac{h^2}{\Lambda^2}\right). \tag{8.38}$$

Canonical normalization $h \to h_{\rm can} = Z_h^{-1/2} h$ rescales the n-point Higgs vertices by $Z_h^{-n/2}$. Keeping first order in $\varepsilon := c_H v^2/\Lambda^2$,

$$\lambda_3^{\rm phys} \;=\; \lambda_3\Big(1 - \tfrac{3}{2}\varepsilon\Big), \qquad \lambda_4^{\rm phys} \;=\; \lambda_4\Big(1 - 2\,\varepsilon\Big). \tag{8.39}$$

The same field renormalization multiplies the hVV and Yukawa couplings by $(1 - \frac{1}{2}\varepsilon)$ per external h leg, preserving gauge relations mandated by custodial symmetry at tree level.

Anisotropic kinetic tensor. If the medium induces a CPT-even anisotropy $\delta K^{\mu\nu}$ (Sec. 8.1), the canonically normalized field is obtained by diagonalizing $Z^{\mu\nu} = g^{\mu\nu} + \delta K^{\mu\nu}$; the net rescaling of n-point scalar vertices is again $Z_h^{-n/2}$ with $Z_h = \sqrt{-\det Z^\mu{}_\nu}^{\,1/2}$ in the aether rest frame, while dispersion deformations are bounded by the (G) gate.

3. One-loop effective potential and spectral matching

Coleman–Weinberg form. In a background h (Landau gauge), the renormalized one-loop correction reads

$$\Delta V_{\rm CW}(h) \;=\; \frac{1}{64\pi^2}\sum_i n_i\, m_i(h)^4 \left[\ln\frac{m_i(h)^2}{\mu^2} - c_i\right], \tag{8.40}$$

where the sum runs over real degrees of freedom i with multiplicities n_i and $c_i = \frac{3}{2}$ for scalars/fermions and $\frac{5}{6}$ for gauge bosons in $\overline{\rm MS}$. Field-dependent masses are

$$m_W^2(h) = \frac{g^2}{4}(v+h)^2, \qquad m_Z^2(h) = \frac{g^2+g'^2}{4}(v+h)^2, \tag{8.41}$$

$$m_t(h) = \frac{y_t}{\sqrt{2}}(v+h), \qquad m_h^2(h) = \frac{\partial^2 V}{\partial h^2}, \qquad m_\pi^2(h) = \frac{1}{v+h}\frac{\partial V}{\partial h} \quad \text{(Goldstones)}. \tag{8.42}$$

The loop-corrected self-couplings follow from

$$\lambda_{3,4}^{\rm 1\text{–}loop} \;=\; \frac{\partial^{3,4}}{\partial h^{3,4}}\Big[V(h) + \Delta V_{\rm CW}(h)\Big]\Big|_{h=0} \;\times\; Z_h^{-3/2,-2}. \tag{8.43}$$

Spectral heavy states contribute additional terms of the form (8.40) with their field-dependent masses $M_\alpha(h)$ and decouple for $\mu \ll M_\alpha$ as per Appelquist–Carazzone (Ch. 3).

Gauge independence. By the Nielsen identity (Sec. 8.1), the extrema of $V_{\rm eff} = V + \Delta V_{\rm CW}$ and the derivatives evaluated at $h = 0$ (after on-shell renormalization of v) are gauge-parameter independent.

4. RG evolution and spectral gates

Running couplings. Let $\lambda(\mu)$ denote the renormalized quartic in the $\overline{\rm MS}$ scheme. To one loop with SM-like light content,

$$16\pi^2\,\beta_\lambda = 24\lambda^2 - 6y_t^4 + \lambda\Big(-9g^2 - 3g'^2 + 12y_t^2\Big) + \frac{9}{8}g^4 + \frac{3}{4}g^2g'^2 + \frac{3}{8}g'^4 \;+\; \Delta\beta_\lambda^{\rm spec}, \tag{8.44}$$

where $\Delta\beta_\lambda^{\rm spec}$ encodes thresholds and interactions with spectral sectors. The running of c_6 and c_H obeys

$$\mu\frac{d}{d\mu}\begin{pmatrix}c_6\\ c_H\end{pmatrix} = \frac{1}{16\pi^2}\,\mathbf{B}\,\begin{pmatrix}c_6\\ c_H\end{pmatrix} + \text{inhomogeneous terms from heavy loops}, \tag{8.45}$$

with a calculable anomalous-dimension matrix $\mathbf{B}$ once the UV completion (spectral sector content) is specified.

Positivity and stability gates. The (S) gate requires that the scalar forward amplitudes be positive: in particular $\lambda(\mu) > 0$ in the validity window of the EFT and higher-derivative coefficients satisfy known dispersion bounds. Metastability is allowed if the lifetime exceeds cosmic times, but then matching onto spectral UV data must ensure no violation of the (TQ) gate along the flow (Ch. 3).

5. From parameters to observables

Physical definitions. After canonical normalization,

$$\lambda_3^{\rm phys} \equiv \left.\frac{\partial^3 V_{\rm eff}}{\partial h_{\rm can}^3}\right|_0, \qquad \lambda_4^{\rm phys} \equiv \left.\frac{\partial^4 V_{\rm eff}}{\partial h_{\rm can}^4}\right|_0. \tag{8.46}$$

The single-Higgs rates fix m_h and v (via G_F), while di-Higgs production constrains $\lambda_3^{\rm phys}$ and, at higher luminosity, $\lambda_4^{\rm phys}$. In the present framework, deviations trace back to $\{c_6, c_H\}$ and to loop/threshold effects encoded in $V_{\rm eff}$.

Summary of leading deviations. Collecting Eqs. (8.34)–(8.39) at $\mathcal{O}(v^2/\Lambda^2)$,

$$\frac{\lambda_3^{\rm phys}}{3m_h^2/v} = 1 \;+\; \underbrace{\frac{2c_6}{\Lambda^2}\frac{v^4}{m_h^2}}_{\text{potential (sextic)}} \;-\; \underbrace{\frac{3}{2}\frac{c_H v^2}{\Lambda^2}}_{\text{wavefunction}} \;+\; \Big[\text{1-loop CW + thresholds}\Big], \tag{8.47}$$

$$\frac{\lambda_4^{\rm phys}}{3m_h^2/v^2} = 1 \;+\; 6\,\frac{2c_6}{\Lambda^2}\frac{v^4}{m_h^2} \;-\; 2\,\frac{c_H v^2}{\Lambda^2} \;+\; \Big[\text{1-loop CW + thresholds}\Big]. \tag{8.48}$$

These relations provide a direct map from spectral UV data to measurable Higgs self-interactions, constrained by positivity, causality, and unitarity gates.

6. Summary

At tree level the spectral potential fixes the Higgs self-couplings with $\lambda_3 = 3m_h^2/v$ and $\lambda_4 = 3m_h^2/v^2$ in the renormalizable limit, while a sextic $(H^\dagger H)^3$ induces correlated enhancements $\delta_4 = 6\delta_3$. Derivative operators and χ^μ-driven anisotropies modify the couplings through wavefunction renormalization, uniformly rescaling n-point vertices by $Z_h^{-n/2}$. Loop corrections are captured by the Coleman–Weinberg potential with field-dependent masses and RG running; all parameters are subject to the spectral S/G/TQ/D gates. These results feed directly into precision bounds via oblique parameters (S, T, U) (Sec. 8.4) and di-Higgs phenomenology (Part VII).

8.4 Oblique parameters (S,T,U) and bounds

Aim. We formulate precision electroweak constraints on the spectral model in the *oblique* approximation, where new physics modifies only gauge-boson two-point functions. We give first-principles definitions of the Peskin–Takeuchi parameters (S,T,U) in terms of vacuum polarizations, introduce the zero-momentum basis $(\hat{S},\hat{T},W,Y)$ that is most convenient for our fiber/constitutive description, derive how these quantities are computed from the neutral and charged gauge quadratic forms obtained in Secs. 7.3–7.4, and state rigorous positivity/causality bounds that follow from the spectral gates. Throughout, $\Delta\Pi \equiv \Pi - \Pi_{\rm SM}$ denotes the new-physics contribution after subtracting the SM reference at the same renormalization point.

1. Gauge-boson vacuum polarizations

Transverse structure. Let $X,Y \in \{\gamma,Z,W\}$ label physical gauge fields after canonicalization (Sec. 7.4). The 1PI two-point function reads

$$i\Pi^{\mu\nu}_{XY}(q) \;=\; i\Big(q^\mu q^\nu - q^2 g^{\mu\nu}\Big)\,\Pi_{XY}(q^2), \qquad q_\mu \Pi^{\mu\nu}_{XY} = 0, \tag{8.49}$$

with analytic $\Pi_{XY}(q^2)$ in a punctured neighborhood of $q^2=0$. We Taylor expand near $q^2=0$:

$$\Delta\Pi_{XY}(q^2) = \Delta\Pi_{XY}(0) \;+\; q^2\,\Delta\Pi'_{XY}(0) \;+\; \tfrac{1}{2}q^4\,\Delta\Pi''_{XY}(0) \;+\cdots. \tag{8.50}$$

2. Peskin–Takeuchi (S,T,U)

Definitions (physical basis). In terms of the *physical* fields (γ,Z,W) and the weak mixing angles $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$ one defines

$$\alpha\, S \;=\; 4s_W^2 c_W^2\Big[\,\Delta\Pi'_{ZZ}(0) - \Delta\Pi'_{\gamma\gamma}(0) - \frac{c_W^2 - s_W^2}{s_W c_W}\,\Delta\Pi'_{Z\gamma}(0)\Big], \tag{8.51}$$

$$\alpha\, T \;=\; \frac{\Delta\Pi_{WW}(0)}{m_W^2} - \frac{\Delta\Pi_{ZZ}(0)}{m_Z^2}, \tag{8.52}$$

$$\alpha\, U \;=\; 4s_W^2\Big[\Delta\Pi'_{WW}(0) - c_W^2\Delta\Pi'_{ZZ}(0) - 2s_W c_W\Delta\Pi'_{Z\gamma}(0) - s_W^2\Delta\Pi'_{\gamma\gamma}(0)\Big], \tag{8.53}$$

where $\alpha = e^2/4\pi$ is evaluated at the Z pole. These combinations isolate universal (oblique) corrections to on-shell observables.

Interpretation. T encodes custodial breaking at $q^2=0$; S is the *slope* (mixing) of neutral currents at $q^2=0$; U measures derivative breaking between charged and neutral channels and is typically suppressed when the new scale is heavy.

3. Zero-momentum basis $(\hat{S},\hat{T},W,Y)$

Definitions (gauge-eigenstate basis). It is technically convenient to work in the (W^1,W^2,W^3,B) basis before diagonalization and define [**?**, **?**]

$$\hat{S} \;:=\; \frac{g}{g'}\,\Delta\Pi'_{3B}(0), \qquad \hat{T} \;:=\; \frac{\Delta\Pi_{33}(0) - \Delta\Pi_{11}(0)}{m_W^2}, \tag{8.54}$$

$$W \;:=\; \frac{m_W^2}{2}\,\Delta\Pi''_{33}(0), \qquad Y \;:=\; \frac{m_W^2}{2}\,\Delta\Pi''_{BB}(0), \tag{8.55}$$

with g, g' the canonically normalized couplings. In the regime where higher-derivative pieces are negligible at LEP1 momenta ($|q^2| \ll \Lambda^2$), one may set $W = Y = 0$ and use the approximate relations

$$S \;\simeq\; \frac{4s_W^2}{\alpha}\,\hat{S}, \qquad T \;\simeq\; \frac{1}{\alpha}\,\hat{T}, \qquad U \;\simeq\; 0 \quad (\text{if } W = Y = 0), \tag{8.56}$$

while retaining W, Y gives the complete LEP1/LEP2 description.

4. Computation in the spectral model

Constitutive tensor and kinetic mixing. The quadratic gauge Lagrangian of Sec. 7.3 is

$$\mathcal{L}^{(2)}_{\text{gauge}} \;=\; -\frac{1}{4}\,\mathcal{K}^{IJ}{}_{\mu\alpha\nu\beta}\,F^{I\,\mu\nu}F^{J\,\alpha\beta}, \qquad I, J \in \{1, 2, 3, Y\}, \tag{8.57}$$

with the isotropic part Z_1^{IJ} defined in (7.49). In momentum space and at small q^2,

$$\Delta\Pi_{IJ}(q^2) \;=\; -\,Z_1^{IJ}\,q^2 + \mathcal{O}(q^4) \;\Longrightarrow\; \Delta\Pi'_{IJ}(0) \;=\; -\,Z_1^{IJ}, \qquad \Delta\Pi''_{IJ}(0) = 0 \;\text{(purely two-derivative)}. \tag{8.58}$$

Hence, after canonical normalization of the *diagonal* entries $Z_1^{YY} = 1/g'^2$, $Z_1^{aa} = \delta^{aa}/g^2$, the *off-diagonal* entry controls $\hat{S}$:

$$\hat{S} \;=\; -\frac{g}{g'}\,Z_1^{3Y}, \qquad \Rightarrow \qquad S \;\simeq\; -\,\frac{4s_W^2}{\alpha}\,Z_1^{3Y}. \tag{8.59}$$

Geometrically, Z_1^{3Y} measures the non-orthogonality of the $SU(2)$ and $U(1)$ orbit metrics on the fiber (Sec. 7.1); it vanishes for perfectly orthogonal orbits.

Custodial breaking and $\hat{T}$. At dimension four, custodial breaking enters via the operator

$$\frac{c_T}{\Lambda^2}\left(H^\dagger \overleftrightarrow{D}_\mu H\right)^2 \;\Rightarrow\; \alpha\,T \;=\; \Delta\rho \;=\; c_T\,\frac{v^2}{\Lambda^2} \;+\; \cdots, \tag{8.60}$$

so that in our conventions $\hat{T} = \alpha T$. In the spectral model, custodial symmetry is exact in the limit described in Sec. 8.2 and c_T appears only through heavy custodial-violating thresholds; at tree level $\hat{T} = 0$.

Higher-derivative pieces W, Y. Four-derivative gauge operators induce nonzero W, Y,

$$\Delta\mathcal{L} \supset -\frac{W}{2m_W^2}\,(D_\rho W^{a\,\rho\mu})^2 - \frac{Y}{2m_W^2}\,(\partial_\rho B^{\rho\mu})^2 \quad\Longleftrightarrow\quad W = \frac{m_W^2}{2}\Delta\Pi''_{33}(0),\;\; Y = \frac{m_W^2}{2}\Delta\Pi''_{BB}(0). \tag{8.61}$$

In the constitutive expansion (7.42), these arise first from the k^4 term and are suppressed by the spectral gap Λ; they are also bounded to be nonnegative by forward-limit dispersion relations (see below).

5. Spectral positivity/causality bounds

Forward dispersion and sum rules. Causality and unitarity (S/G/TQ gates) imply a Källén–Lehmann representation with nonnegative spectral densities, giving

$$\Delta\Pi'_{IJ}(0) \;\le\; 0 \quad (\text{for } I = J), \qquad \Delta\Pi''_{IJ}(0) \;\ge\; 0, \qquad W \;\ge\; 0, \quad Y \;\ge\; 0, \tag{8.62}$$

and, for the off-diagonal entry, a Cauchy–Schwarz inequality

$$\left|\Delta\Pi'_{3B}(0)\right| \;\le\; \sqrt{\Delta\Pi'_{33}(0)\,\Delta\Pi'_{BB}(0)} \quad\Rightarrow\quad |\hat{S}| \;\le\; \frac{g}{g'}\sqrt{\frac{1}{g^2 g'^2}} \quad \text{(after canonicalization)}. \tag{8.63}$$

In practice the canonical normalization ensures that $\Delta\Pi'_{33}(0) = -1/g^2$, $\Delta\Pi'_{BB}(0) = -1/g'^2$ from the SM piece, so any *additional* new-physics contribution must satisfy $|Z_1^{3Y}| \lesssim \mathcal{O}(10^{-2})$ to remain within precision fits; see the phenomenology note below.

Hyperbolicity. Anisotropic χ^μ-induced deformations (Sec. 7.3) modify dispersion at order k^2 but leave the zero-momentum definitions (8.51)–(8.53) unchanged. Hyperbolicity demands that the principal symbol eigencones remain subluminal with respect to g_{eff}, which translates into W, Y sufficiently small and $|\hat{S}|$ bounded by (8.63).

6. SMEFT/operator mapping (tree level)

Relevant operators. At dimension six, the Warsaw-basis operators that contribute at tree level are

$$\mathcal{O}_{HWB} = (H^\dagger \sigma^a H)\, W^a_{\mu\nu} B^{\mu\nu}, \qquad \mathcal{O}_{HD} = (H^\dagger \overleftrightarrow{D}_\mu H)^2, \tag{8.64}$$

$$\mathcal{O}_{2W} = -(D^\rho W^a_{\rho\mu})^2, \qquad \mathcal{O}_{2B} = -(\partial^\rho B_{\rho\mu})^2. \tag{8.65}$$

Writing $\Delta\mathcal{L} = \sum C_i\, \mathcal{O}_i/\Lambda^2$ and expanding about $H = (0, (v+h)/\sqrt{2})^\top$,

$$\hat{S} \;=\; -\frac{g}{g'}\,\frac{v^2}{\Lambda^2}\,C_{HWB}\,, \qquad \hat{T} \;=\; +\frac{v^2}{\Lambda^2}\,C_{HD}\,, \qquad W \;=\; +\frac{g^2 v^2}{2\Lambda^2}\,C_{2W}\,, \qquad Y \;=\; +\frac{g'^2 v^2}{2\Lambda^2}\,C_{2B}\,, \tag{8.66}$$

in the convention where the kinetic terms are canonical before inserting $\mathcal{O}_{HWB}$.[1] These expressions provide a direct bridge between spectral UV data (encoded in the Wilsons) and precision observables.

7. Phenomenology and admissible region

Global-fit template. Precision data constrain a *convex* region in the $(\hat{S}, \hat{T}, W, Y)$ space with strong positive correlation between $(\hat{S}, \hat{T})$ and tight bounds on (W, Y). A robust working regime for the spectral model is

$$|\hat{S}| \;\lesssim\; \mathcal{O}(10^{-2})\,, \qquad |\hat{T}| \;\lesssim\; \mathcal{O}(10^{-2})\,, \qquad W, Y \;\lesssim\; \mathcal{O}(10^{-3})\,, \tag{8.67}$$

which keeps all shifts within current per-mille accuracy of electroweak pseudoobservables. (Concrete numbers depend on the fit scheme, input set, and treatment of correlations.)

Spectral predictions and control knobs.

- **T** is protected by custodial symmetry; heavy custodial-violating thresholds or large Yukawa isospin breaking generate $\hat{T} > 0$; Eq. (8.60) shows how to translate a bound on T into a lower limit on the custodial-breaking scale $\Lambda/\sqrt{|c_T|}$.

[1] Signs follow from $\Delta\Pi'_{IJ}(0) = -$ coefficient of $F^I_{\mu\nu} F^{J\,\mu\nu}$; the overall conventions of Eq. (8.66) match the definitions (8.54)–(8.55).

- **S** is controlled by the *geometric* off-diagonal kinetic entry Z_1^{3Y} (Eq. (8.59)) or, in SMEFT language, by C_{HWB}; orthogonal orbits on the fiber ($Z_1^{3Y} = 0$) give $\hat{S} = 0$ at tree level.
- **W** and **Y** are generated by four-derivative gauge operators and are *positive* (Eq. (8.62)); they are suppressed by the spectral gap and by the (G) hyperbolicity gate.

8. Practical recipe for a given spectral fiber

Algorithm.

1. Canonicalize the quadratic neutral gauge sector as in Eqs. (7.66)–(7.67).
2. Read off $\Delta\Pi_{IJ}(q^2)$ at small q^2 from the constitutive tensor; extract $\Delta\Pi'_{3B}(0)$, $\Delta\Pi_{33}(0) - \Delta\Pi_{11}(0)$, $\Delta\Pi''_{33}(0)$, $\Delta\Pi''_{BB}(0)$.
3. Compute $(\hat{S}, \hat{T}, W, Y)$ using (8.54)–(8.55); if desired, map to (S, T, U) using (8.51)–(8.53).
4. Check the positivity/hyperbolicity constraints (8.62) and custodial consistency (Sec. 8.2).
5. Compare with global-fit ellipses to derive allowed ranges of fiber-metric deformations and Wilson coefficients via (8.59), (8.60), (8.61), (8.66).

9. Summary

Oblique parameters provide a sharp, basis-independent interface between spectral fiber geometry and precision data. In our framework:

- S is a geometric slope (kinetic mixing) fixed by the non-orthogonality of $SU(2)$ and $U(1)$ orbits;
- T vanishes at tree level by custodial symmetry and is a clean probe of custodial-violating thresholds;
- U is generically small, while the more complete set $(\hat{S}, \hat{T}, W, Y)$ captures higher-derivative effects with $W, Y \geq 0$.

These quantities, constrained by positivity and hyperbolicity, delimit the admissible parameter volume of the spectral electroweak sector and will be employed in the global fits of Part VII.

Chapter 9

Precision Electroweak Phenomenology

9.1 LEP/LHC observables and mapping to spectral couplings

Aim. We assemble the precision electroweak *pseudo-observables* (LEP/SLD line shape, asymmetries; m_W and widths; Drell–Yan at the LHC) and provide a systematic, gauge-invariant map from the spectral parameters of the model to these quantities. At leading order this map is expressed in terms of (i) the geometric gauge couplings (g, g') and the vacuum scale v (Secs. 7.4, 8.2), (ii) the oblique parameters $(\hat{S}, \hat{T}, W, Y)$ or (S, T, U) that encode universal two-point distortions (Sec. 8.4), and (iii) controlled higher-dimensional operators suppressed by the spectral gap Λ. Anisotropies induced by the director χ^μ enter only through the constitutive tensors already subsumed in the oblique set at zero momentum (Sec. 7.3).

1. Input schemes and tree-level map

Schemes. We adopt the standard $\{\alpha, m_Z, G_F\}$ input set:

$$\alpha \equiv \frac{e^2}{4\pi}, \qquad m_Z, \qquad G_F = \frac{1}{\sqrt{2}\, v^2}, \tag{9.1}$$

with $e = g s_W = g' c_W$ and $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$. At tree level,

$$m_W^2 = \frac{g^2 v^2}{4} = m_Z^2 c_W^2, \qquad s_W^2 c_W^2 = \frac{\pi\alpha}{\sqrt{2}\, G_F\, m_Z^2}, \tag{9.2}$$

and the $Zf\bar{f}$ vector/axial couplings are

$$g_A^f = T_3^f, \qquad g_V^f = T_3^f - 2Q_f s_W^2. \tag{9.3}$$

In the spectral framework (g, g') are fixed at the matching scale by vertical fiber metrics (Sec. 7.4), while v is fixed by the Higgs susceptibilities (Sec. 8.1).

2. Oblique deformations of on-shell quantities

Definitions. Let $\Delta\Pi$ denote new-physics vacuum polarizations beyond the SM reference. The Peskin–Takeuchi parameters (S, T, U) are given by Eqs. (8.51)–(8.53); equivalently use the zero-momentum basis $(\hat{S}, \hat{T}, W, Y)$ of Eqs. (8.54)–(8.55). In the $\{\alpha, m_Z, G_F\}$ scheme, the leading *shifts* of key on-shell quantities read (for small S, T, U)

$$\frac{\delta m_W^2}{m_W^2} = \frac{\alpha}{c_W^2 - s_W^2}\left(-\tfrac{1}{2}S + c_W^2 T + \frac{c_W^2 - s_W^2}{4s_W^2}\,U\right), \tag{9.4}$$

$$\delta s^2_{\rm eff} \equiv s^2_{\rm eff} - s_W^2 = \frac{\alpha}{c_W^2 - s_W^2}\left(\tfrac{1}{4}S - s_W^2 c_W^2 T\right), \tag{9.5}$$

where $s^2_{\rm eff}$ is the *effective* weak mixing angle that controls Z-pole asymmetries. At dimension six with $(\hat{S}, \hat{T}, W, Y)$, one may use

$$S \simeq \frac{4s_W^2}{\alpha}\,\hat{S}, \qquad T \simeq \frac{1}{\alpha}\,\hat{T}, \qquad U \simeq 0 \text{ if } W = Y = 0. \tag{9.6}$$

3. Z-pole pseudo-observables (LEP/SLD)

Partial widths and total width. For a fermion f with color factor N_c^f,

$$\Gamma_f = \frac{N_c^f\, G_F m_Z^3}{6\pi\sqrt{2}}\;\rho_f\;\left[(g_V^f)^2 + (g_A^f)^2\right]\left(1 + \delta^f_{\rm QED} + \delta^f_{\rm QCD}\right), \tag{9.7}$$

where $\rho_f = 1 + \Delta\rho_f$ (custodial factor) and radiative QED/QCD corrections are standard. In the *oblique* limit $\Delta\rho_f = \Delta\rho = \alpha T$ is universal and

$$g_A^f = \sqrt{\rho}\,T_3^f, \qquad g_V^f = \sqrt{\rho}\left(T_3^f - 2Q_f s^2_{\rm eff}\right). \tag{9.8}$$

The hadronic peak cross-section and ratios are

$$\sigma^0_{\rm had} = \frac{12\pi}{m_Z^2}\,\frac{\Gamma_e \Gamma_{\rm had}}{\Gamma_Z^2}, \qquad R^0_\ell = \frac{\Gamma_{\rm had}}{\Gamma_\ell}, \qquad R^0_q = \frac{\Gamma_q}{\Gamma_{\rm had}}. \tag{9.9}$$

Asymmetries. Define polarization asymmetries

$$\mathcal{A}_f = \frac{2g_V^f g_A^f}{(g_V^f)^2 + (g_A^f)^2}, \qquad A^{0,f}_{\rm FB} = \frac{3}{4}\,\mathcal{A}_e\,\mathcal{A}_f. \tag{9.10}$$

To first order in oblique parameters, the dominant dependence enters through $s^2_{\rm eff}$ in (9.5) and through the universal factor $\rho = 1 + \alpha T$.

Spectral map. Given (g, g') (hence s_W^2) at μ_* and the oblique set at m_Z, compute $s^2_{\rm eff}$ via (9.5); then evaluate (9.7)–(9.10). Non-orthogonality of fiber orbits (Sec. 7.1) gives $\hat{S} = -\frac{g}{g'} Z_1^{3Y}$ (Eq. (8.59)); custodial violation from heavy thresholds gives $\hat{T} = (v^2/\Lambda^2) C_{HD}$ (Eq. (8.66)). Four-derivative gauge operators induce (W, Y) but do not affect the Z-pole at linear order in q^2.

4. m_W, W width, and leptonic decays

Mass and width. The shift δm_W follows from (9.4). The leptonic partial width

$$\Gamma(W \to \ell\nu_\ell) = \frac{G_F m_W^3}{6\pi\sqrt{2}} \Big(1 + \Delta r_W\Big), \tag{9.11}$$

is modified by the same oblique set through the renormalization of m_W and by contact terms only if nonuniversal operators are present (absent in the pure-oblique spectral limit). The total width adds quark channels with QCD corrections.

Muon decay and Δr. Equating the G_F extraction from muon decay to the Lagrangian value $1/(\sqrt{2}v^2)$ yields the Sirlin combination Δr; in the oblique limit, Δr is a linear function of (S, T, U) and reproduces (9.4) when recast in the $\{\alpha, m_Z, G_F\}$ scheme.

5. Drell–Yan at the LHC and effective couplings

Neutral-current Drell–Yan. Near the Z pole, the dilepton angular coefficient $A_{\rm FB}(M_{\ell\ell})$ in $pp \to \ell^+\ell^- + X$ determines $s^2_{\rm eff}$ through (9.10), with PDFs and QCD resummation under control. Off-pole bins probe $\Pi'_{Z\gamma}(q^2)$ and hence $\hat{S}$.

Charged-current Drell–Yan. The m_T spectrum in $pp \to \ell\nu$ is primarily sensitive to m_W; thus (9.4) maps the spectral $(\hat{S}, \hat{T}, U)$ into the template used by ATLAS/CMS. In the spectral model, χ^μ-induced anisotropies do not enter these inclusive observables at linear order (they modify wavefronts but preserve zero-momentum polarizations; Sec. 7.3).

6. Operator (SMEFT) correspondence at tree level

Warsaw basis. At dimension six the universal subset affecting LEP/LHC electroweak precision is

$$\mathcal{O}_{HWB},\ \mathcal{O}_{HD},\ \mathcal{O}_{2W},\ \mathcal{O}_{2B} \quad \Longleftrightarrow \quad (\hat{S}, \hat{T}, W, Y) \text{ via Eq. (8.66)}. \tag{9.12}$$

Thus the *entire* LEP1/LEP2 input is captured by four Wilson coefficients in the universal limit. In the spectral model these Wilsons are calculable functions of the fiber vertical metric (for $\hat{S}$), custodial-breaking thresholds (for $\hat{T}$), and the k^4 coefficients of the constitutive tensor (for W, Y).

7. Worked linearized formulas for pseudo-observables

Master shifts. For quick phenomenology at first order (keeping only S, T and setting $U = W = Y = 0$),

$$\frac{\delta\Gamma_Z}{\Gamma_Z} \simeq \alpha\, T \ + \ \kappa_Z\, \delta s^2_{\rm eff}, \qquad \kappa_Z := \frac{2\sum_f N_c^f Q_f (T_3^f - 2Q_f s_W^2)}{\sum_f N_c^f [(T_3^f - 2Q_f s_W^2)^2 + (T_3^f)^2]}\,, \tag{9.13}$$

$$\frac{\delta\sigma^0_{\rm had}}{\sigma^0_{\rm had}} \simeq -2\,\frac{\delta\Gamma_Z}{\Gamma_Z} + \frac{\delta\Gamma_e}{\Gamma_e} + \frac{\delta\Gamma_{\rm had}}{\Gamma_{\rm had}}, \tag{9.14}$$

$$\delta A_{\rm FB}^{0,f} = \frac{3}{4}\Big(\mathcal{A}_f\, \delta\mathcal{A}_e + \mathcal{A}_e\, \delta\mathcal{A}_f\Big), \qquad \delta\mathcal{A}_f \simeq \Big(\partial_{s^2_{\rm eff}} \mathcal{A}_f\Big)\, \delta s^2_{\rm eff}, \tag{9.15}$$

with $\delta s^2_{\rm eff}$ from (9.5). These relations suffice to propagate (S, T) constraints to the spectral parameters.

8. Practical mapping recipe (spectral ⇒ data)

Algorithm.

1. **Geometric step:** compute (g, g') and thus s_W^2 from the fiber vertical metric at μ_* (Sec. 7.4); fix v from the Higgs potential (Sec. 8.1).
2. **RG step:** run $(g, g') \to (m_Z)$ using the spectral RG (Ch. 3); include threshold decoupling.
3. **Oblique step:** compute $(\hat{S}, \hat{T}, W, Y)$ from the constitutive tensor and custodial-breaking operators (Sec. 8.4); translate to (S, T, U) if desired.
4. **Prediction step:** evaluate m_W via (9.4), s^2_{eff} via (9.5), then all Z-pole pseudo-observables using (9.7)–(9.10).
5. **Fit step:** compare to LEP/SLD/LHC inputs; the allowed region returns bounds on non-orthogonality (Z_1^{3Y}), custodial breaking (C_{HD}), and k^4 constitutive entries (through W, Y).

9. Remarks on nonuniversal effects

Beyond oblique. Vertex corrections (e.g. $Z\bar{b}b$) or four-fermion operators arise only if the spectral UV completion couples nonuniversally to flavors; these are absent in the universal limit treated here and will be discussed in Sec. 9.3 for loop-induced rare processes.

10. Summary

Precision observables at LEP/SLD and the LHC admit a compact, gauge-invariant description in terms of a few pseudo-observables. In the spectral model, these are determined by geometric gauge couplings (g, g'), the vacuum scale v, and the oblique set $(\hat{S}, \hat{T}, W, Y)$ derived from the constitutive tensor and custodial structure. The linearized formulas (9.4)–(9.15) provide a ready-to-use pipeline from fiber geometry to data, constrained by the positivity and causality gates of Sec. 8.4.

9.2 Triple/quartic gauge vertices and EFT coefficients

Aim. We give a gauge-invariant parameterization of anomalous triple-gauge couplings (TGCs) and quartic-gauge couplings (QGCs), derive their embedding in the spectral effective field theory, and map them to Wilson coefficients of a convenient operator basis. We also state custodial and gauge constraints, RG properties, unitarity growth, and positivity/causality bounds relevant for diboson and vector-boson scattering (VBS) measurements.

1. TGC parameterization (HPZH form)

Conventions. Define $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$, $V_\mu \in \{\gamma_\mu, Z_\mu\}$, and $g_{WW\gamma} = e$, $g_{WWZ} = gc_W$. The most general CP-even, Lorentz- and $U(1)_{\text{em}}$-invariant TGC Lagrangian up to

one derivative beyond the Yang–Mills form is [**?**]

$$\mathcal{L}_{\rm TGC} = i g_{WWV}\Big\{ g_1^V\big(W^+_{\mu\nu}W^{-\mu}V^\nu - W^-_{\mu\nu}W^{+\mu}V^\nu\big) + \kappa_V\, W^+_\mu W^-_\nu V^{\mu\nu} + \frac{\lambda_V}{m_W^2}\, W^{+\mu}{}_\nu W^{-\nu}{}_\rho V^\rho{}_\mu \Big\}, \tag{9.16}$$

with $V^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$, $W^\pm_{\mu\nu} = \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu$. Electromagnetic gauge invariance fixes

$$g_1^\gamma = 1, \qquad \kappa_\gamma = 1 + \Delta\kappa_\gamma, \qquad g_1^Z = 1 + \Delta g_1^Z, \qquad \kappa_Z = 1 + \Delta\kappa_Z, \qquad \lambda_\gamma = \lambda_Z \equiv \lambda. \tag{9.17}$$

At the SM tree level: $\Delta g_1^Z = \Delta\kappa_V = \lambda = 0$.

Custodial/weak-mixing relation (dim-6). For $SU(2)_L \times U(1)_Y$-invariant deformations at dimension six,

$$\Delta\kappa_Z \;=\; \Delta g_1^Z - \tan^2\theta_W\, \Delta\kappa_\gamma, \qquad \lambda_Z = \lambda_\gamma, \tag{9.18}$$

so only three independent *CP-even* TGC parameters exist: $\{\Delta g_1^Z,\, \Delta\kappa_\gamma,\, \lambda\}$.

2. SMEFT bases and clean matching

HISZ/Warsaw operators. A convenient gauge-invariant set controlling TGCs at $\mathcal{O}(1/\Lambda^2)$ is

$$\mathcal{O}_W = (H^\dagger \sigma^a \overleftrightarrow{D}_\mu H)\, D_\nu W^{a\,\mu\nu}, \qquad \mathcal{O}_B = (H^\dagger \overleftrightarrow{D}_\mu H)\, \partial_\nu B^{\mu\nu}, \qquad \mathcal{O}_{3W} = \epsilon^{abc}\, W^{a\,\mu}{}_\nu W^{b\,\nu}{}_\rho W^{c\,\rho}{}_\mu, \tag{9.19}$$

together with their CP-odd duals $\tilde{\mathcal{O}}_{3W}$, $\tilde{\mathcal{O}}_W$, $\tilde{\mathcal{O}}_B$. We write the Lagrangian deformation as

$$\Delta\mathcal{L} \;=\; \frac{C_W}{\Lambda^2}\mathcal{O}_W + \frac{C_B}{\Lambda^2}\mathcal{O}_B + \frac{C_{3W}}{\Lambda^2}\mathcal{O}_{3W} + (CP\text{-odd duals}) + (\text{oblique set } \mathcal{O}_{HWB}, \mathcal{O}_{HD}, \mathcal{O}_{2W}, \mathcal{O}_{2B}). \tag{9.20}$$

Linearized mapping to HPZH. After canonical normalization and rotation to (A_μ, Z_μ) (Sec. 7.4), the *CP-even* TGCs at linear order in v^2/Λ^2 are

$$\Delta g_1^Z = \frac{v^2}{2\Lambda^2}\, C_W, \tag{9.21}$$

$$\Delta\kappa_\gamma = \frac{v^2}{2\Lambda^2}\,(C_W + C_B), \qquad \Delta\kappa_Z = \frac{v^2}{2\Lambda^2}\Big(C_W - \tan^2\theta_W\, C_B\Big), \tag{9.22}$$

$$\lambda_\gamma = \lambda_Z = \frac{3 g^2 v^2}{2\Lambda^2}\, C_{3W}, \tag{9.23}$$

while the *CP-odd* partners read

$$\tilde\lambda_\gamma = \tilde\lambda_Z = \frac{3 g^2 v^2}{2\Lambda^2}\, \tilde C_{3W}, \qquad \tilde\kappa_\gamma = \frac{v^2}{2\Lambda^2}(\tilde C_W + \tilde C_B), \qquad \tilde\kappa_Z = \frac{v^2}{2\Lambda^2}\Big(\tilde C_W - \tan^2\theta_W\, \tilde C_B\Big). \tag{9.24}$$

Oblique operators $(\mathcal{O}_{HWB}, \mathcal{O}_{HD})$ shift two-point functions (Sec. 8.4) and must be included consistently in experimental fits; at linear order they do *not* generate independent TGC structures beyond Eqs. (9.21)–(9.23).

Spectral interpretation. In the spectral model:

$$C_W,\ C_B \propto \text{ derivatives of the neutral/charged current polarizations w.r.t. the Higgs background } H, \tag{9.25}$$

$$C_{3W} \propto \text{ three-current (adjoint) Kubo correlator } \langle J^a J^b J^c \rangle \text{ on the fiber orbits,} \tag{9.26}$$

normalized by the gauge stiffness κ_G and the spectral gap; the precise proportionality constants follow from matching trilinear vertices in the canonical basis.

3. Quartic gauge couplings: dim-6 correlations and dim-8 freedom

Correlated QGC at dim-6. Operators $\mathcal{O}_W, \mathcal{O}_B$ and $\mathcal{O}_{3W}$ also shift QGCs in a way *correlated* with the TGC shifts (9.21)–(9.23). Hence, within dimension six, diboson and triboson data constrain the same parameters.

Independent QGC at dim-8. Genuine QGC deformations without TGC shifts first arise at dimension eight. A useful (CP-even) subset is

$$\mathcal{O}_{S,0} = \Big[(D_\mu H)^\dagger D_\nu H\Big]\Big[(D^\mu H)^\dagger D^\nu H\Big], \qquad \mathcal{O}_{S,1} = \Big[(D_\mu H)^\dagger D^\mu H\Big]\Big[(D_\nu H)^\dagger D^\nu H\Big], \tag{9.27}$$

$$\mathcal{O}_{M,0} = (D_\mu H)^\dagger \sigma^a D_\nu H\, W^{a\,\mu\rho} B^\nu{}_\rho, \quad \mathcal{O}_{M,2} = (D_\mu H)^\dagger \sigma^a D_\nu H\, W^{a\,\mu\rho} W^\nu{}_\rho, \ \ldots \tag{9.28}$$

$$\mathcal{O}_{T,0} = \Big[\mathrm{Tr}\, W_{\mu\nu} W^{\mu\nu}\Big]^2, \qquad \mathcal{O}_{T,1} = \mathrm{Tr}\, W_{\alpha\nu} W^{\mu\beta}\, \mathrm{Tr}\, W_{\mu\beta} W^{\alpha\nu}, \ \ldots \tag{9.29}$$

with coefficients $f_{S,i}, f_{M,i}, f_{T,i}$ (normalized as $\Delta\mathcal{L} = \sum f_i\, \mathcal{O}_i/\Lambda^4$). These operators control longitudinal (S), mixed (M), and transverse (T) VBS channels, respectively, and enter directly $pp \to VVjj$ kinematics at high invariant mass.

4. RG and running to experimental scales

Anomalous dimensions (sketch). At one loop with SM light fields, the running takes the form

$$\mu \frac{d}{d\mu} \begin{pmatrix} C_W \\ C_B \\ C_{3W} \end{pmatrix} = \frac{1}{16\pi^2} \begin{pmatrix} \gamma_{WW} & \gamma_{WB} & 0 \\ \gamma_{BW} & \gamma_{BB} & 0 \\ 0 & 0 & \gamma_{3W} \end{pmatrix} \begin{pmatrix} C_W \\ C_B \\ C_{3W} \end{pmatrix} + \text{ mixing from } \{C_{HWB}, C_{2W}, C_{2B}\}, \tag{9.30}$$

with $\gamma_{3W} \propto b_2 g^2$ and $(\gamma_{WW}, \gamma_{BB}, \gamma_{WB})$ calculable numbers depending on g, g', y_t, λ. The spectral matching provides boundary conditions at μ_* that are then evolved to the diboson/VBS scales.

5. Energy growth and unitarity estimates

Scaling. With anomalous couplings, representative helicity amplitudes scale as

$$\mathcal{M}(VV \to VV) \ \sim \ \begin{cases} \lambda\, \dfrac{E^2}{m_W^2} & \text{(from } \mathcal{O}_{3W}\text{)}, \\ \Delta g_1^Z,\ \Delta\kappa_V & \text{(energy independent at leading power)}, \\ f_{S,M,T}\, \dfrac{E^4}{\Lambda^4} & \text{(from dim-8 QGC)}. \end{cases} \tag{9.31}$$

Partial-wave unitarity, $|a_0| < \frac{1}{2}$, implies an EFT validity scale

$$E_{\max} \;\sim\; \min\left\{ \frac{4\pi\, m_W}{\sqrt{|\lambda|}\, g} \;,\quad \left(\frac{16\pi\, \Lambda^4}{|f_{S,M,T}|} \right)^{1/4} \right\} \tag{9.32}$$

(up to $\mathcal{O}(1)$ factors from angular/projector details). Analyses should impose kinematic cuts $E < E_{\max}$ or include unitarization models when fitting high-mass tails.

6. Positivity/causality bounds (forward VBS)

Dimension-8 positivity. Analyticity and unitarity of forward elastic amplitudes lead to *model-independent* inequalities on QGC Wilsons. For custodially symmetric, CP-even EFTs,

$$f_{S,0} \;\geq 0, \qquad f_{S,0} + f_{S,1} \;\geq 0, \tag{9.33}$$

$$f_{T,0} \;\geq 0, \qquad f_{T,1} + f_{T,2} \;\geq 0, \tag{9.34}$$

$$\text{and mixed-channel bounds of the form } f_{M,i}^2 \;\leq\; \alpha_i\, f_{S,\cdot}\, f_{T,\cdot}, \quad \alpha_i > 0, \tag{9.35}$$

with additional linear inequalities following from projecting definite-helicity states. These constraints must be enforced when sampling parameter space; they coincide with the spectral (TQ) gate applied to gauge two-to-two scattering (Ch. 17).

7. Spectral constitutive origin and χ^μ effects

Constitutive matching. In the spectral framework, trilinear and quartic gauge couplings arise from the expansion of the constitutive tensor $\mathcal{K}^{IJ}{}_{\mu\alpha\nu\beta}$ (Sec. 7.3) in the gauge fields and in the background Higgs profile:

$$\mathcal{K} \;=\; \mathcal{K}^{(0)} \;+\; \frac{1}{\Lambda^2}\,\mathcal{K}^{(2)}[H] \;+\; \frac{1}{\Lambda^4}\,\mathcal{K}^{(4)}[H] \;+\; \cdots . \tag{9.36}$$

The tensors $\mathcal{K}^{(2)}$ generate $\mathcal{O}_W, \mathcal{O}_B$ after integration by parts, while cubic pieces of the non-Abelian field strength produce $\mathcal{O}_{3W}$ with a coefficient fixed by a three-current spectral integral [Eqs. (9.25)–(9.26)]. Mild anisotropies aligned with χ^μ correct helicity projectors but, being CPT-even and gauge-index blind at leading order, do not introduce new independent TGC structures beyond Eqs. (9.21)–(9.23).

8. Observables and fit strategy (diboson & VBS)

Diboson (TGC). The parameters $\{\Delta g_1^Z, \Delta\kappa_\gamma, \lambda\}$ are constrained by $pp \to W^+W^-$, $W^\pm Z$, $W^\pm\gamma$ with polarized differential distributions (e.g. p_T^V, $\Delta\phi_{\ell\ell}$, m_{VV}). Helicity-selection observables isolate λ (transverse–transverse enhancement) vs. $\Delta g_1^Z, \Delta\kappa_\gamma$ (mixed helicities).

VBS (QGC). Same-sign $W^\pm W^\pm jj$, $WZjj$, $ZZjj$ in the VBS topology probe $f_{S,i}, f_{M,i}, f_{T,i}$ at high m_{VV} and large rapidity separation. Analyses must impose $E < E_{\max}$ (Eq. (9.32)) and enforce positivity (9.33)–(9.35).

9. Summary

Anomalous TGCs are fully captured at dimension six by three CP-even parameters $\{\Delta g_1^Z, \Delta\kappa_\gamma, \lambda\}$, mapped cleanly to the gauge-invariant coefficients (C_W, C_B, C_{3W}) via Eqs. (9.21)–(9.23); custodial symmetry enforces the relation (9.18). Independent QGCs first appear at dimension eight through $(\mathcal{O}_S, \mathcal{O}_M, \mathcal{O}_T)$ and are bounded by forward-limit positivity and unitarity. In the spectral model these coefficients have a microscopic Kubo origin and are constrained by the same S/G/TQ/D gates that govern the gauge sector, providing a consistent, predictive interface to diboson and VBS data.

9.3 Rare decays and loop-induced effects

Aim. We organize electroweak *loop-induced* and *rare* processes in a basis that interfaces cleanly with the spectral EFT. Our emphasis is on $h \to \gamma\gamma$, $h \to Z\gamma$ (and $h \to gg$ for completeness), rare Z channels, and diboson radiative tails where one-loop sensitivity to the TGC/QGC sector enters. We provide (i) a gauge-invariant definition of the effective hVV form factors, (ii) the matching to dimension-six Wilson coefficients generated by the spectral constitutive tensors, (iii) the role of anomalous TGCs at one loop, and (iv) RG/threshold rules consistent with the gates of Secs. 3.3, 8.4.

1. Gauge-invariant hVV form factors

Definition. Write the on-shell $h \to \gamma\gamma$ and $h \to Z\gamma$ amplitudes in terms of gauge-invariant scalar form factors:

$$\begin{aligned}\mathcal{L}_{hVV}^{\text{eff}} = {}& \frac{\alpha}{4\pi}\frac{1}{v}\Big[\kappa_{\gamma\gamma}\, h\, F_{\mu\nu}F^{\mu\nu} + \tilde\kappa_{\gamma\gamma}\, h\, F_{\mu\nu}\tilde F^{\mu\nu}\Big] \\ & + \frac{\alpha}{4\pi}\frac{1}{v}\Big[\kappa_{Z\gamma}\, h\, Z_{\mu\nu}F^{\mu\nu} + \tilde\kappa_{Z\gamma}\, h\, Z_{\mu\nu}\tilde F^{\mu\nu}\Big] \;+\; \frac{\alpha_s}{4\pi}\frac{1}{v}\Big[\kappa_{gg}\, h\, G^A_{\mu\nu}G^{A\,\mu\nu} + \tilde\kappa_{gg}\, h\, G^A_{\mu\nu}\tilde G^{A\,\mu\nu}\Big],\end{aligned} \tag{9.37}$$

with $F_{\mu\nu}$, $Z_{\mu\nu}$, $G^A_{\mu\nu}$ the EM, Z, and gluon field strengths, and tildes indicating duals. The SM at one loop produces $\kappa_{\gamma\gamma}^{\text{SM}} = \mathcal{A}_1(\tau_W) + \sum_f N_c^f Q_f^2 \mathcal{A}_{1/2}(\tau_f)$, $\kappa_{Z\gamma}^{\text{SM}}$ in terms of W, t loops, and $\kappa_{gg}^{\text{SM}} = \sum_q \mathcal{A}_{1/2}(\tau_q)$, where

$$\tau_i = \frac{4m_i^2}{m_h^2}, \qquad \mathcal{A}_{1/2}(\tau) = 2\Big[\tau + (\tau-1)f(\tau)\Big]\tau^{-2}, \qquad \mathcal{A}_1(\tau) = -\Big[2\tau^2 + 3\tau + 3(2\tau-1)f(\tau)\Big]\tau^{-2}, \tag{9.38}$$

$$f(\tau) = \begin{cases} \arcsin^2(\tau^{-1/2}), & \tau \ge 1, \\ -\frac{1}{4}\left[\ln\frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}} - i\pi\right]^2, & \tau < 1. \end{cases} \tag{9.39}$$

We normalize observables by $\mu_X \equiv \Gamma(h\to X)/\Gamma(h\to X)_{\text{SM}}$.

2. Dimension-six matching in the universal (electroweak) sector

Operator basis. In the Warsaw/HISZ language the gauge-invariant operators that *directly* feed hVV at tree level are

$$\mathcal{O}_{HB} = (H^\dagger H)B_{\mu\nu}B^{\mu\nu}, \quad \mathcal{O}_{HW} = (H^\dagger H)W^a_{\mu\nu}W^{a\,\mu\nu}, \quad \mathcal{O}_{HWB} = (H^\dagger\sigma^a H)W^a_{\mu\nu}B^{\mu\nu}, \tag{9.40}$$

and their CP-odd duals $\tilde{\mathcal{O}}_{HB/HW/HWB}$. Writing $\Delta\mathcal{L} = \sum C_i\,\mathcal{O}_i/\Lambda^2$, the EWSB rotation gives linear combinations

$$\kappa_{\gamma\gamma}^{(6)} = \mathcal{N}_\gamma\,\frac{v^2}{\Lambda^2}\Big(c_W^2\,C_{HB} + s_W^2\,C_{HW} - s_W c_W\,C_{HWB}\Big), \tag{9.41}$$

$$\kappa_{Z\gamma}^{(6)} = \mathcal{N}_{Z\gamma}\,\frac{v^2}{\Lambda^2}\Big[2 s_W c_W\,(C_{HW} - C_{HB}) + (c_W^2 - s_W^2)\,C_{HWB}\Big], \tag{9.42}$$

$$\kappa_{gg}^{(6)} = \mathcal{N}_g\,\frac{v^2}{\Lambda^2}\,C_{HG}, \qquad \mathcal{O}_{HG} = (H^\dagger H) G^A_{\mu\nu} G^{A\,\mu\nu}, \tag{9.43}$$

with analogous expressions for tilded (CP-odd) coefficients. The real, scheme-dependent normalization factors $\mathcal{N}_\gamma, \mathcal{N}_{Z\gamma}, \mathcal{N}_g$ are chosen once the effective Lagrangian is put in the form (9.37); we keep them explicit to avoid convention clashes.[1] Equations (9.41)–(9.43) provide the *tree-level* EFT contributions on top of the SM loops.

Spectral origin. In the spectral model the C's are Kubo integrals of two-current correlators *in the presence of a Higgs background*, obtained from the k^2 piece of the constitutive tensor expanded to linear order in $H^\dagger H$:

$$C_{HW/HB/HWB} \;\propto\; \frac{1}{\kappa_G}\,\frac{\partial\, Z_1^{IJ}}{\partial (H^\dagger H)}\Big|_{H=v/\sqrt{2}}, \qquad C_{HG} \;\propto\; \frac{1}{\kappa_G^{(c)}}\,\frac{\partial\, Z_1^{AA}}{\partial (H^\dagger H)}\Big|_{H=v/\sqrt{2}}, \tag{9.44}$$

with $I, J \in \{3, Y\}$ and A a color index; $\kappa_G^{(c)}$ denotes the color stiffness (Sec. 10).

3. Anomalous TGCs at one loop

Structure. The TGC deformations of Sec. 9.2 mix into $h \to \gamma\gamma$ and $h \to Z\gamma$ at one loop via W loops with one anomalous vertex:

$$\kappa_{\gamma\gamma}^{\rm TGC} \;=\; \frac{\alpha}{4\pi}\Big[a_\lambda\,\lambda + a_1\,\Delta g_1^Z + a_\kappa\,\Delta\kappa_\gamma\Big], \qquad \kappa_{Z\gamma}^{\rm TGC} \;=\; \frac{\alpha}{4\pi}\Big[\bar{a}_\lambda\,\lambda + \bar{a}_1\,\Delta g_1^Z + \bar{a}_\kappa\,\Delta\kappa_\gamma\Big], \tag{9.45}$$

where $a_i(\tau_W, \theta_W)$, $\bar{a}_i(\tau_W, \theta_W)$ are loop functions (finite, gauge-parameter independent) determined by the same Ward/Nielsen identities used in Ch. 2. Numerically these contributions are $\mathcal{O}\big((\alpha/4\pi) \times \text{TGC}\big)$ and can be competitive with (9.41)–(9.43) when $C_i v^2/\Lambda^2$ is small.

Gauge independence. The sum $\kappa^{(6)} + \kappa^{\rm TGC}$ is gauge-parameter independent provided the input set is consistent (Sec. 9.1) and the BRST structure is respected (Ch. 2).

4. Rates and linearized deviations

Higgs diphoton and $Z\gamma$. To first order in EFT corrections, the signal strength shifts are

$$\mu_{\gamma\gamma} = \frac{\Gamma(h \to \gamma\gamma)}{\Gamma(h \to \gamma\gamma)_{\rm SM}} \simeq 1 + 2\ \mathrm{Re}\left[\frac{\kappa_{\gamma\gamma}^{(6)} + \kappa_{\gamma\gamma}^{\rm TGC}}{\kappa_{\gamma\gamma}^{\rm SM}}\right], \tag{9.46}$$

$$\mu_{Z\gamma} \simeq 1 + 2\ \mathrm{Re}\left[\frac{\kappa_{Z\gamma}^{(6)} + \kappa_{Z\gamma}^{\rm TGC}}{\kappa_{Z\gamma}^{\rm SM}}\right]. \tag{9.47}$$

[1] A common choice is $\mathcal{N}_\gamma = \mathcal{N}_{Z\gamma} = \mathcal{N}_g = 1$ when one defines κ's exactly as the shift of the SM loop amplitude; here we allow for general normalizations.

CP-odd pieces $\tilde{\kappa}$ do not interfere with the SM in total rates but are accessible via photon polarization/azimuthal asymmetries (future lepton colliders, converted-photon techniques).

Higgs digluon. Similarly,

$$\mu_{gg} \equiv \frac{\Gamma(h \to gg)}{\Gamma(h \to gg)_{\rm SM}} \simeq 1 + 2\,\mathrm{Re}\left[\frac{\kappa_{gg}^{(6)}}{\kappa_{gg}^{\rm SM}}\right], \tag{9.48}$$

supplemented by threshold corrections from colored spectral states and by the well-known QCD K-factors.

5. Rare Z channels and radiative tails

$Z \to \ell^+\ell^-\gamma$ and $Z \to \nu\bar{\nu}\gamma$. Off-shell $Z\gamma^*$ and γZ^* transitions probe the same effective $Z\gamma\gamma$ and $ZZ\gamma$ form factors that enter $h \to Z\gamma$ at linear order in the EFT. The dominant linear sensitivity is to C_{HWB} via $\hat{S}$ (Sec. 8.4) and to $C_{W,B}$ through their impact on TGCs (Sec. 9.2), folded with known radiator functions.

$Z \to \gamma\gamma$, $Z \to 3\gamma$. These are forbidden/ultra-suppressed in the SM; gauge-invariant dim-6 does not generate an on-shell $Z\gamma\gamma$ contact term, but dim-8 operators with three field strengths do. Spectral positivity (forward limits) implies nontrivial sign constraints on those coefficients (Sec. 17); any observation at current sensitivities would point to Λ close to the electroweak scale.

6. RG evolution and thresholds

Running. The Wilsons $\{C_{HB}, C_{HW}, C_{HWB}, C_{HG}\}$ mix under SM loops:

$$\mu\frac{d}{d\mu}\begin{pmatrix} C_{HB} \\ C_{HW} \\ C_{HWB} \\ C_{HG} \end{pmatrix} = \frac{1}{16\pi^2}\,\boldsymbol{\gamma}\begin{pmatrix} C_{HB} \\ C_{HW} \\ C_{HWB} \\ C_{HG} \end{pmatrix} + \text{ inhomogeneous terms from } \{C_W, C_B, C_{3W}\}, \tag{9.49}$$

where $\boldsymbol{\gamma}$ depends on $(g, g', g_s, y_t, \lambda)$ and is known in the SMEFT literature. Boundary conditions are set at the geometric matching scale μ_* (Sec. 7.4) by (9.44) and evolved to $\mu \sim m_h, m_Z$.

Threshold decoupling. Heavy spectral states with mass $M \gg m_Z$ produce finite threshold shifts $\Delta C_i(\mu < M)$ consistent with Appelquist–Carazzone; chiral thresholds also leave WZW/inflow remnants respected by anomaly matching (Part II).

7. Consistency gates and gauge/BRST checks

Positivity and hyperbolicity. Forward-limit dispersion enforces positivity on the hVV forward amplitudes built from (9.37), implying (schematically) nonnegative combinations of $\{C_{HB}, C_{HW}, C_{HWB}\}$ once expressed in the $(\hat{S}, \hat{T}, W, Y)$ basis; higher-derivative terms are bounded as in Sec. 8.4. Anisotropic χ^μ deformations alter propagation but not the zero-momentum matching that defines κ's.

Ward/Nielsen identities. Gauge-parameter independence of $\kappa_{\gamma\gamma}, \kappa_{Z\gamma}$ follows from the Nielsen identity applied to $V_{\rm eff}$ and from the BRST structure of Sec. 2, provided inputs are chosen consistently (e.g. $\{\alpha, m_Z, G_F\}$).

8. Practical recipe for phenomenology

Algorithm.

1. Compute the spectral constitutive response $\partial Z_1^{IJ}/\partial(H^\dagger H)$ to obtain $\{C_{HB}, C_{HW}, C_{HWB}\}$ and, if needed, C_{HG} at μ_*.
2. Evolve to $\mu \sim m_h$ with (9.49); include heavy thresholds.
3. If anomalous TGCs are present, determine $\{\Delta g_1^Z, \Delta\kappa_\gamma, \lambda\}$ at μ_* (Sec. 9.2) and add the one-loop contributions (9.45).
4. Form κ's via (9.41)–(9.43) and predict $\mu_{\gamma\gamma}, \mu_{Z\gamma}, \mu_{gg}$ with (9.46)–(9.48).
5. Enforce positivity/hyperbolicity gates and custodial constraints; confront with LHC measurements.

9. Summary

Loop-induced electroweak decays provide a clean, gauge-invariant window on the spectral EFT. Tree-level dimension-six operators $(\mathcal{O}_{HB/HW/HWB}, \mathcal{O}_{HG})$ shift $h \to \gamma\gamma$, $h \to Z\gamma$, and $h \to gg$ via (9.41)–(9.43), while anomalous TGCs feed these channels at one loop as in (9.45). All coefficients descend from Kubo correlators of the spectral medium, evolve under controlled RG, and obey positivity, hyperbolicity, and BRST/Nielsen consistency conditions. The resulting framework enables a tightly constrained, data-driven bridge from fiber geometry to precision Higgs and Z observables.

9.4 Global fits and kill criteria

Aim. We formulate a statistically rigorous framework to confront the spectral electroweak sector with precision data and to define *falsifiable* ("kill") criteria. The fit consistently incorporates: (i) the geometric inputs (g, g') and v (Secs. 7.4, 8.2); (ii) universal oblique deformations $(\hat{S}, \hat{T}, W, Y)$ (Sec. 8.4); (iii) anomalous TGC/QGC parameters mapped to SMEFT Wilsons (Sec. 9.2); (iv) loop-induced Higgs form factors and self-couplings (Sec. 9.3); (v) RG running with thresholds (Ch. 3); and (vi) the spectral S/G/TQ/D gates as *priors/constraints*. We then state quantitative, model-level kill criteria.

1. Parameter layers and prediction map

Parameter vector. At a geometric matching scale μ_* we collect the *spectral* parameters

$$\boldsymbol{\theta}_{\rm spec}(\mu_*) = \Big(g,\ g',\ v;\ \hat{S},\ \hat{T},\ W,\ Y;\ C_W,\ C_B,\ C_{3W};\ f_{S,i}, f_{M,i}, f_{T,i};\ c_6,\ c_H;\ \dots\Big)_{\mu_*}, \tag{9.50}$$

with the understanding that $(\hat{S}, \hat{T}, W, Y)$, C's and f's are calculable from the fiber constitutive tensors and Kubo correlators (Secs. 7.3, 9.2, 9.3).

RG and thresholds. Predictions at an experimental scale μ follow from

$$\boldsymbol{\theta}(\mu) \;=\; \mathcal{R}(\mu,\mu_*)\;\circ\;\mathcal{M}_{\text{spec}\to\text{EFT}}\,\Big[\boldsymbol{\theta}_{\text{spec}}(\mu_*)\Big], \tag{9.51}$$

where $\mathcal{M}_{\text{spec}\to\text{EFT}}$ implements the geometric/constitutive matching (e.g. $Z_1^{3Y}\to\hat{S}$) and $\mathcal{R}$ is the RG flow with decoupling of heavy thresholds (Ch. 3). The observable vector $\mathbf{O}_{\text{th}}$ (LEP/SLD pseudo-observables, m_W, diboson/VBS, Higgs rates, ...) is then a function $\mathbf{O}_{\text{th}} = \mathcal{P}\Big(\boldsymbol{\theta}(\mu)\Big)$; compact linearized forms were given in Secs. 9.1–9.3.

2. Likelihood, nuisances, and spectral gates

Gaussian core. Let $\mathbf{d}$ denote the measured observables and Σ their covariance. For a set of nuisance parameters $\boldsymbol{\eta}$ (PDFs, α_s, m_t, hadronic running $\Delta\alpha_{\text{had}}$, ...) with priors $\boldsymbol{\eta}\sim\mathcal{N}(\bar{\boldsymbol{\eta}},\Lambda_\eta)$, we define

$$-2\ln\mathcal{L}_{\text{data}}(\boldsymbol{\theta},\boldsymbol{\eta}) = \Big[\mathbf{d}-\mathbf{O}_{\text{th}}(\boldsymbol{\theta},\boldsymbol{\eta})\Big]^{\top}\Sigma^{-1}\Big[\mathbf{d}-\mathbf{O}_{\text{th}}(\boldsymbol{\theta},\boldsymbol{\eta})\Big] + (\boldsymbol{\eta}-\bar{\boldsymbol{\eta}})^{\top}\Lambda_\eta^{-1}(\boldsymbol{\eta}-\bar{\boldsymbol{\eta}}). \tag{9.52}$$

EFT truncation and validity. Dimension-D truncation induces a correlated theory uncertainty Σ_{EFT} estimated as

$$\Big(\Sigma_{\text{EFT}}\Big)_{ij} \;\sim\; \sum_k \left|\frac{E_i}{\Lambda}\right|^{D_k-4}\left|\frac{E_j}{\Lambda}\right|^{D_k-4}\rho_{ij}^{(k)}, \qquad E_i:\ \text{typical energy of observable } i. \tag{9.53}$$

We use $\Sigma\to\Sigma+\Sigma_{\text{EFT}}$ and enforce $E<E_{\max}(\boldsymbol{\theta})$ from partial-wave unitarity (Eq. (9.32)) as a hard cut.

Spectral gates as priors/constraints. We impose

Positivity/causality (TQ): $W\ge 0,\ Y\ge 0,\ f_{S,0}\ge 0,\ f_{S,0}+f_{S,1}\ge 0,\ f_{T,0}\ge 0,\ \ldots$ (9.54)

Custodial (approx.): $\hat{T}=0$ (tree) $\Rightarrow$ $\hat{T}\sim\mathcal{N}(0,\sigma_T^2)$ with σ_T set by loop/threshold expectation (9.55)

TGC custodial relation: $\Delta\kappa_Z-\Delta g_1^Z+\tan^2\theta_W\,\Delta\kappa_\gamma=0$ at dim-6, Eq. (9.18), (9.56)

Higgs self-coupling correlation: $\delta_4-6\delta_3=0$ if only c_6 active (Sec. 8.3); (9.57)

Hyperbolicity (G): principal symbols within g_{eff} cone $\Rightarrow$ upper bounds on $|Z_1^{3Y}|$, W, Y. (9.58)

The equalities can be implemented as narrow Gaussians or as hard linear constraints in the sampler.

3. Linear–Gaussian solution and Fisher geometry

Linearization. Expanding $\mathbf{O}_{\text{th}}\approx\mathbf{O}_0+J\,\delta\boldsymbol{\theta}$ around a reference point, the profile-likelihood estimator and covariance for $\delta\boldsymbol{\theta}$ are

$$\widehat{\delta\boldsymbol{\theta}} = (J^{\top}\Sigma^{-1}J)^{-1}J^{\top}\Sigma^{-1}(\mathbf{d}-\mathbf{O}_0), \qquad \mathcal{C}_\theta = (J^{\top}\Sigma^{-1}J)^{-1}, \tag{9.59}$$

with $J_{i\alpha}=\partial O_i/\partial\theta_\alpha$. Eqs. (9.4), (9.5) and the TGC/QGC mappings (9.21)–(9.23) furnish J analytically for the universal fit.

Model comparison. For nested hypotheses $H_0 \subset H_1$ (e.g. spectral relations on Wilsons), Wilks' theorem gives $\Delta\chi^2 \simeq \chi^2_k$ with k the number of constrained directions. Bayesian evidence $\mathcal{Z}$ is approximated by Laplace's method,

$$\ln \mathcal{Z} \;\simeq\; \ln \mathcal{L}_{\max} - \tfrac{1}{2} \ln \det \Big(2\pi\, \mathcal{C}_\theta\Big) + \ln \pi(\boldsymbol{\theta}_{\max}), \tag{9.60}$$

to form Bayes factors against agnostic EFT baselines.

4. Data sets and observable blocks

Core blocks. (i) Z-pole pseudo-observables and m_W (Sec. 9.1); (ii) diboson TGC-sensitive spectra (W^+W^-, WZ, $W\gamma$) (Sec. 9.2); (iii) VBS ($W^\pm W^\pm jj$, $WZjj$, $ZZjj$) for QGC (Sec. 9.2); (iv) Higgs loop-induced decays ($h \to \gamma\gamma, Z\gamma, gg$) (Sec. 9.3); (v) optional low-energy inputs (Møller, APV) that map to $(\hat{S}, \hat{T}, W, Y)$ at $q^2 \to 0$.

5. Quantitative kill criteria (spectral electroweak)

We define "kill" as the failure of the *minimal* spectral electroweak hypothesis to describe data under gates and EFT validity. Let χ^2_{spec} be the profiled minimum with constraints (9.54)–(9.58) enforced.

1. **Oblique inconsistency (K1).** The spectral geometry predicts at tree level $\hat{T} = 0$ and $W, Y \geq 0$ with $|\hat{S}| = |g/g'| \cdot |Z_1^{3Y}|$ controlled by fiber non-orthogonality. *Kill* if any holds at $> 3\sigma$:

$$\hat{T} \neq 0 \;\; \text{(beyond loop/threshold prior)}, \qquad W < 0 \text{ or } Y < 0, \qquad |\hat{S}| > \hat{S}^{\max}_{\text{geom}}, \tag{9.61}$$

   where $\hat{S}^{\max}_{\text{geom}}$ follows from the hyperbolicity/orthogonality bounds of Secs. 7.4, 8.4.

2. **TGC custodial violation (K2).** At dim-6, the relation (9.18) must hold. *Kill* if the fitted combination

$$\Delta_{\text{cust}} \;:=\; \Delta\kappa_Z - \Delta g_1^Z + \tan^2\theta_W\, \Delta\kappa_\gamma \tag{9.62}$$

   differs from zero by $> 3\sigma$ after profiling oblique effects.

3. **Higgs self-coupling correlation (K3).** If only the sextic $(H^\dagger H)^3$ is active at leading power, Eq. (9.57) applies. *Kill* that minimal scalar completion if

$$\Delta_{h,\,\text{corr}} \;:=\; \delta_4 - 6\,\delta_3 \;\neq\; 0 \quad \text{at} \;\; > 3\sigma, \tag{9.63}$$

   or if the sign/size violates forward-positivity (Sec. 8.3).

4. **EFT breakdown (K4).** *Kill* if the preferred region requires

$$E_{\text{ana}} \;\geq\; E_{\max}(\boldsymbol{\theta}) \quad \text{or} \quad \operatorname{Tr}\Big[\Sigma_{\text{EFT}}\, \Sigma^{-1}\Big] \;\gtrsim\; 1, \tag{9.64}$$

   i.e. the analysis energy enters the non-perturbative/invalid EFT regime.

5. **RG inconsistency (K5).** Using (9.51), *kill* if no $(\mu_*, \boldsymbol{\theta}_{\text{spec}}(\mu_*))$ exists such that the same fiber geometry fits simultaneously LEP/SLD ($\mu \sim m_Z$) and LHC diboson/VBS scales within thresholds and gates.

6. **Global fit failure (K6).** *Kill* if the profiled goodness-of-fit is poor:

$$p\text{-value} = P\Big(\chi^2_{n_{\rm dof}} > \chi^2_{\rm spec}\Big) \;<\; 10^{-3} \quad \text{or} \quad \ln B \equiv \ln(\mathcal{Z}_{\rm spec}/\mathcal{Z}_{\rm agnostic}) < -5, \tag{9.65}$$

where the baseline "agnostic" EFT frees the constrained spectral relations.

7. **Positivity/hyperbolicity violation (K7).** *Kill* if the best-fit Wilsons lie outside the convex positivity cone (9.54) or demand superluminal principal symbols, contradicting the (TQ)/(G) gates.

8. **Geometric Weinberg-angle misfit (K8).** If θ_W is *predicted* at μ_* by the vertical metric ratio (Sec. 7.4), *kill* that geometric ansatz if the RG-evolved value disagrees with the effective angle from data ($s^2_{\rm eff}$) at $> 3\sigma$ after marginalizing $(\hat{S}, \hat{T})$.

6. Forecasts and Fisher projections

Fisher forecasts. For a future covariance $\Sigma_{\rm fut}$ and Jacobian $J_{\rm fut}$ (including new lepton-collider inputs on $s^2_{\rm eff}$, Γ_Z, TGC polarimetry, and VBS with polarization), the forecasted errors and discovery reach follow from Eq. (9.59). Convex positivity/causality constraints shrink the credible region and should be enforced at the sampling level (projected-gradient or hit-and-run within the cone).

7. Practical workflow

1. **Assemble d** and Σ (block-diagonal by channel with published correlations).
2. **Match** spectral tensors $\to (\hat{S}, \hat{T}, W, Y)$, (C_W, C_B, C_{3W}), $(C_{HB/HW/HWB})$, c_6, c_H at μ_*.
3. **Run** to analysis scales with thresholds (Ch. 3); impose $E < E_{\rm max}$.
4. **Predict** $\mathbf{O}_{\rm th}$ using the formulas of Secs. 9.1–9.3.
5. **Fit** by profiling nuisances and enforcing gates (9.54)–(9.58).
6. **Diagnose** kill criteria (K1)–(K8); if not killed, report allowed spectral parameter volumes and correlations.

8. Summary

We have provided a coherent global-fit machinery that preserves gauge/BRST consistency, RG matching, and the spectral S/G/TQ/D gates. The kill criteria (K1)–(K8) render the framework falsifiable: they target (i) oblique geometry, (ii) custodial/TGC correlations, (iii) Higgs self-coupling structure, (iv) EFT/positivity validity, (v) RG coherence, and (vi) overall goodness of fit. Together they define a sharp, quantitative boundary between *admissible* spectral electroweak realizations and those decisively excluded by precision data.

Part IV

Strong Sector: Confinement and Chiral Breaking

Chapter 10

Emergent QCD and Color Dynamics

10.1 Color $SU(3)$ from fiber sub-isometries

Aim. We derive QCD color as an *emergent* local symmetry from sub-isometries of the internal spectral fiber $(\mathcal{F}, h)$ introduced in Part I. The eight gluons arise as the Ehresmann connection associated with an $SU(3)$-action by fiber isometries; their coupling g_s is fixed by the vertical (orbit) metric, and their interactions follow from the same horizontality principle and Kubo response used for the electroweak sector (Secs. 1.2–1.4, 7.1). We also define color matter bundles and the role of the Z_3 center, setting the stage for confinement and chiral breaking (Secs. 10.2–10.4 and Part IV).

1. Sub-isometry decomposition and color algebra

Isometry content. Assume the fiber admits a subgroup of isometries

$$\mathrm{Iso}(\mathcal{F}, h) \;\supset\; SU(3)_c \times G_{\mathrm{ew}} \times \cdots , \tag{10.1}$$

where $SU(3)_c$ is the *color* factor and G_{ew} contains the electroweak isometries discussed in Part III. Let $\{\xi_a\}_{a=1}^{8}$ be the eight Killing vectors generating the $SU(3)_c$-action,

$$\mathcal{L}_{\xi_a} h = 0, \qquad [\xi_a, \xi_b] = f^{c}{}_{ab}\, \xi_c, \tag{10.2}$$

with $f^{c}{}_{ab}$ the $su(3)$ structure constants (we use the Gell-Mann basis with $\mathrm{Tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$).

Orbit geometry. At a point $y \in \mathcal{F}$ the $SU(3)_c$-orbit $\mathcal{O}_y = \{g\cdot y \,|\, g \in SU(3)_c\}$ carries the *vertical* metric

$$\kappa^{(c)}_{ab}(y) \;:=\; h_{AB}(y)\, \xi^A_a(y)\, \xi^B_b(y). \tag{10.3}$$

We will use the orbit-averaged metric $\overline{\kappa}^{(c)}_{ab} := \langle \kappa^{(c)}_{ab} \rangle_{\mathcal{O}}$ (any $SU(3)$-invariant averaging) and, when convenient, choose the t^a basis so that $\overline{\kappa}^{(c)}_{ab} = \kappa_c\, \delta_{ab}$.

2. Color principal bundle and Ehresmann connection

Bundle and section. The data $(M, g_{\mathrm{eff}}; \mathcal{F}, h; SU(3)_c)$ define a principal $SU(3)_c$-bundle $P_c \to M$ and the associated bundle $E_c = P_c \times_{SU(3)} \mathcal{F} \to M$. A field configuration is a section $y : M \to E_c$ as in Eq. (**??**).

Horizontality principle. As in Sec. 1.2, we split $T\mathcal{F} = \mathcal{H} \oplus \mathcal{V}$ by demanding that *physical* fiber gradients of $y(x)$ be horizontal: $h\big(\mathcal{D}_\mu y, \xi_a\big) = 0$. This fixes the *pure-gauge* Ehresmann connection on P_c:

$$\boxed{\mathcal{A}^a_{\mu,\,\mathrm{pg}}(x) \;=\; \omega^a{}_A(y)\,\partial_\mu y^A, \qquad \omega^a{}_A \text{ satisfies } \; h_{AB}\,\xi^B_b\,\omega^a{}_A = \kappa^{(c)}_{ba}\;.} \tag{10.4}$$

Local color transformations $g(x) \in SU(3)_c$ act as $y \mapsto g\cdot y$ and

$$\mathcal{A}_\mu \;\mapsto\; \mathcal{A}'_\mu = g\,\mathcal{A}_\mu\,g^{-1} - (\partial_\mu g)g^{-1}. \tag{10.5}$$

Dynamical gluons. The physical gauge potential is the sum of the pure-gauge piece and a dynamical adjoint field,

$$\mathcal{A}^a_\mu \;=\; \mathcal{A}^a_{\mu,\,\mathrm{pg}} \;+\; a^a_\mu, \qquad F^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + f^a{}_{bc}\,\mathcal{A}^b_\mu \mathcal{A}^c_\nu. \tag{10.6}$$

The curvature of $\mathcal{A}_{\mu,\,\mathrm{pg}}$ vanishes identically ($F_{\mu\nu}[\mathcal{A}_{\mathrm{pg}}] = 0$); all dynamics reside in a_μ.

3. Yang–Mills kinetic term and coupling normalization

Constitutive form. Integrating out fast fiber fluctuations (or via the current–current Kubo kernel) generates the two-derivative gauge quadratic form

$$\mathcal{L}^{(2)}_{\mathrm{gluon}} \;=\; -\frac{1}{4}\,Z^{ab}_{1,c}\,F^a_{\mu\nu}F^{b\,\mu\nu}, \qquad Z^{ab}_{1,c} \;=\; \kappa^{(c)}_G\,\overline{\kappa}^{(c)}_{ab}, \tag{10.7}$$

with a positive stiffness $\kappa^{(c)}_G > 0$ determined by the spectral medium. In the basis $\overline{\kappa}^{(c)}_{ab} = \kappa_c\,\delta_{ab}$ one has $Z^{ab}_{1,c} = \kappa^{(c)}_G\kappa_c\,\delta^{ab}$ and we define the strong coupling by

$$\boxed{\frac{1}{g_s^2} \;:=\; Z_{1,c} \;\equiv\; \kappa^{(c)}_G\,\kappa_c} \tag{10.8}$$

so that, after the rescaling $\mathcal{A}^a_\mu = g_s\,G^a_\mu$, the Lagrangian becomes the canonical $-\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu}$ with $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^a{}_{bc}G^b_\mu G^c_\nu$.

Block-diagonality. Orthogonality of orbits implies $Z^{ab}_{1,c}$ has no mixing with electroweak entries at leading order; any small Z^{3Y}-like cross term between color and electroweak indices vanishes by gauge-index parity and fiber factorization (cf. Sec. 8.4).

4. Color matter bundles and covariant couplings

Associated bundles. Let $\rho_{\mathbf{3}} : SU(3) \to GL(\mathbb{C}^3)$ be the fundamental representation. Quark-like fields are sections of

$$\mathcal{V}_{\mathbf{3}} \;:=\; P_c \times_{\rho_{\mathbf{3}}} \mathbb{C}^3 \;\longrightarrow\; M, \tag{10.9}$$

while anti-quarks are sections of $\mathcal{V}_{\bar{\mathbf{3}}}$. Their covariant derivatives read

$$D_\mu\psi \;=\; \partial_\mu\psi - ig_s\,G^a_\mu\,t^a\,\psi, \qquad D_\mu\bar\psi \;=\; \partial_\mu\bar\psi + ig_s\,\bar\psi\,G^a_\mu t^a. \tag{10.10}$$

In the spectral model, ψ are composite excitations built from fiber textures and the spectral phase Φ (Vols. I–III); Eq. (10.10) is enforced by the local $SU(3)_c$ isometry.

Color currents. Variation of the action yields the adjoint color current sourced by matter and the fiber map y:

$$J_a^\mu \;=\; \bar\psi\gamma^\mu t_a\psi \;+\; \kappa_y\, h_{AB}\,\xi_a^A(y)\,\mathcal{D}^\mu y^B, \qquad D_\nu G_a^{\nu\mu} \;=\; g_s\, J_a^\mu. \tag{10.11}$$

(The second term parallels Eq. (**??**); it vanishes if y sits at a color-invariant locus.)

5. Center symmetry, triality, and topological sectors

Center and representations. The center $Z_3 \subset SU(3)_c$ acts trivially in the adjoint and by phases $\exp(2\pi ik/3)$ on **3**, defining *triality*. In the absence of fundamentals, the theory enjoys a 1-form Z_3 symmetry; fundamental matter explicitly breaks it. These structures are inherited from the fiber action and will control the confining string sectors in Sec. 10.2.

Instantons and bundles. Topological sectors are classified by the second Chern number

$$Q \;=\; \frac{1}{32\pi^2}\int_M \mathrm{Tr}\big(G\wedge G\big) \;\in\; \mathbb{Z}, \qquad G := G^a t^a, \tag{10.12}$$

which count fiber-winding of the $SU(3)_c$ bundle. Their spectral imprint enters the θ-term and chiral anomalies (Part II); dynamical consequences (topological susceptibility, η') are discussed in Part IV.

6. Examples of color-bearing fibers

Homogeneous color orbits. Two instructive realizations:

1. *Free action:* an 8D submanifold $\mathfrak{C} \subset \mathcal{F}$ on which $SU(3)_c$ acts freely; then $\mathcal{O}_y \simeq SU(3)$ and $\overline{\kappa}_{ab}^{(c)} \propto \mathrm{Tr}(t_a t_b)$.
2. *Coadjoint orbit:* $\mathcal{O}_y \simeq \mathcal{O}_\lambda = SU(3)/U(1)^2$ (flag manifold) of real dimension 6; the full set of eight ξ_a still closes the algebra and defines the same $Z_{1,c}$ through the projector to the orbit tangent bundle.

In both cases, Eq. (10.8) determines g_s from the orbit metric.

7. Consistency gates (color sector)

(S) stability and (G) hyperbolicity. $Z_{1,c} > 0$ ensures a positive-definite gluon quadratic form; the principal symbol of gluon waves must lie within the g_{eff} light cone (no superluminal constitutive entries).

(TQ) positivity/unitarity. Forward-limit positivity of $VV \to VV$ amplitudes in the color sector bounds higher-derivative operators (to be analyzed in Sec. 10.2); asymptotic freedom will follow from the sign of the one-loop β-function (Sec. 10.3).

(D) dissipation gate. Spectral dissipation cannot generate a gluon mass nor violate BRST; any medium-induced friction appears only through gauge-invariant higher-derivative terms compatible with unitarity.

8. Summary

Color emerges as a *local* redundancy from $SU(3)$ sub-isometries of the fiber: the Ehresmann connection (10.4) produces the gluon potential, its dynamics are governed by the Yang–Mills term (10.7), and the coupling is geometrically fixed by (10.8). Quark fields are sections of the fundamental associated bundle (10.9) and couple via (10.10). Center symmetry and topological charges (10.12) prepare the ground for the *spectral string tension* (Sec. 10.2), running α_s and hadronic scales (Sec. 10.3), and collider imprints (Sec. 10.4).

10.2 Gluon sector and spectral string tension

Aim. We develop the nonperturbative infrared (IR) structure of the color sector in the spectral framework, derive confinement diagnostics (Wilson and 't Hooft loops), and obtain a *spectral* expression for the (fundamental) string tension σ_f in terms of gauge–invariant field–strength correlators and dual (Kalb–Ramond) response. We also state representation dependence (Casimir scaling at intermediate distances vs. N-ality at asymptotia), center-symmetry implications, finite-temperature behavior, and positivity/hyperbolicity gates for confining dynamics.

1. Wilson loop, static potential, and area law

Definitions. For a closed contour C and representation R of $SU(3)_c$,

$$W_R(C) \;:=\; \frac{1}{d_R}\Big\langle \mathrm{Tr}_R\,\mathcal{P}\exp\Big(ig_s\oint_C G_\mu\,dx^\mu\Big)\Big\rangle, \qquad G_\mu := G_\mu^a t^a, \tag{10.13}$$

with $d_R = \dim R$. For a rectangular loop $C_{L\times T}$ of spatial extent L and temporal extent T,

$$V_R(L) \;=\; -\lim_{T\to\infty}\frac{1}{T}\,\ln W_R(C_{L\times T}). \tag{10.14}$$

Confinement in representation R is signaled by an *area law*

$$W_R(C) \;\simeq\; \exp\Big(-\sigma_R\,\mathcal{A}[C] - \mu_R\,\mathcal{P}[C] + \cdots\Big), \qquad \Rightarrow \qquad V_R(L) \;\simeq\; \sigma_R L - \frac{\pi}{12\,L} + \cdots, \tag{10.15}$$

where $\mathcal{A}[C]$ is the minimal area, $\mathcal{P}[C]$ the perimeter. The $-\pi/(12L)$ term is the universal Lüscher correction from string fluctuations.

2. Gauge–invariant field–strength correlators and Kubo form

Correlator with Schwinger line. Let $F^a_{\mu\nu}$ be the color field strength and $\mathcal{U}(x,0)$ a straight Schwinger line in the adjoint. Define the gauge–invariant connected correlator

$$\mathcal{G}^{ab}_{\mu\nu,\rho\sigma}(x) \;:=\; \Big\langle \mathrm{Tr}\Big[t^a\,F_{\mu\nu}(x)\,\mathcal{U}(x,0)\,t^b\,F_{\rho\sigma}(0)\,\mathcal{U}(0,x)\Big]\Big\rangle_c. \tag{10.16}$$

Rotational/isotropy (with respect to $g_{\rm eff}$) implies the decomposition

$$\mathcal{G}^{ab}_{0i,0j}(x) \;=\; \delta^{ab}\Big[\delta_{ij}\,\mathcal{D}_E(x^2) + \partial_i\partial_j\,\mathcal{D}_{E,1}(x^2)\Big], \qquad \mathcal{G}^{ab}_{ij,kl}(x) \;=\; \delta^{ab}\Big[(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\,\mathcal{D}_B(x^2)+\cdots\Big], \tag{10.17}$$

with electric/magnetic scalar form factors $\mathcal{D}_E, \mathcal{D}_{E,1}, \mathcal{D}_B$.

Spectral string tension (Kubo expression). Using the non-Abelian Stokes theorem and a cumulant expansion truncated at the Gaussian level (justified by cluster decomposition in the confining phase), the area term of $\ln W_R$ is

$$\sigma_R \;=\; C_R\,\sigma_{\rm spec} \;+\; \mathcal{O}\Big(\frac{1}{L}\Big), \qquad \boxed{\sigma_{\rm spec} \;:=\; \frac{g_s^2}{2(N_c^2-1)}\int_{\mathbb{R}^2} d^2\zeta\;\mathcal{D}_E(\zeta^2)} \tag{10.18}$$

where ζ spans the *minimal* surface, C_R is the quadratic Casimir, and $N_c = 3$. In momentum space (with $\mathbf{k}_\perp$ transverse to the surface),

$$\sigma_{\rm spec} \;=\; \frac{g_s^2}{4\pi}\int_0^\infty \frac{dk_\perp}{2\pi}\,\widetilde{\mathcal{D}}_E(k_\perp^2), \qquad \widetilde{\mathcal{D}}_E(k_\perp^2) := \int d^2\zeta\; e^{i\mathbf{k}_\perp\cdot\boldsymbol{\zeta}}\,\mathcal{D}_E(\zeta^2). \tag{10.19}$$

Equations (10.18)–(10.19) *define* the **spectral string tension** in terms of a gauge–invariant correlator. Positivity of the spectral density implies $\sigma_{\rm spec} > 0$ in the confining phase (see Sec. 10.2 7).

Parametric estimate. If $\mathcal{D}_E(\zeta^2) \sim \mathcal{D}_0 \exp(-|\zeta|/\xi_c)$ with correlation length ξ_c (set by the lowest 0^{++} glueball), then

$$\sigma_{\rm spec} \;\simeq\; g_s^2\,\mathcal{D}_0\,\xi_c^2, \qquad \sigma_R \;\simeq\; C_R\,g_s^2\,\mathcal{D}_0\,\xi_c^2, \tag{10.20}$$

displaying the geometric control by the gauge stiffness ($g_s^{-2} = Z_{1,c}$) and the IR correlation length.

3. Dual 2-form (Kalb–Ramond) representation and disorder operator

Hubbard–Stratonovich dualization. Introduce an adjoint 2-form $B^a_{\mu\nu}$ to linearize the F^2 term and integrate out G_μ at long distances. The leading IR dual action takes the form

$$\mathcal{L}_{\rm dual} \;=\; \frac{1}{12}\,H^a_{\mu\nu\rho}H^{a\,\mu\nu\rho} \;+\; \frac{m_B^2}{4}\,B^a_{\mu\nu}B^{a\,\mu\nu} \;+\; \cdots, \qquad H^a_{\mu\nu\rho} = \partial_{[\mu}B^a_{\nu\rho]}, \tag{10.21}$$

where m_B encodes the *magnetic* (disorder) screening generated by condensed topological defects (monopoles/vortices) of the spectral fiber (Sec. 12). The Wilson loop couples to B through the worldsheet $\Sigma[C]$,

$$\mathcal{L}_\Sigma \;=\; i\,\frac{g_s}{2}\,J_\Sigma^{a\,\mu\nu}\,B^a_{\mu\nu}, \qquad J_\Sigma^{a\,\mu\nu}(x) \propto t_R^a\int_{\Sigma[C]} d^2\sigma^{\mu\nu}\,\delta^{(4)}(x - X(\sigma)). \tag{10.22}$$

Integrating out B yields a Nambu–Goto area term with

$$\boxed{\sigma_R \;=\; C_R\,\frac{g_s^2}{8\pi}\,m_B^2 \;+\; \cdots} \tag{10.23}$$

(up to scheme-dependent $\mathcal{O}(1)$ factors from the UV completion of the dualization). Equations (10.18) and (10.23) are two faces of the same physics: m_B^{-1} coincides with the correlation length ξ_c that controls $\mathcal{D}_E$.

't Hooft loop and center 1-form symmetry. The disorder operator $\widetilde{W}_k(C)$ creating Z_3 magnetic flux measures the 1-form center symmetry. *Confinement* (area law for W_R) corresponds to an unbroken *electric* 1-form Z_3 symmetry and a *perimeter* law for $\widetilde{W}_k$; deconfinement interchanges the behaviors.

4. Representation dependence: Casimir scaling and N-ality

Intermediate distances. For L larger than the correlation length but below string breaking (or gluon screening) scales,

$$\sigma_R \;\approx\; \frac{C_R}{C_F}\,\sigma_f, \tag{10.24}$$

(*Casimir scaling*), a direct consequence of the adjoint-index structure in (10.18).

Asymptotia and N-ality. At very large L, representations with the same N-ality $k \in \{0,1,2\}$ in $SU(3)$ share the same asymptotic string tension; adjoint sources ($k=0$) are screened, while $k=1,2$ strings are stable in the pure gauge theory. With dynamical fundamentals, the $k=1$ (and $k=2$) string breaks at $L \gtrsim L_{\rm br}$ via $q\bar{q}$ pair creation, leading to $V(L) \to$ const.

5. Finite temperature and Polyakov loop

Order parameter. Let $P(\mathbf{x}) = \mathcal{P}\exp\!\Big(ig_s\!\int_0^{1/T} G_0(\tau,\mathbf{x})\,d\tau\Big)$ be the Polyakov loop. In the pure gauge theory

$$\langle \mathrm{Tr}\,P\rangle = 0 \;\; \text{(center-symmetric, confined)}, \qquad \langle \mathrm{Tr}\,P\rangle \neq 0 \;\; \text{(center-broken, deconfined)}. \tag{10.25}$$

The spatial string tension $\sigma_s(T)$ is extracted from spacelike Wilson loops and, in the dual description, relates to the temperature dependence of $m_B(T)$; phenomenologically $\sigma_f(T)\downarrow$ and vanishes at $T=T_c$ as center symmetry breaks.

6. Spectral origin and parameter control

Constitutive control of $\mathcal{D}_E$. In the spectral medium the gauge–invariant correlator (10.17) arises from the *nonlocal* two-current response of the fiber:

$$\mathcal{D}_E(\zeta^2) \;\sim\; \frac{1}{\kappa_G^{(c)}}\Big\langle J^a_{0i}(x)\,J^a_{0i}(0)\Big\rangle_c, \qquad J^a_{0i} \equiv \frac{\delta\mathcal{L}}{\delta G^{a\,0i}} \;\propto\; \xi_a\cdot\mathcal{D}_0 y \;+\; \bar{\psi}\gamma_0 t^a\psi \;+\; \cdots, \tag{10.26}$$

so that $g_s^{-2} = Z_{1,c} = \kappa_G^{(c)}\kappa_c$ (Sec. 10.1) and the IR falloff scale of $\mathcal{D}_E$ (hence m_B) are fixed by the spectrum of color–singlet excitations (glueballs/flux–tube thickness). Mild anisotropies tied to χ^μ may split electric vs. magnetic correlators but do not change the area law so long as hyperbolicity holds (Sec. 10.2 7).

From $\sigma_{\rm spec}$ to physical σ_f. Combining (10.18) and (10.20), the *fundamental* tension reads

$$\boxed{\;\sigma_f \;=\; C_F\,g_s^2(\mu)\,\mathcal{D}_0(\mu)\,\xi_c^2(\mu) \;\times\; \Big[1+\Delta_{\rm NL}(\mu)\Big], \qquad C_F = \frac{4}{3}\,,\;} \tag{10.27}$$

where $\Delta_{\rm NL}$ collects non-Gaussian and higher–derivative corrections and the μ–dependence obeys spectral RG (Ch. 3).

7. Positivity, hyperbolicity, and gates

(TQ) positivity. The Källén–Lehmann representation of the color–electric correlator implies $\mathcal{D}_E(\zeta^2) \geq 0$ and $\widetilde{\mathcal{D}}_E(k_\perp^2) \geq 0$, hence $\sigma_{\text{spec}} \geq 0$ in (10.18). Forward-limit positivity for $VV \to VV$ with adjoint external states further constrains higher–derivative color operators (Part VI).

(G) hyperbolicity. The principal symbol of gluon propagation must lie inside the g_{eff} cone; anisotropic constitutive entries aligned with χ^μ may renormalize string *thickness* but cannot flip the sign of σ_{spec} without violating causality.

(S)/(D) stability and dissipation. No mass term for G_μ is allowed; dissipative terms compatible with BRST cannot erase the area law as long as the static correlator (10.17) remains short–ranged and positive.

8. Lattice/spectral matching and observables

Matching strategy.

1. Compute $\mathcal{G}_{0i,0j}$ in the spectral medium (either analytically in controlled limits or numerically via the spectral–lattice protocol of Sec. 12.3) to extract $\mathcal{D}_E(\zeta^2)$ and ξ_c.
2. Obtain σ_{spec} from (10.18); check Casimir scaling (10.24) at intermediate L and N-ality at asymptotia.
3. Predict the static potential and flux–tube profile; confront with lattice data and heavy–quark spectroscopy (string slopes, Lüscher term, excitation spectrum).

9. Summary

The gluon sector of the spectral model is confining when the gauge–invariant color–electric correlator is short–ranged and positive. The *spectral string tension* admits two equivalent, first–principles expressions: a Kubo formula (10.18) in terms of $\mathcal{D}_E$, and a dual 2-form mass relation (10.23) via m_B. Representation dependence follows Casimir scaling at intermediate distances and N-ality at asymptotia; finite temperature ties $\sigma_f(T)$ to center symmetry through the Polyakov loop. All statements are controlled by the spectral gates (positivity, hyperbolicity, stability) and furnish concrete, gauge–invariant handles for lattice/spectral validation.

10.3 Running α_s and hadronic scales

Aim. We derive the renormalization–group (RG) flow of the strong coupling in the spectral framework, define the QCD scale Λ and its scheme dependence, implement heavy–flavor decoupling across thresholds, and relate the RG scale to *hadronic* scales (string tension σ_f, Sommer/gradient–flow scales) that are natural in the confining regime. All statements are aligned with the constitutive (Kubo) origin of the gluon kinetic term (Sec. 10.1) and the confining correlators (Sec. 10.2).

1. β function from spectral loops

Definition. Let $\alpha_s(\mu) \equiv g_s^2(\mu)/(4\pi)$ in a mass–independent scheme (e.g. $\overline{\text{MS}}$). The RG equation reads

$$\mu \frac{d\alpha_s}{d\mu} \;=\; \beta(\alpha_s) \;=\; -2\,\alpha_s^2 \left[\beta_0 \;+\; \beta_1 \frac{\alpha_s}{4\pi} \;+\; \beta_2 \left(\frac{\alpha_s}{4\pi}\right)^2 \;+\; \cdots\right], \tag{10.28}$$

with universal one– and two–loop coefficients for $SU(3)$ with n_f light Dirac flavors in the fundamental:

$$\beta_0 \;=\; \frac{11}{3} C_A - \frac{4}{3} T_F n_f \;=\; 11 - \frac{2}{3} n_f, \qquad \beta_1 \;=\; \frac{34}{3} C_A^2 - \frac{20}{3} C_A T_F n_f - 4 C_F T_F n_f \;=\; 102 - \frac{38}{3} n_f. \tag{10.29}$$

Here $C_A = 3$, $C_F = \tfrac{4}{3}$, $T_F = \tfrac{1}{2}$. The spectral origin is the logarithmic UV sensitivity of the current–current kernel that defines $Z_{1,c}$ in Eq. (10.7):

$$\mu \frac{d}{d\mu}\left(\frac{1}{g_s^2}\right) \;=\; -\frac{\beta(\alpha_s)}{2\pi\,\alpha_s^2} \;=\; \frac{1}{8\pi^2}\left(\beta_0 + \beta_1 \frac{\alpha_s}{4\pi} + \cdots\right), \tag{10.30}$$

where the *sign* of $\beta_0 > 0$ (for $n_f < 16.5$) follows from the S/TQ gates (positivity + gauge invariance) applied to the adjoint two–point function.

Asymptotic freedom. For $n_f \le 6$ (light flavors up to top threshold) we have $\beta_0 > 0$, hence $\alpha_s(\mu) \to 0$ as $\mu \to \infty$.

2. Solution and the Λ parameter

Two–loop solution. Integrating Eq. (10.28) to two loops yields

$$\alpha_s(\mu) \;=\; \frac{4\pi}{\beta_0\, L}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln L}{L}\right] \;+\; \mathcal{O}\!\left(\frac{1}{L^3}\right), \qquad L \equiv \ln \frac{\mu^2}{\Lambda^2}, \tag{10.31}$$

which *defines* the RG–invariant QCD scale Λ in the chosen scheme. Equivalently,

$$\Lambda \;=\; \mu\, \exp\!\left(-\frac{2\pi}{\beta_0\,\alpha_s(\mu)}\right) \left[\beta_0\,\alpha_s(\mu)\right]^{-\beta_1/\beta_0^2} \;\times\; \Big[1 + \mathcal{O}(\alpha_s)\Big]. \tag{10.32}$$

Scheme transformations. If $\alpha_s' = \alpha_s + c_1 \alpha_s^2/(4\pi) + \cdots$, then

$$\Lambda' \;=\; \Lambda\, \exp\!\left(\frac{c_1}{2\beta_0}\right), \tag{10.33}$$

while β_0, β_1 are scheme–independent.

3. Matching at heavy–flavor thresholds

Appelquist–Carazzone decoupling. Across $\mu \simeq m_h$ (heavy quark mass), match the (n_f+1)–flavor and n_f–flavor theories:

$$\alpha_s^{(n_f)}(\mu) \;=\; \alpha_s^{(n_f+1)}(\mu)\left[1 + \sum_{k\ge 1} c_k \left(\frac{\alpha_s^{(n_f+1)}(\mu)}{4\pi}\right)^k\right], \qquad \mu \sim m_h, \tag{10.34}$$

with known c_k (start at two loops in $\overline{\text{MS}}$). This implies piecewise $\Lambda^{(n_f)}$ with continuity of α_s at the threshold. In the spectral model, the same rule follows from integrating out heavy composite flavors as in Sec. 3.2 with the universality–class map.

4. Coupling schemes: $\overline{\rm MS}$, V–scheme, and MOM

V–scheme (static potential). Define $V(Q)$ via the Fourier transform of the short–distance static potential,

$$V(Q) \;=\; -\,\frac{4\pi C_F}{Q^2}\,\alpha_V(Q)\,\Big[1+\mathcal{O}\Big(\frac{\Lambda^2}{Q^2}\Big)\Big], \tag{10.35}$$

with perturbative relation $\alpha_V = \alpha_{\overline{\rm MS}}\Big[1 + a_1\frac{\alpha_{\overline{\rm MS}}}{4\pi} + a_2\Big(\frac{\alpha_{\overline{\rm MS}}}{4\pi}\Big)^2 + \cdots\Big]$ (known $a_{1,2}$). The V–scheme is convenient to connect to the lattice static force and Sommer scales below.

MOM schemes. Momentum–subtraction couplings defined from three–gluon or ghost–gluon vertices in Landau gauge are useful in spectral–lattice calculations; they relate to $\overline{\rm MS}$ by finite redefinitions as in Eq. (10.33).

5. Hadronic scales: σ_f, Sommer and gradient–flow

String tension. The fundamental string tension σ_f is defined by the area law (Sec. 10.2). In the UV it correlates with Λ up to a scheme– and n_f–dependent constant:

$$\sqrt{\sigma_f} \;=\; c_\sigma^{(n_f)}\,\Lambda\,\Big[1+\mathcal{O}\big(\alpha_s\big)\Big]. \tag{10.36}$$

In the spectral description $c_\sigma^{(n_f)}$ is calculable from the IR correlator $\mathcal{D}_E$ and the short–distance normalization of g_s via Eqs. (10.18), (10.32).

Sommer scales. Define r_0 and r_1 by the force $F(r) = dV/dr$:

$$r_0^2\,F(r_0) = 1.65, \qquad r_1^2\,F(r_1) = 1.0. \tag{10.37}$$

At intermediate distances where $V(r) \simeq -\frac{4\pi C_F}{(2\pi)^3}\int d^3Q\, e^{i\mathbf{Q}\cdot\mathbf{r}}\frac{\alpha_V(Q)}{Q^2} + \sigma_f r$, one finds

$$r_0\,\Lambda \;=\; c_{r_0}^{(n_f)}\Big[1+\mathcal{O}(\alpha_s)\Big], \qquad r_0\,\sqrt{\sigma_f} \;=\; \sqrt{1.65-\delta_{\rm pert}}, \tag{10.38}$$

where $\delta_{\rm pert}$ encodes the perturbative Coulomb piece (small relative to the linear term near r_0).

Gradient–flow scales. With the flowed energy density $E(t) = \frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu}$,

$$t_0:\quad t^2\langle E(t)\rangle\Big|_{t=t_0} = c_0, \qquad w_0:\quad t\,\frac{d}{dt}\Big[t^2\langle E(t)\rangle\Big]\Big|_{t=w_0^2} = c_w, \tag{10.39}$$

with conventional choices (c_0, c_w). At small t, $t^2\langle E\rangle = \frac{3}{128\pi^2}g_s^2(1+k_1\alpha_s+\cdots)$, so that

$$\frac{1}{\sqrt{8t_0}\,\Lambda} \;=\; c_{t_0}^{(n_f)}\Big[1+\mathcal{O}(\alpha_s)\Big], \qquad \frac{1}{w_0\,\Lambda} \;=\; c_{w_0}^{(n_f)}\Big[1+\mathcal{O}(\alpha_s)\Big], \tag{10.40}$$

with $c_{t_0}^{(n_f)}, c_{w_0}^{(n_f)}$ determined by matching to the perturbative tail and the spectral–lattice result for $\langle E\rangle$.

6. Trace anomaly, RG invariants, and OPE links

Trace anomaly. The breaking of classical scale invariance is

$$\Theta^{\mu}{}_{\mu} \;=\; \frac{\beta(g_s)}{2g_s}\, G^{a}_{\mu\nu} G^{a\,\mu\nu} \;+\; (1+\gamma_m) \sum_q m_q\, \bar{q} q, \tag{10.41}$$

which generates Λ and connects to vacuum condensates. RG–invariant combinations (e.g. the RGI quark mass $\widehat{m}_q$) and the gluon condensate appear in OPE analyses of short–distance observables and in QCD sum rules; in the spectral model they descend from the same constitutive tensors that set $Z_{1,c}$.

Heavy–quark potential and α_V. Short–distance $V(r)$ determines $\alpha_V(Q)$ and hence Λ; long–distance $V(r)$ measures σ_f. The smooth interpolation across $r \sim 1/\Lambda$ is a stringent consistency gate for the spectral–lattice implementation.

7. Positivity/causality gates along the flow

UV and IR constraints.

- **UV (asymptotic freedom):** $\beta_0 > 0$ is required by the sign of the adjoint two–point function's logarithm; violation would contradict spectral positivity (TQ) of the gauge kernel.
- **IR (confinement):** positivity of the color–electric correlator implies $\sigma_{\rm spec} > 0$ (Sec. 10.2); hence $\Lambda > 0$ must correlate with $\sigma_f > 0$ via (10.36).
- **Hyperbolicity (G):** running higher–derivative entries must keep the principal symbol within the $g_{\rm eff}$ cone; this bounds the growth of MOM–scheme couplings at low μ and enforces $E < E_{\rm max}$ in any fit (Sec. 9.4).

8. Spectral matching and practical recipe

Top–down (spectral ⇒ hadronic).

1. Compute $Z_{1,c}(\mu_*)$ from the fiber orbit metric and stiffness (Eq. (10.8)); extract $\alpha_s(\mu_*)$.
2. Evaluate the spectral one– and two–loop kernels to obtain (β_0, β_1) (they match Eq. (10.29) for fundamentals).
3. Run to μ with Eq. (10.31), applying threshold matching (10.34); read off Λ from (10.32).
4. Predict σ_f via (10.27) or (10.36); predict r_0, r_1, t_0, w_0 via (10.38), (10.40).

Bottom–up (hadronic ⇒ spectral).

1. Fix a hadronic scale (e.g. r_0 or $\sqrt{\sigma_f}$) from spectral–lattice; infer Λ from (10.36) or (10.38).
2. Run $\alpha_s(\mu)$ upward with (10.31) to electroweak scales, ensuring consistency with $\alpha_s(m_Z)$ block in the global fit (Sec. 9.1).
3. Deduce $Z_{1,c}(\mu_*)$ and hence the vertical orbit metric combination $\kappa_G^{(c)}\kappa_c$ via (10.8).

9. Summary

The running of α_s follows from the spectral flow of the gluon kinetic kernel, reproducing the universal β function coefficients and asymptotic freedom. The RG–invariant scale Λ organizes the connection between UV couplings and IR hadronic quantities; in the spectral framework it is tightly linked to the confining correlator $\mathcal{D}_E$ and to the string tension σ_f. Sommer and gradient–flow scales provide scheme–robust bridges across regimes and are natural calibration points for spectral–lattice simulations. Together with threshold matching and the S/G/TQ/D gates, these relations yield a closed, predictive map from fiber geometry to QCD phenomena.

10.4 Jets and event-shape imprints

Aim. We develop the perturbative–nonperturbative interface of QCD jets in the spectral framework. We (i) define global event shapes and jet substructure observables; (ii) state factorization theorems and Sudakov resummation; (iii) incorporate leading power corrections through universal shape functions and relate their first moments to spectral IR data (σ_f, correlators $\mathcal{D}_E$, and the dual mass m_B); (iv) discuss Casimir scaling, non-global logarithms, grooming, and collider implementations; (v) formulate positivity/hyperbolicity gates for jet observables.

1. Global event shapes and definitions

Thrust and C-parameter. In $e^+e^- \to$ hadrons at c.m. energy Q,

$$T \;:=\; \max_{\mathbf{n}} \frac{1}{Q}\sum_i |\mathbf{p}_i\cdot\mathbf{n}|\,, \qquad \tau \;:=\; 1-T, \tag{10.42}$$

and

$$C \;:=\; \frac{3}{2Q^2}\sum_{i,j}\frac{(\mathbf{p}_i\cdot\mathbf{p}_j)^2}{|\mathbf{p}_i||\mathbf{p}_j|}\,\sin^2\theta_{ij}\,. \tag{10.43}$$

More generally, *angularities* $e_\alpha = \sum_i (E_i/Q)\,\theta_i^\alpha$ with $0<\alpha\le 2$ interpolate among event shapes.

Jet mass and energy correlators. For a jet of radius R and transverse momentum p_T, define the (normalized) jet mass

$$\rho \;:=\; \frac{m_J^2}{p_T^2 R^2}\,, \tag{10.44}$$

and 2-point energy correlators $e_2^{(\beta)} = \sum_{i<j} z_i z_j\,\theta_{ij}^\beta$ with $z_i = E_i/\sum_k E_k$.

2. Factorization and Sudakov resummation

SCET factorization (global shapes). For $\tau \ll 1$ (two-jet limit),

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = H(Q,\mu)\int ds_n\, ds_{\bar n}\, dk\; J(s_n,\mu)\,J(s_{\bar n},\mu)\,S(k,\mu)\,\delta\!\left(\tau-\frac{s_n+s_{\bar n}+Q\,k}{Q^2}\right)\Big[1+\mathcal{O}(\tau)\Big], \tag{10.45}$$

with hard, jet, and soft functions H, J, S and σ_0 the Born cross section. Renormalization-group (RG) evolution from natural scales $\mu_H \sim Q$, $\mu_J \sim Q\sqrt{\tau}$, $\mu_S \sim Q\tau$ resums large logarithms $\ln \tau$.

Sudakov radiator. In Laplace (or N-moment) space, the resummed spectrum reads

$$\tilde{\sigma}(N) \;\equiv\; \int_0^\infty d\tau\, e^{-N\tau}\, \frac{1}{\sigma_0}\frac{d\sigma}{d\tau} \;=\; H(Q,\mu)\, e^{-R(\ln N, Q, \mu)}\, \tilde{S}(\partial_{\ln N}, \mu)\, \tilde{J}^2(\partial_{\ln N}, \mu), \tag{10.46}$$

with the Sudakov radiator

$$R(L, Q, \mu) \;=\; \int_{\mu^2}^{Q^2} \frac{dk^2}{k^2} \left[A(\alpha_s(k))\, \ln \frac{Q^2}{k^2\, N^2} + B(\alpha_s(k)) \right], \qquad L \equiv \ln N, \tag{10.47}$$

where $A(\alpha_s) = \sum_{n\geq 1} A_n\, [\alpha_s/(4\pi)]^n$ and $A_1 = 4C_F\, \Gamma^{(1)}_{\rm cusp} = 4C_F$ is set by the lightlike cusp anomalous dimension; $B(\alpha_s)$ is non-cusp. Analogous formulae hold for C-parameter and angularities with calculable replacements of kinematic kernels.

Hadron-collider jets. For small $\rho \ll 1$,

$$\frac{d\sigma}{d\rho} \;=\; \sum_{i=q,g} H_i(p_T, \eta, \mu) \left[B_a \otimes B_b \otimes J_i(\rho, \mu) \otimes S_i(\rho, \mu) \right] \left[1 + \mathcal{O}(\rho) \right], \tag{10.48}$$

with beam functions $B_{a,b}$, jet function J_i, and (global) soft function S_i; resummation proceeds via RG between hard, collinear, and soft scales.

3. Leading power corrections and universal shape functions

Shape-function convolution. In the two-jet limit, nonperturbative (NP) soft radiation enters as a convolution in the soft momentum k:

$$S(k, \mu) \;\longrightarrow\; \int_0^\infty dk'\; S_{\rm pert}(k - k', \mu)\; F_e(k')\,, \qquad \int_0^\infty dk'\; F_e(k') = 1, \tag{10.49}$$

with $e \in \{\tau, C, e_\alpha, \dots\}$ labeling the observable. Expanding for $k' \ll Qe$ yields a *shift* of the spectrum,

$$e \;\mapsto\; e \;-\; \frac{c_e\, \Omega_1}{Q} \;+\; \mathcal{O}\!\left(\frac{1}{Q^2}\right), \qquad \Omega_1 \;:=\; \int_0^\infty dk'\; k'\, F_e(k'), \tag{10.50}$$

where c_e is a calculable observable-dependent coefficient (fixed by soft Wilson-line geometry) and Ω_1 is the universal first moment of the soft shape function in the hemisphere scheme. For hadron-collider jet mass,

$$\rho \;\mapsto\; \rho \;-\; \frac{\Omega_1^{(i)}}{p_T R} \;+\; \mathcal{O}\!\left(\frac{1}{p_T^2 R^2}\right), \qquad \Omega_1^{(i)} \;=\; C_i\, \Omega_1 \left[1 + \mathcal{O}(\alpha_s) \right], \quad C_i \in \{C_F, C_A\}, \tag{10.51}$$

showing Casimir scaling at leading power.

Renormalon-safe definitions. A short-distance scheme (Rgap or MSR) removes the $u = \frac{1}{2}$ renormalon from Ω_1 by defining a subtracted $\Omega_1(R_{\rm gap})$ and compensating shifts in $S_{\rm pert}$; we will adopt a renormalon-free Ω_1 throughout.

4. Spectral origin of Ω_1 and link to σ_f

Kubo/dual representation. The NP soft momentum k' in (10.49) is carried by large-angle soft gluons sourced by lightlike Wilson lines. In the spectral framework,

$$\Omega_1 \;\propto\; \int d^2\zeta\; \mathcal{D}_E(\zeta^2)\; \mathcal{W}_e(\zeta)\,, \tag{10.52}$$

where $\mathcal{D}_E$ is the color-electric correlator in Eq. (10.17) and $\mathcal{W}_e$ is an observable-dependent weight that encodes the geometry of soft radiation into the hemispheres (or jet region). Using $\mathcal{D}_E(\zeta^2)\sim\mathcal{D}_0 e^{-|\zeta|/\xi_c}$ (Sec. 10.2), dimensional analysis and the dual B-field picture (10.21) give

$$\boxed{\Omega_1 \;=\; \kappa_e\,\sqrt{\sigma_f}\,\Big[1+\mathcal{O}\Big(\frac{\Lambda}{Q}\Big)\Big], \qquad \sigma_f \;\simeq\; C_F\, g_s^2\, \mathcal{D}_0\, \xi_c^2\,,} \tag{10.53}$$

with $\kappa_e = \mathcal{O}(1)$ calculable from $\mathcal{W}_e$ and the relation between $m_B = 1/\xi_c$ and $\mathcal{D}_E$ (cf. Eq. (10.23)). Equation (10.53) ties the *jet* NP shift to the *confining* string tension—a central spectral prediction.

Casimir scaling. Because $\mathcal{D}_E$ is adjoint-index blind and Wilson-line color enters linearly,

$$\Omega_1^{(i)} \;=\; \frac{C_i}{C_F}\,\Omega_1 \;+\; \mathcal{O}\Big(\frac{\alpha_s}{\pi}\Big), \tag{10.54}$$

reproducing (10.51) and linking to the representation dependence in Sec. 10.2.

5. Non-global logarithms (NGLs) and clustering effects

NGLs. Restricted angular regions (hemisphere masses, gaps between jets, groomed observables) generate non-global logarithms of the form $\alpha_s^n L^m$ with $m \le n$. They exponentiate via the Banfi–Marchesini–Smye (BMS) equation at large N_c; beyond large N_c one uses dressed-gluon expansions. The spectral framework does not alter the *perturbative* NGL kernel; NP effects enter only through Ω_1-type boundary contributions and remain power suppressed.

Clustering logs. Jet algorithms (anti-k_T, C/A) induce additional logarithms that are under control in resummed predictions and do not spoil universality of Ω_1 at leading power.

6. Grooming and reduction of nonperturbative sensitivity

Soft drop grooming. For parameters $(\beta, z_{\rm cut})$, soft drop removes soft wide-angle radiation, yielding groomed jet mass ρ_g with factorization

$$\frac{d\sigma}{d\rho_g} \;=\; H\,\Big[B_a \otimes B_b \otimes J_i^{\rm gr}(\rho_g,\mu) \otimes S_i^{\rm gr}(\rho_g,\mu)\Big]. \tag{10.55}$$

The leading NP shift scales as

$$\rho_g \;\mapsto\; \rho_g \;-\; \frac{\Omega_1^{\rm gr}(\beta, z_{\rm cut})}{p_T R}\,, \qquad \Omega_1^{\rm gr} \;=\; C_i\, f_{\rm gr}(\beta, z_{\rm cut})\;\Omega_1, \tag{10.56}$$

with $f_{\rm gr} < 1$ calculable and strongly suppressing NP sensitivity for $\beta \lesssim 1$. Equation (10.53) then implies a direct prediction for groomed shifts in terms of $\sqrt{\sigma_f}$.

7. Jet substructure discriminants and spectral imprints

N-subjettiness and energy correlators. Discriminants such as $\tau_{21} = \tau_2/\tau_1$ and $D_2 = e_3^{(\beta)}/(e_2^{(\beta)})^3$ admit SCET$_+$ factorization with collinear-soft functions. Their NP shifts are again proportional to $C_i\,\Omega_1/Q_{\rm eff}$, where $Q_{\rm eff}$ is the relevant hard scale (jet p_T, groomed scale $z_{\rm cut}\,p_T R$, etc.). Casimir scaling in (10.54) implies quark/gluon differences that can be calibrated against *independent* σ_f determinations.

Director anisotropy χ^μ. Small anisotropies of the spectral medium alter collinear splitting kernels only at higher twist (respecting hyperbolicity), leaving global-shape leading logs intact. Their leading imprint is a mild azimuthal modulation in wide-angle soft radiation—absorbed into κ_e at $\mathcal{O}(\Omega_1/Q)$ and bounded by the same (G)/(TQ) gates.

8. Positivity, hyperbolicity, and consistency gates

(TQ) positivity. Exponentiation requires $R(L) \geq 0$; this follows from $\Gamma_{\rm cusp} > 0$ and reflection positivity of Wilson-line correlators. The universal sign of Ω_1 is fixed by $\mathcal{D}_E \geq 0$, hence the NP shift reduces event shapes ($e \to e - {\rm const}/Q$), consistent with data.

(G) hyperbolicity. Principal symbols of collinear and soft modes must lie within the $g_{\rm eff}$ cone; this forbids acausal reshaping of angular distributions and constrains any constitutive corrections to remain subleading in the resummation hierarchy.

(S)/(D) stability. No mass term for gluons or light quarks; dissipative effects cannot spoil factorization at leading power and appear only via higher-twist power corrections.

9. Practical workflows and predictions

From spectral IR to jets.

1. Determine σ_f (or ξ_c) from lattice/spectral correlators (Sec. 10.2).
2. Compute κ_e for the chosen observable using Wilson-line geometry; then predict $\Omega_1 = \kappa_e\sqrt{\sigma_f}$ via (10.53).
3. Insert Ω_1 into resummed spectra using the shift (10.50) or a full shape-function convolution (10.49); include RG-improved perturbative ingredients at NLL/NNLL.

From jets to spectral IR.

1. Fit Ω_1 (and, where needed, $\Omega_1^{\rm gr}$) from thrust/C at multiple Q and from groomed jet mass at the LHC, enforcing Casimir scaling and universality across observables.
2. Use (10.53) to infer $\sqrt{\sigma_f}$ and compare to independent confining determinations (static potential, Regge slopes).

10. Summary

Jet observables furnish a sharp, calculable bridge from perturbative color radiation to the confining IR of the spectral medium. Factorization and resummation control logarithms; universal shape functions encode leading $1/Q$ power corrections with first moment Ω_1. In the spectral model Ω_1 is tied to gauge–invariant correlators and, remarkably, to the string tension, $\Omega_1 = \kappa_e\sqrt{\sigma_f}$, while maintaining Casimir scaling for quark vs. gluon jets. Grooming reduces NP sensitivity in a controlled way and enables precision extractions at colliders. All ingredients are consistent with the spectral S/G/TQ/D gates, rendering jets an incisive, overconstrained test of the emergent QCD sector.

Chapter 11

Chiral Symmetry Breaking and Pions as Phasons

11.1 $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$

Aim. We formulate spontaneous chiral symmetry breaking in the spectral (quantum–aether) framework and identify the light pseudoscalars as *phasons*—Goldstone modes of flavor–phase textures of the spectral medium. We construct the coset $G/H = \Big(SU(N_f)_L \times SU(N_f)_R\Big)/SU(N_f)_V$, derive the leading chiral Lagrangian with external sources, and express its low–energy constants in terms of spectral stiffness and Kubo correlators. Explicit breaking by quark masses is embedded via spurions, preparing the GMOR relations (Sec. 11.2) and anomalous WZW terms (Sec. 11.4).

1. Chiral group, currents, and the spectral order parameter

Global symmetry. For N_f light flavors (with color gauged as in Sec. 10.1), massless QCD enjoys

$$G_\chi \;=\; SU(N_f)_L \times SU(N_f)_R \times U(1)_V \;, \tag{11.1}$$

while $U(1)_A$ is anomalous (Part II). Let $\{T^a\}$, $a = 1, \ldots, N_f^2 - 1$, be the $su(N_f)$ generators with $\mathrm{Tr}[T^a T^b] = \frac{1}{2}\,\delta^{ab}$. Vector and axial currents are $V_\mu^a = \bar{q}\gamma_\mu T^a q$, $A_\mu^a = \bar{q}\gamma_\mu\gamma_5 T^a q$.

Spectral order parameter. In the spectral medium, chiral breaking corresponds to the condensation of a flavor–matrix order parameter

$$\Sigma(x) \;\sim\; \langle \bar{q}_R q_L \rangle \quad\leftrightarrow\quad \Sigma_0\, U(x), \qquad U(x) \in SU(N_f), \tag{11.2}$$

where Σ_0 is a stiffness–controlled magnitude and $U(x)$ is a *phase* field capturing slow flavor rotations of the spectral phase Φ along broken directions. Spontaneous symmetry breaking selects a vacuum alignment with

$$\langle \Sigma \rangle \;=\; \Sigma_0\, \mathbf{1}_{N_f} \;:\quad SU(N_f)_L \times SU(N_f)_R \;\longrightarrow\; SU(N_f)_V. \tag{11.3}$$

The associated Goldstones π^a are *phasons*—long–wavelength modulations of flavor phases supported by the medium's spectral stiffness.

2. Coset parameterization and transformation law

Coset field. We parameterize

$$U(x) \;=\; \exp\!\Big(\frac{i\,\pi^a(x)\,T^a}{F}\Big) \;\in\; SU(N_f), \qquad \pi^a \in \mathbb{R}, \tag{11.4}$$

where F is the decay constant (to be related to spectral response below). Under $g_L \in SU(N_f)_L$, $g_R \in SU(N_f)_R$,

$$U(x) \;\mapsto\; U'(x) = g_R\,U(x)\,g_L^\dagger. \tag{11.5}$$

It is often convenient to use $U = u^2$ with $u \in SU(N_f)$ and the CCWZ connection/ vielbein

$$\Gamma_\mu := \tfrac{1}{2}\Big[u^\dagger(\partial_\mu - i r_\mu)u + u(\partial_\mu - i l_\mu)u^\dagger\Big], \tag{11.6}$$

$$u_\mu := i\Big[u^\dagger(\partial_\mu - i r_\mu)u - u(\partial_\mu - i l_\mu)u^\dagger\Big], \tag{11.7}$$

transforming as a gauge field (Γ_μ) and as a tensor (u_μ) under the unbroken $SU(N_f)_V$. Here l_μ, r_μ are external left/right sources (to be specified below).

3. External sources and covariant derivatives

Sources and gauging. Introduce Hermitian sources $l_\mu(x), r_\mu(x)$ and a scalar/pseudoscalar spurion $\chi(x)$ with the chiral transformation rules

$$l_\mu \;\mapsto\; g_L\,l_\mu\,g_L^\dagger + i\,g_L\partial_\mu g_L^\dagger, \quad r_\mu \;\mapsto\; g_R\,r_\mu\,g_R^\dagger + i\,g_R\partial_\mu g_R^\dagger, \quad \chi \;\mapsto\; g_R\,\chi\,g_L^\dagger. \tag{11.8}$$

The U-covariant derivative is

$$D_\mu U \;:=\; \partial_\mu U - i\,r_\mu U + i\,U l_\mu. \tag{11.9}$$

Electroweak gauging follows by setting $l_\mu = r_\mu = v_\mu + a_\mu$ with v_μ containing $SU(2)_L \times U(1)_Y$ gauge fields projected on flavor charges.

4. Leading chiral Lagrangian and phason kinematics

Nonlinear sigma model at $\mathcal{O}(p^2)$. The most general invariant at two derivatives (and one spurion insertion) is

$$\boxed{\mathcal{L}_2 \;=\; \frac{F^2}{4}\,\langle D_\mu U\, D^\mu U^\dagger\rangle + \frac{F^2}{4}\,\langle \chi U^\dagger + U\chi^\dagger\rangle, \qquad \chi \;:=\; 2B\,\mathcal{M}\;,} \tag{11.10}$$

with $\langle\cdots\rangle \equiv \mathrm{Tr}_{N_f}$, $\mathcal{M} = \mathrm{diag}(m_u, \ldots)$ the quark mass matrix, F the decay constant and B proportional to the condensate density. Expanding U gives the canonical kinetic term $\frac{1}{2}\,\partial_\mu\pi^a\partial^\mu\pi^a + \cdots$, confirming that F fixes the phason normalization.

Currents and PCAC. Varying $\mathcal{L}_2$ w.r.t. l_μ, r_μ yields vector/axial currents; in particular

$$\langle 0|\,A_\mu^a(0)\,|\,\pi^b(p)\rangle \;=\; i\,\delta^{ab}\,F\,p_\mu \;+\; \mathcal{O}(p^3), \tag{11.11}$$

so F is the residue of the axial current pole and governs the phason sound speed within g_{eff}.

5. Spectral interpretation of F and B

Axial Kubo formula (stiffness). In the spectral medium, F^2 is the *stiffness* of flavor–phase textures, expressible in terms of axial current correlators:

$$F^2\,\delta^{ab} \;=\; -\lim_{q\to 0}\frac{q^\mu q^\nu}{q^2}\,\Pi^{ab}_{A,\mu\nu}(q)\;, \qquad \Pi^{ab}_{A,\mu\nu}(q) \;:=\; i\int d^4x\, e^{iqx}\,\langle 0|\,T\{A^a_\mu(x)A^b_\nu(0)\}\,|0\rangle. \tag{11.12}$$

Equivalently, in a spectral representation $\rho^{ab}_A(s)$, F^2 is the pion–pole residue in ρ_A (chiral limit). Microscopically, $F^2 = Z_A\,\kappa_{\rm ph}$ with $\kappa_{\rm ph}$ the spectral phase stiffness and Z_A a (finite) axial renormalization.

Condensate mapping. The constant B measures the linear response of the vacuum energy to explicit chiral breaking:

$$\langle \bar q q\rangle \;=\; -\,F^2 B\;\times\;\mathbf{1}_{N_f}\;+\;\mathcal{O}(m_q)\,, \tag{11.13}$$

so that the quark masses enter $\mathcal{L}_2$ through $\chi = 2B\mathcal{M}$. In the spectral picture $B \propto \partial\mathcal{E}_{\rm vac}/\partial\mathcal{M}$ and scales with the IR correlation length set by the confining sector (Sec. 10.2). The GMOR relation $m_\pi^2 F^2 = -(m_u+m_d)\langle\bar q q\rangle + \cdots$ will follow in Sec. 11.2.

6. Electroweak gauging and covariant phasons

Minimal coupling. Setting $l_\mu = r_\mu = e\,Q\,A_\mu$ (EM) reproduces the standard charged–pion covariant derivative; more generally, with $SU(2)_L \times U(1)_Y$ gauging, $l_\mu = g\,W^i_\mu T^i_L + g' B_\mu Y_L$, $r_\mu = g' B_\mu Y_R$, and $D_\mu U$ in (11.9) ensures gauge invariance. The phason kinetic term thus predicts $V\pi\pi$ and $VV\pi\pi$ couplings fixed by F and group theory, feeding precision EW observables via loops (Part III).

7. Power counting and next orders

Chiral expansion. The derivative/mass counting assigns $U \sim \mathcal{O}(1)$, $\partial_\mu \sim \mathcal{O}(p)$, $\chi \sim \mathcal{O}(p^2)$. At $\mathcal{O}(p^4)$ one adds the Gasser–Leutwyler terms with low–energy constants $L_{1\ldots10}$; in the spectral framework they are Kubo integrals of four–current correlators and differences of vector/axial polarizations (e.g. $L_{10} \propto -\frac{1}{4}\Pi'_{V-A}(0)$). Anomalous $\mathcal{O}(p^4)$ structures are captured by the WZW functional (Sec. 11.4).

8. Consistency gates (chiral sector)

(S) stability and (G) hyperbolicity. $F^2 > 0$ ensures positive phason kinetic energy; the principal symbol $\propto g^{\mu\nu}_{\rm eff}$ enforces subluminal propagation in the effective cone.

(TQ) positivity/unitarity. Axial spectral positivity guarantees $F^2 \geq 0$ and bounds $L_{1,2,3}$ combinations by forward $\pi\pi$ dispersion relations.

(D) dissipation gate. Medium dissipation cannot endow Goldstones with a mass; it may generate controlled widths via higher–order, gauge–invariant operators consistent with unitarity.

9. Summary

Chiral symmetry breaking in the spectral model arises from the alignment of a flavor–matrix order parameter $\Sigma = \Sigma_0 U$ with $U \in SU(N_f)$. The pions are *phasons*, i.e. phase Goldstones of the spectral medium, governed at leading order by the chiral Lagrangian (11.10). The decay constant F and condensate parameter B descend from axial Kubo stiffness and vacuum energetics, respectively, linking low–energy constants to confining IR data (Sec. 10.2). This provides a first–principles bridge from spectral microphysics to current algebra, setting up GMOR relations (Sec. 11.2), Skyrmions (Sec. 11.3), and anomalous WZW couplings (Sec. 11.4).

11.2 GMOR/GOR relations from spectral energetics

Aim. We derive the Gell–Mann–Oakes–Renner (GMOR/GOR) relations from two complementary angles: (i) axial Ward identities saturated by the phason (pion) pole, and (ii) the *spectral effective potential* for the order parameter $\Sigma = \Sigma_0 U$ with explicit breaking by the quark mass spurion. Both routes express pseudoscalar masses in terms of the chiral stiffness F and the condensate parameter B of Eq. (11.10), thereby linking meson masses to spectral Kubo data (Sec. 11.1) and to confinement correlators (Sec. 10.2).

1. Axial Ward identity and pion-pole saturation

Ward identity with masses. Let T^a be flavor generators with $\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$ and define vector/axial currents $V^a_\mu = \bar{q}\gamma_\mu T^a q$, $A^a_\mu = \bar{q}\gamma_\mu\gamma_5 T^a q$, and pseudoscalar density $P^a = \bar{q}\, i\gamma_5 T^a q$. For the (Hermitian) mass matrix $\mathcal{M}$ one has the exact identity

$$\partial^\mu A^a_\mu \;=\; 2\,\mathrm{Tr}(T^a\mathcal{M})\;\bar{q}i\gamma_5 q \;+\; 2\,\bar{q}\,i\gamma_5\,[T^a,\mathcal{M}]\,q \;=:\; 2\,\bar{q}\,\{T^a,\mathcal{M}\}\,i\gamma_5\,q\;, \tag{11.14}$$

where the commutator term vanishes for degenerate flavors or for a aligned with the mass matrix.

Matrix elements. Introduce

$$\langle 0|\,A^a_\mu(0)\,|\,\pi^b(p)\rangle \;=\; i\,\delta^{ab}F\,p_\mu, \qquad \langle 0|\,P^a(0)\,|\,\pi^b(p)\rangle \;=\; \delta^{ab}G_\pi\;, \tag{11.15}$$

with F the decay constant of Sec. 11.1. Taking the divergence of the first matrix element and using the second, the pion pole implies

$$F\,m_\pi^2\,\delta^{ab} \;=\; 2\,\langle 0|\,\bar{q}\,\{T^a,\mathcal{M}\}\,i\gamma_5\,q\,|\,\pi^b\rangle \;=\; 2\,G_\pi\,\delta^{ab}\,\mathrm{Tr}(\mathcal{M}) \quad \text{(degenerate case)}. \tag{11.16}$$

Thus $Fm_\pi^2 = 2\,\overline{m}\,G_\pi$ for $\mathcal{M} = \overline{m}\,\mathbf{1}_{N_f}$. Matching to the chiral Lagrangian at leading order (LO) gives $G_\pi = FB$ and hence

$$m_\pi^2 \;=\; 2\,\overline{m}\,B \qquad \text{(degenerate masses)}. \tag{11.17}$$

GMOR (two flavors). For $N_f = 2$ with $\hat{m} = \frac{1}{2}(m_u + m_d)$ and using the vacuum relation $\langle\bar{q}q\rangle = -F^2 B\,\mathbf{1}_2 + \mathcal{O}(m_q)$ (Eq. (11.13)),

$$\boxed{F_\pi^2\,m_\pi^2 \;=\; -\,(m_u+m_d)\,\langle\bar{q}q\rangle \;+\; \mathcal{O}(m_q^2)\;.} \tag{11.18}$$

This is the standard GMOR relation, here derived as a consequence of spectral stiffness (for F) and vacuum energetics (for B).

2. Mass matrices from the spectral effective potential

Quadratic expansion. From Eq. (11.10) with $\chi = 2B\mathcal{M}$ and $U = \exp(i\pi^a T^a/F)$,

$$\mathcal{L}^{(2)}_{\chi,\mathrm{mass}} = \frac{F^2}{4}\left\langle \chi U^\dagger + U\chi^\dagger \right\rangle\Big|_{\pi^2} = -\frac{B}{2}\,\mathrm{Tr}\big(\mathcal{M}\,\pi^2\big) = -\frac{1}{2}\,(M^2)_{ab}\;\pi^a\pi^b, \tag{11.19}$$

whence the pseudoscalar mass matrix is

$$\boxed{(M^2)_{ab} \;=\; B\;\mathrm{Tr}\Big[\mathcal{M}\,\{T^a, T^b\}\Big]\;.} \tag{11.20}$$

For $N_f = 2$ this yields $m_\pi^2 = B(m_u + m_d)$ (all three pions degenerate at LO). For $N_f = 3$ one obtains the standard tree-level pattern

$$m_\pi^2 = B(m_u{+}m_d), \qquad m_{K^+}^2 = B(m_u{+}m_s), \quad m_{K^0}^2 = B(m_d{+}m_s), \qquad m_{\eta_8}^2 = \tfrac{B}{3}(m_u{+}m_d{+}4m_s), \tag{11.21}$$

up to η_8–η_0 mixing and anomaly effects (Part II).

Energetic curvature. Let $\mathcal{E}_{\mathrm{vac}}(\mathcal{M}, U)$ denote the vacuum energy density from $\mathcal{L}_2$:

$$\mathcal{E}_{\mathrm{vac}} \;=\; -\frac{F^2}{4}\,\langle \chi U^\dagger + U\chi^\dagger\rangle \;+\; \cdots \;=\; -F^2 B\;\mathrm{Re}\,\langle \mathcal{M}U^\dagger\rangle \;+\; \cdots\,. \tag{11.22}$$

Parametrize a small axial twist $U(\alpha) = \exp(i\alpha^a T^a/F)$ around the aligned vacuum $U = \mathbf{1}$. Then

$$\frac{\partial \mathcal{E}_{\mathrm{vac}}}{\partial \alpha^a}\Big|_{\alpha=0} = 0, \qquad \frac{\partial^2 \mathcal{E}_{\mathrm{vac}}}{\partial \alpha^a\,\partial \alpha^b}\Big|_{\alpha=0} = \frac{F^2}{2}\,(M^2)_{ab}, \tag{11.23}$$

i.e. the *curvature* of the spectral vacuum energy in axial directions equals the pion mass matrix times $F^2/2$. Thus the GMOR masses are second derivatives (Hessian) of the explicit–breaking energy with respect to chiral angles.

3. Feynman–Hellmann and the condensate

Condensate as a slope. By Feynman–Hellmann,

$$\frac{\partial \mathcal{E}_{\mathrm{vac}}}{\partial m_q} \;=\; \langle \bar{q}q\rangle_q\;. \tag{11.24}$$

Equations (11.22)–(11.24) reproduce $\langle \bar{q}q\rangle = -F^2B + \mathcal{O}(m_q)$ and, together with (11.23), yield the GMOR identity (11.18) without invoking current algebra.

4. NLO corrections and chiral logs

Two-flavor case. At next-to-leading order (NLO) in the chiral expansion,

$$m_\pi^2 \;=\; 2\hat{m}\,B\left[1 + \frac{2\hat{m}\,B}{32\pi^2F^2}\,\ln\frac{2\hat{m}\,B}{\mu^2} + \frac{2\hat{m}\,B}{F^2}\,\bar{\ell}_3(\mu) + \mathcal{O}(\hat{m}^2)\right], \tag{11.25}$$

where $\bar{\ell}_3$ is a renormalized low-energy constant (LEC). In the spectral framework, $\bar{\ell}_3$ is a Kubo integral of four-current correlators (Sec. 11.1).

Three flavors. For $N_f = 3$ the Gasser–Leutwyler LECs L_6, L_7, L_8 correct Eq. (11.21) with known chiral logs. All NLO pieces are calculable spectral responses of the medium, preserving the LO sign relations fixed by positivity (see below).

5. Spectral mapping of F and B

Kubo formulas. From Eq. (11.12),

$$F^2\,\delta^{ab} \;=\; -\lim_{q\to 0}\frac{q^\mu q^\nu}{q^2}\,\Pi^{ab}_{A,\mu\nu}(q), \qquad B \;=\; -\frac{1}{F^2 N_f}\,\mathrm{Tr}\,\langle\bar{q}q\rangle\;, \tag{11.26}$$

with Π_A the axial–current correlator. Spectral positivity of Π_A enforces $F^2 > 0$, while confinement fixes the magnitude of $\langle\bar{q}q\rangle$ through IR correlators ($\mathcal{D}_E$, Sec. 10.2). Consequently, GMOR acquires the *spectral* form

$$F^2 m_\pi^2 \;=\; 2\hat{m}\,F^2 B \;=\; -\,2\hat{m}\;\mathrm{Tr}\,\langle\bar{q}q\rangle/N_f \;+\; \cdots\,, \tag{11.27}$$

with the ellipsis denoting higher-order corrections.

6. Isospin breaking, EM, and general mass matrices

Non-degenerate $\mathcal{M}$. Using (11.20),

$$m^2_{\pi^\pm} = B(m_u + m_d) + \Delta_{\mathrm{EM}}, \qquad m^2_{\pi^0} = B(m_u + m_d) + \Delta_{\mathrm{EM}} + \mathcal{O}\Big((m_u - m_d)^2\Big), \tag{11.28}$$

where Δ_{EM} arises from gauging the sources l_μ, r_μ and integrating photons (counted as $\mathcal{O}(e^2)$). The leading isospin-breaking effects enter only at second order in $(m_u - m_d)$ in QCD.

7. Finite temperature/density (brief)

Thermal shifts. For $T \ll 4\pi F$, chiral perturbation theory yields

$$F(T) \;=\; F\left[1 - \frac{N_f}{12}\frac{T^2}{F^2} + \cdots\right], \qquad \langle\bar{q}q\rangle_T \;=\; \langle\bar{q}q\rangle\left[1 - \frac{N_f^2 - 1}{12}\frac{T^2}{F^2} + \cdots\right], \tag{11.29}$$

so GMOR holds with $F \to F(T)$, $\langle\bar{q}q\rangle \to \langle\bar{q}q\rangle_T$ to this order. In the spectral language these follow from the temperature dependence of axial and scalar susceptibilities.

8. Consistency gates (chiral GMOR)

(TQ) positivity. $F^2 \geq 0$ and $\mathrm{Re}\,B > 0$ guarantee $m_\pi^2 \geq 0$ for $m_q \geq 0$ at LO; the sign is protected by the spectral representation of Π_A and by convexity of $\mathcal{E}_{\mathrm{vac}}$ in $\mathcal{M}$.

(G) hyperbolicity. Phason propagation inherits the g_{eff} cone; GMOR does not alter causal structure, and any anisotropy from χ^μ can only appear beyond LO in derivatives.

(S)/(D) stability. Explicit breaking shifts the vacuum but cannot introduce tachyonic modes; dissipative corrections do not generate mass at $m_q = 0$.

9. Summary

GMOR emerges equivalently from (i) axial Ward identities saturated by the pion pole and (ii) the curvature of the spectral vacuum energy in axial directions. At LO, $m_\pi^2 = 2\hat{m}B$ and $F_\pi^2 m_\pi^2 = -(m_u + m_d)\langle\bar{q}q\rangle$, with F^2 and B fixed by axial Kubo stiffness and condensate energetics. General mass matrices follow from $(M^2)_{ab} = B\,\mathrm{Tr}[\mathcal{M}\{T^a, T^b\}]$, reproducing the SU(3) pattern, while NLO chiral logs and LECs are spectral correlators. Thus the GMOR relation is a *quantitative* bridge between confining IR physics and chiral order encoded in the spectral medium.

11.3 Skyrmions as baryons and quantization

Aim. We construct baryons as topological solitons (Skyrmions) of the chiral phason field $U(x) \in SU(N_f)$, derive their conserved topological charge, energy functional and Euler–Lagrange equations, embed the anomaly-quantized WZW term, and perform collective–coordinate quantization for $SU(2)$ and $SU(3)$ flavor. We express the Skyrme low-energy constants (LECs) in terms of *spectral* Kubo data introduced in Sec. 11.1, relate the soliton scales to the confining correlators of Sec. 10.2, and state stability/positivity gates.

1. Topological charge and baryon number

Baryon current. For $U(x) \in SU(N_f)$ define the left current $L_\mu := U^\dagger \partial_\mu U$. The winding-number current

$$B^\mu \;:=\; \frac{1}{24\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big(L_\nu L_\rho L_\sigma\Big) \tag{11.30}$$

is identically conserved, $\partial_\mu B^\mu = 0$, and its charge

$$B \;:=\; \int d^3x\, B^0 \;\in\; \pi_3\Big(SU(N_f)\Big) \;\simeq\; \mathbb{Z} \tag{11.31}$$

is the *baryon number*. With finite-energy boundary conditions $U(\mathbf{x}) \to \mathbf{1}$ as $|\mathbf{x}| \to \infty$, field configurations compactify to $S^3 \to SU(N_f)$ maps classified by B.

2. Energy functional: $\mathcal{O}(p^2)+\mathcal{O}(p^4)+$ mass and sextic terms

Lagrangian. Augmenting the $\mathcal{O}(p^2)$ chiral Lagrangian (11.10) by the (stabilizing) Skyrme term and an optional BPS–sextic term,

$$\mathcal{L} \;=\; \frac{F^2}{4}\,\langle \partial_\mu U\, \partial^\mu U^\dagger\rangle \;+\; \frac{1}{32e^2}\,\langle [U^\dagger\partial_\mu U,\, U^\dagger \partial_\nu U]^2\rangle \;+\; \frac{F^2}{4}\,\langle \chi U^\dagger + U\chi^\dagger\rangle \;-\; \lambda^2\, B_\mu B^\mu \;+\; \mathcal{L}_{\mathrm{WZW}},$$
$$\chi = 2B\,\mathcal{M}, \qquad B_\mu \text{ as in (11.30)}. \tag{11.32}$$

Here F and B are the spectral stiffness and condensate parameters; e is the Skyrme coupling; λ controls the sextic (BPS) term; $\mathcal{L}_{\mathrm{WZW}}$ is the anomaly functional (Sec. 11.4).

Static energy. For time-independent fields ($\dot{U} = 0$) and two degenerate light flavors with $m_\pi^2 = 2\hat{m}\, B$, the energy reads

$$E[U] \;=\; \int d^3x \left\{ \frac{F^2}{4}\,\mathrm{Tr}(L_i L_i) \;+\; \frac{1}{32e^2}\,\mathrm{Tr}\Big([L_i, L_j]^2\Big) \;+\; F^2 m_\pi^2 \left(1 - \tfrac{1}{2}\mathrm{Tr}\, U\right) \;+\; \lambda^2\, B_0^2 \right\}. \tag{11.33}$$

The $\mathcal{O}(p^4)$ term prevents Derrick collapse; the m_π term encodes explicit breaking; the sextic term yields BPS-like control for large B.

3. Hedgehog ansatz, Euler–Lagrange equation, and lower bound

Hedgehog. For $N_f = 2$ the $B = 1$ ground state is spherically symmetric:

$$U_0(\mathbf{x}) \;=\; \cos F(r) \;+\; i\,\hat{\mathbf{r}}\cdot\boldsymbol{\tau}\;\sin F(r), \qquad F(0) = \pi, \quad F(\infty) = 0. \tag{11.34}$$

Inserting into (11.33) gives

$$\begin{aligned} E[F] \;=\; 4\pi \int_0^\infty dr\, \Bigg\{ & F^2\Big(r^2 F'^2 + 2\sin^2 F\Big) \;+\; \frac{1}{e^2}\Big(\sin^2 F\, F'^2 + \frac{\sin^4 F}{2r^2}\Big) \\ & +\; F^2 m_\pi^2\, r^2\,(1 - \cos F) \;+\; \frac{\lambda^2}{2}\,\frac{\sin^4 F\, F'^2}{\pi^4 r^2} \Bigg\}. \end{aligned} \tag{11.35}$$

The last term uses $B_0 = \frac{1}{2\pi^2}\frac{\sin^2 F\, F'}{r^2}$ for the hedgehog.

Profile equation. Extremizing (11.35) yields a second-order ODE for $F(r)$, regular at $r = 0$ and decaying exponentially for $m_\pi > 0$. In the chiral limit $m_\pi \to 0$, $F(r) \sim c/r^2$ asymptotically.

Topological lower bound. From algebraic inequalities,

$$E \;\geq\; \frac{12\pi^2\, F}{e}\,|B| \qquad (\text{for } \lambda = m_\pi = 0), \tag{11.36}$$

saturated only in special BPS limits (with suitable λ). This exhibits the canonical soliton scale $R_\star \sim 1/(eF)$.

Dimensionless variables. With $\tilde{r} = eFr$,

$$E \;=\; \frac{F}{e}\,\mathcal{E}[\tilde{F}] \;+\; \frac{4\pi F^2 m_\pi^2}{(eF)^3}\,\mathcal{V}[\tilde{F}] \;+\; \frac{\lambda^2 e^3 F^3}{(2\pi^2)^2}\,\mathcal{B}[\tilde{F}], \tag{11.37}$$

exhibiting the scaling of each contribution.

4. Spectral origin of the Skyrme and sextic couplings

Matching to p^4 LECs. Expanding the commutator term gives the $SU(2)$ relations

$$\frac{1}{32e^2}\,\langle [L_\mu, L_\nu]^2\rangle \;\equiv\; L_2\,\langle D_\mu U^\dagger D_\nu U\, D^\mu U^\dagger D^\nu U\rangle \;+\; L_3\,\langle (D_\mu U^\dagger D^\mu U)^2\rangle, \tag{11.38}$$

with the *Skyrme* prediction $L_1 = \frac{1}{2}L_2$, $L_3 = -3L_2$ and $L_2 = \frac{1}{16e^2}$ for two flavors. In the spectral framework, L_i are Kubo integrals of four-current correlators; hence

$$\boxed{\;e^{-2} \;=\; 16\,L_2 \;=\; 16\,\mathcal{K}^{(4)}_{VA,VA}\,, \qquad F^2 \;=\; -\lim_{q\to 0}\frac{q^\mu q^\nu}{q^2}\,\Pi_{A,\mu\nu}(q)\;} \tag{11.39}$$

with $\mathcal{K}^{(4)}_{VA,VA}$ a positive-definite spectral integral (Sec. 11.1).

Sextic (BPS) term. The λ-term is $\propto B_\mu B^\mu$ and arises upon integrating out a massive dual 2-form (Kalb–Ramond) with mass m_B (Sec. 10.2); matching gives $\lambda^2 \sim g_s^2/m_B^2 \propto \sigma_f^{-1}$, tying the BPS stiffness to the confining scale.

5. Collective–coordinate quantization: $SU(2)$

Rotor ansatz. Promote a time-dependent isorotation $A(t) \in SU(2)$,

$$U(\mathbf{x},t) \;=\; A(t)\,U_0(\mathbf{x})\,A^\dagger(t), \qquad A^\dagger\dot{A} \;=\; \tfrac{i}{2}\,\Omega_a\,\tau_a. \tag{11.40}$$

Inserting into the action yields a rigid-rotor Lagrangian

$$L_{\rm coll} \;=\; \tfrac{1}{2}\,\mathcal{I}\,\boldsymbol{\Omega}^2, \qquad \mathcal{I} \;=\; \frac{8\pi}{3}\int_0^\infty dr\;\sin^2\!F\left[r^2F^2 \;+\; \frac{1}{e^2}\Big(r^2F'^2 + \sin^2\!F\Big) \;+\; \frac{\lambda^2}{2\pi^4}\,\frac{\sin^2\!F\,F'^2}{r^2}\right]. \tag{11.41}$$

The hedgehog locks spatial and isospin rotations, enforcing $J = I$.

Quantization and spectrum. Canonical quantization with body-fixed isospin generators R_a gives

$$H_{\rm coll} \;=\; \frac{\mathbf{R}^2}{2\mathcal{I}}, \qquad \mathbf{R}^2 = I(I+1) = J(J+1), \tag{11.42}$$

subject to Finkelstein–Rubinstein (FR) constraints that render the $B = 1$ Skyrmion a *fermion* for odd N_c: allowed lowest states are

$$|N\rangle:\; J = I = \tfrac{1}{2}, \qquad |\Delta\rangle:\; J = I = \tfrac{3}{2}, \tag{11.43}$$

with $M_\Delta - M_N = \frac{3}{2\mathcal{I}}$.

6. $SU(3)$ embedding, WZW quantization, and multiplets

Embedding and coordinates. Embed $U_0 \in SU(2)$ into $SU(3)$ as $\mathrm{diag}(U_0, 1)$ and let $A(t) \in SU(3)$ act by $U \mapsto AU_0A^\dagger$. Write the collective Lagrangian as

$$L^{SU(3)}_{\rm coll} \;=\; \tfrac{1}{2}\,\mathcal{I}_1\sum_{a=1}^{3}\Omega_a^2 \;+\; \tfrac{1}{2}\,\mathcal{I}_2\sum_{a=4}^{7}\Omega_a^2 \;+\; \frac{N_cB}{2\sqrt{3}}\,\Omega_8, \tag{11.44}$$

where Ω_a are body-fixed angular velocities and the last term is induced by the WZW functional (quantized by the QCD anomaly, Sec. 11.4).

WZW constraint and representations. Canonical quantization imposes the constraint on the *right* hypercharge

$$Y_R \;=\; \frac{2}{\sqrt{3}}\,R_8 \;=\; \frac{N_c\,B}{3}, \tag{11.45}$$

selecting allowed $SU(3)$ representations for $N_c = 3$, $B = 1$: the octet **8** and decuplet **10** (and higher ones for excited states). The collective Hamiltonian reads

$$H_{\rm coll}^{SU(3)} \;=\; \frac{1}{2\mathcal{I}_1}\,\mathbf{R}_{SU(2)}^2 \;+\; \frac{1}{2\mathcal{I}_2}\left(C_2[SU(3)] - \mathbf{R}_{SU(2)}^2 - \tfrac{N_c^2 B^2}{12}\right), \tag{11.46}$$

with C_2 the quadratic Casimir of the chosen representation. Explicit $SU(3)$ breaking (strange quark mass) enters via symmetry-breaking operators linear in the collective coordinates.

7. Currents, axial couplings, and form factors

Noether currents. Vector and axial currents follow from variation of (11.32); their matrix elements in quantized Skyrmion states yield static observables. In particular, the isovector axial coupling is

$$g_A \;=\; \frac{16\pi}{3}\int_0^\infty dr\;\sin^2\!F\,\left[F' + \frac{\sin 2F}{2r}\right]\left[1 + \frac{2}{e^2F^2}\left(F'^2 + \frac{\sin^2\!F}{r^2}\right)\right]^{-1} \;+\; \cdots, \tag{11.47}$$

where the ellipsis indicates corrections from $m_\pi \neq 0$ and $\lambda \neq 0$. Electromagnetic form factors follow from gauging and minimal substitution in (11.32).

8. Spectral matching, scales, and phenomenology

LECs from spectral data. Using Secs. 11.1 and 10.2,

$$F^2 \;\sim\; \text{axial Kubo stiffness}, \qquad e^{-2} \;\sim\; \text{four-current Kubo integral}, \qquad \lambda^2 \;\sim\; \sigma_f^{-1}, \qquad m_\pi^2 \;=\; 2\hat{m}B. \tag{11.48}$$

Therefore the *classical* soliton size $R_\star \sim (eF)^{-1}$ and mass $M_{\rm cl} \sim F/e$ are calculable from spectral inputs; $1/N_c$ and chiral corrections enter via loops and collective quantization.

Consistent windows. A controlled semiclassical description requires $eF \gg m_\pi$ (thin-wall limit suppressed), $\mathcal{I}\,M_{\rm cl} \gg 1$ (rigid-rotor validity), and EFT cutoffs above the intrinsic scales of the soliton ($\Lambda_\chi \gtrsim 4\pi F$). These conditions are enforced in the global-fit gates of Sec. 9.4.

9. Consistency gates (Skyrmion sector)

(S) stability. $F^2 > 0$, $e^2 > 0$, and $\lambda^2 \geq 0$ ensure a bounded energy functional; the Skyrme term (or sextic term) prevents collapse.

(G) hyperbolicity. The principal symbol of small phason fluctuations around U_0 remains within the $g_{\rm eff}$ cone; anisotropies from χ^μ appear only in higher-derivative, strictly subluminal operators.

(TQ) positivity/unitarity. The p^4 LECs obey forward-positivity inequalities; the WZW coefficient is quantized ($\propto N_c$) by anomaly inflow (Part II), guaranteeing consistent fermionic quantization for odd N_c.

(D) dissipation gate. Dissipative couplings cannot change B; they only generate controlled widths for excited Skyrmions through higher-twist operators consistent with unitarity.

10. Summary

Baryons emerge as topological phason solitons whose charge is the winding number $B \in \mathbb{Z}$, identified with baryon number through the WZW anomaly. Their structure is governed by spectral LECs (F, e, B, λ) fixed by axial and four-current Kubo correlators and by the confining string scale σ_f. Collective quantization reproduces the N/Δ spectrum ($SU(2)$) and, with the WZW constraint $Y_R = N_c B/3$, the $SU(3)$ octet/decuplet pattern. All ingredients obey the spectral S/G/TQ/D gates, providing a first-principles, anomaly-safe realization of baryons in the emergent chiral medium.

11.4 Anomalous couplings and WZW terms

Aim. We construct the Wess–Zumino–Witten (WZW) functional that encodes the *parity-odd* (anomalous) sector of the chiral phason theory, derive its gauge variation and anomaly, present the gauged form with external sources in the Bardeen (vector-conserving) scheme, and extract benchmark anomalous couplings such as $\pi^0 \to \gamma\gamma$ and $\gamma \to 3\pi$. We also show how the WZW term ties Skyrmion baryon number to the microscopic quark anomaly, and we map its quantized coefficient to spectral 't Hooft matching.

1. Ungauged WZW functional and quantization

Five–form and bulk extension. Let $U(x) \in SU(N_f)$ be the phason field (Sec. 11.1) on spacetime M ($\partial B = M$). Define the left-invariant one-form $\beta := U^{-1}dU$ and the closed 5-form

$$\omega_5(U) \;:=\; \frac{i}{240\pi^2}\,\big\langle \beta^5 \big\rangle, \qquad d\,\omega_5 = 0, \tag{11.49}$$

with $\langle \cdots \rangle \equiv \mathrm{Tr}_{N_f}$. The *ungauged* WZW action is

$$\boxed{\Gamma^{(0)}_{\mathrm{WZW}}[U] \;=\; -\,N_c \int_B \omega_5(U)\;, \qquad N_c \in \mathbb{Z}\;.} \tag{11.50}$$

Independence of the bulk extension B fixes the overall coefficient to be an integer (quantization by $\pi_5(SU(N_f)) = \mathbb{Z}$); anomaly matching identifies it with the number of colors N_c.

Local 4D density. On M one may write $\Gamma^{(0)}_{\mathrm{WZW}} = \int_M \mathcal{L}^{(0)}_{\mathrm{WZW}}$ with $\mathcal{L}^{(0)}_{\mathrm{WZW}}$ a parity-odd 4-form whose exterior derivative reproduces (11.49). Any two such 4-forms differ by a total derivative; physical vertices are unambiguous.

2. Gauged WZW in the Bardeen scheme

External sources and field strengths. Introduce left/right sources as one-forms

$$A_L := l_\mu dx^\mu, \qquad A_R := r_\mu dx^\mu, \qquad F_{L,R} := dA_{L,R} - i\,A_{L,R}^2, \tag{11.51}$$

with U transforming as $U \mapsto g_R U g_L^\dagger$. It is convenient to switch to vector/axial sources

$$v := \tfrac{1}{2}(A_R + A_L), \qquad a := \tfrac{1}{2}(A_R - A_L), \qquad A_{R,L} = v \pm a. \tag{11.52}$$

Gauged functional and anomaly. The *gauged* WZW functional $\Gamma_{\rm WZW}[U; A_L, A_R]$ is the unique local completion such that its infinitesimal gauge variation reproduces the *consistent* chiral anomaly while preserving *vector* gauge invariance (Bardeen choice):

$$\delta_g \Gamma_{\rm WZW} = -\frac{N_c}{24\pi^2}\int_M \Big\langle \epsilon_L\, d\Big(A_L dA_L + \tfrac{i}{2}A_L^3\Big) \;-\; \epsilon_R\, d\Big(A_R dA_R + \tfrac{i}{2}A_R^3\Big)\Big\rangle, \tag{11.53}$$

where $\epsilon_{L,R}$ are Lie-algebra parameters. In components this yields the standard Bardeen anomaly

$$\partial^\mu J^a_{5\,\mu} \;=\; \frac{N_c}{16\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big[T^a(F_{L\,\mu\nu}F_{L\,\rho\sigma} - F_{R\,\mu\nu}F_{R\,\rho\sigma})\Big], \qquad \partial^\mu J^b_{V\,\mu} = 0, \tag{11.54}$$

ensuring conserved vector currents and anomalous axial currents, as required by QCD.

4D form of the gauged action. A compact differential-form representation is (Kaymakcalan–Rajeev–Schechter form)

$$\begin{aligned}\Gamma_{\rm WZW}[U; A_L, A_R] = \;& \Gamma^{(0)}_{\rm WZW}[U] \;+\; \frac{iN_c}{48\pi^2}\int_M \Big\langle A_L\,\beta^3 + A_R\,\alpha^3 - (A_L + A_R)\Big(\beta\alpha - \alpha\beta\Big) \\ & +\; A_L\,U A_R U^{-1}(\alpha+\beta) + A_L dA_L\,\beta + A_R dA_R\,\alpha + \tfrac{i}{2}A_L^3\beta - \tfrac{i}{2}A_R^3\alpha\Big\rangle,\end{aligned} \tag{11.55}$$

with $\alpha := dU\,U^{-1}$ and $\beta := U^{-1}dU$. Equation (11.55) reproduces (11.53) upon gauge variation and is equivalent (up to exact forms) to other standard representations.

3. Electromagnetic specialization and benchmark amplitudes

EM embedding. Set $A_L = A_R = e\,Q\,A$ with the charge matrix $Q = \mathrm{diag}(q_u, \ldots)$ and photon one-form $A = A_\mu dx^\mu$ ($F = dA$). Expanding $U = \exp(i\pi^a T^a/F)$ to leading nontrivial order yields the classic anomalous vertices:

$\pi^0 \to \gamma\gamma$.

$$\boxed{\mathcal{L}_{\pi^0\gamma\gamma} \;=\; \frac{N_c\,e^2}{24\pi^2 F}\,\pi^0\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \quad\Longrightarrow\quad \mathcal{A}(\pi^0\to\gamma\gamma) = \frac{N_c\,\alpha}{\pi F}\,\epsilon^{\mu\nu\rho\sigma}\,\varepsilon_\mu k_\nu \varepsilon'_\rho k'_\sigma.} \tag{11.56}$$

$\gamma \to 3\pi$.

$$\boxed{\mathcal{L}_{\gamma 3\pi} \;=\; -\,\frac{e\,N_c}{12\pi^2 F^3}\,\epsilon^{\mu\nu\rho\sigma}\,A_\mu\,\partial_\nu\pi^a\,\partial_\rho\pi^b\,\partial_\sigma\pi^c\,\mathrm{Tr}\Big(Q\,T^aT^bT^c\Big)\;.} \tag{11.57}$$

Analogous expressions hold for η_8, η_0 with mixing and for weak/electroweak sources by choosing $A_{L,R}$ accordingly.

4. Baryon number from gauged WZW

$U(1)_B$ probe and Skyrmion charge. Couple a background baryon gauge field B_μ via $A_L \to A_L + \frac{1}{3}B_\mu\,\mathbf{1}\,dx^\mu$ and $A_R \to A_R + \frac{1}{3}B_\mu\,\mathbf{1}\,dx^\mu$. Varying (11.55) with respect to B_μ at $A_{L,R} = 0$ yields the baryon current

$$J_B^\mu \;=\; \frac{\delta\Gamma_{\rm WZW}}{\delta B_\mu}\Big|_{A=0} \;=\; N_c\,\frac{1}{24\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big(L_\nu L_\rho L_\sigma\Big) \;=\; N_c\,B^\mu, \tag{11.58}$$

which integrates to $B_{\rm quark} = N_c \times B_{\rm Skyrmion}$, precisely identifying the topological Skyrmion charge with baryon number for $N_c = 3$ (Sec. 11.3).

5. Spectral origin and anomaly matching

Descent from the anomaly polynomial. In the spectral framework the UV chiral fermions (composites of the medium) generate a 6-form anomaly polynomial

$$\mathcal{I}_6 \;=\; \frac{N_c}{24\pi^2}\Big\langle \mathcal{F}_L^3 - \mathcal{F}_R^3 \Big\rangle, \qquad \mathcal{F}_{L,R} = d\mathcal{A}_{L,R} - i\,\mathcal{A}_{L,R}^2, \tag{11.59}$$

whose descent produces (11.53). Because $\mathcal{I}_6$ is a *closed, quantized* characteristic class, its coefficient is topological and insensitive to spectral deformations: it is fixed by 't Hooft matching to be N_c (Part II and Ch. 6). Hence the WZW term is robust across phases and media, modulo local counterterms that reshuffle vector/axial assignments (Bardeen shifts).

6. Parity, time reversal, and CP

Discrete structure. $\Gamma_{\rm WZW}$ is odd under parity and time reversal but even under CP for real $\mathcal{M}$ and vanishing θ-angle. Its variation yields P-odd amplitudes (e.g. $\pi^0 F\tilde{F}$) required by the anomaly, with no conflict with unitarity (it is linear in derivatives and does not alter the positive-definite kinetic sector).

7. Practical vertices beyond EM

Electroweak sources. For $A_L = g\,W^i T_L^i + g'\,B\,Y_L$ and $A_R = g'\,B\,Y_R$, (11.55) generates anomalous VVP, AVP, $VVVP$, … vertices. Examples include $Z \to \pi\pi\gamma$ contact terms and $\omega \to 3\pi$ once vector mesons are introduced as external gauge fields of hidden local symmetry (HLS); the overall normalizations are inherited from N_c and group traces.

8. Consistency gates (anomalous sector)

(TQ) positivity/unitarity. The WZW term does not modify the quadratic kinetic operator; optical-theorem checks pass provided the non-anomalous sector obeys positivity (Secs. 11.1–11.2).

(G) hyperbolicity. Being first order in derivatives and topological, Γ_{WZW} does not affect the principal symbol; causal cones remain those of g_{eff}.

(S)/(D) stability. No mass terms or dissipative operators arise from Γ_{WZW}; scheme changes (Bardeen counterterms) only redistribute currents while keeping physical amplitudes fixed.

9. Summary

The WZW functional is the unique, quantized parity-odd action for phasons whose gauge variation reproduces the QCD chiral anomaly with vector-current conservation. Its coefficient equals N_c by 't Hooft matching, and it fixes benchmark processes such as $\pi^0 \to \gamma\gamma$ and $\gamma \to 3\pi$ with absolute normalizations. Coupling a $U(1)_B$ probe extracts the baryon current, equating Skyrmion winding to baryon number. All features are topological and robust under spectral deformations, providing an anomaly-safe bridge between microscopic chiral fermions and macroscopic phason dynamics.

Chapter 12

Glueballs, Topological Defects, and Lattice Benchmarks

12.1 Glueball spectra from composite Φ–χ^μ modes

Aim. We formulate glueballs as gauge–invariant composite excitations of the color sector that are *hybridized* with spectral medium modes—the phase Φ and the director χ^μ introduced in Vol. I. We build a minimal operator basis for the lowest $J^{PC} = \{0^{++}, 2^{++}, 0^{-+}, 1^{+-}, \dots\}$ channels, derive the quadratic mixing Lagrangian from the constitutive dependence of the Yang–Mills kernel on (Φ, χ^μ), obtain the pole equations from two–point correlators, and give sum rules and dimensionless benchmarks in units of $\sqrt{\sigma_f}$ (Sec. 10.2). All constructions are fully gauge invariant and constrained by the S/G/TQ/D gates.

1. Glueball operator basis and spectral composites

Gluonic primaries. Let $G_{\mu\nu} = G^a_{\mu\nu} t^a$ be the color field strength. The standard gauge–invariant local primaries are

$$\mathcal{O}_S := \operatorname{Tr} G_{\mu\nu} G^{\mu\nu} \qquad (0^{++}), \tag{12.1}$$

$$\mathcal{O}_P := \operatorname{Tr} G_{\mu\nu} \tilde{G}^{\mu\nu}, \quad \tilde{G}^{\mu\nu} := \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma} \qquad (0^{-+}), \tag{12.2}$$

$$\mathcal{O}^{(2)}_{\mu\nu} := \operatorname{Tr}(G_{\mu\alpha} G_\nu{}^\alpha) - \tfrac{1}{4}\, g_{\mu\nu}\, \mathcal{O}_S \qquad (2^{++}). \tag{12.3}$$

Spectral composites. The Yang–Mills stiffness enters through the constitutive factor $Z_{1,c}(\mathcal{S})$ (Sec. 10.1), writing

$$\mathcal{L}_{\rm YM} \;=\; -\frac{1}{4}\, Z_{1,c}(\Phi, \rho, \eta, \chi^\mu, \dots)\, \operatorname{Tr} G_{\mu\nu} G^{\mu\nu}. \tag{12.4}$$

Expanding around a homogeneous background $(\bar{\Phi}, \bar{\chi}^\mu, \dots)$,

$$Z_{1,c} \;=\; Z^{(0)}_{1,c} \Big[1 + \alpha_\Phi\, \varphi + \tfrac{1}{2} \beta_\Phi\, \varphi^2 + \alpha_\chi\, (\chi \cdot u)^2 + \gamma_\chi\, \chi_\mu \chi_\nu \Pi^{\mu\nu} + \cdots \Big], \quad \varphi := \Phi - \bar{\Phi}, \tag{12.5}$$

where u^μ selects the aether rest frame and $\Pi^{\mu\nu} := g^{\mu\nu}_{\rm eff} - u^\mu u^\nu$ projects to the spatial slice. Equation (12.4) then induces gauge–invariant *composite* operators built from (φ, χ^μ) that

transform with the same J^{PC} as the gluonic primaries:

$$S_{\rm spec} := \varphi, \qquad (0^{++}), \tag{12.6}$$

$$P_{\rm spec} := \epsilon_{\mu\nu\rho\sigma}\,\partial^\mu\chi^\nu\partial^\rho\chi^\sigma \quad \text{or } \partial_\mu J_5^\mu[\Phi]\ , \qquad (0^{-+}), \tag{12.7}$$

$$T^{\rm spec}_{\mu\nu} := \nabla_{\langle\mu}\chi_{\nu\rangle}\ , \qquad (2^{++}), \tag{12.8}$$

with $\langle\cdots\rangle$ indicating the symmetric tracefree (STF) projector using $g_{\rm eff}$.

2. Quadratic mixing Lagrangian and J^{PC} projectors

Mixing structure. To leading order in fluctuations the parity–even sector reads

$$\mathcal{L}_{0^{++},\,2^{++}} = -\frac{1}{4}\,Z^{(0)}_{1,c}\,\mathrm{Tr}\,G^2 - \frac{1}{2}\,s\,\mathcal{O}_S + \frac{1}{2}\,(\partial s)^2 - \frac{1}{2}m_s^2 s^2 + \frac{1}{2}\,T^{\mu\nu}\,\mathcal{O}^{(2)}_{\mu\nu} + \mathcal{L}_{\rm kin}[T] - \frac{1}{2}m_T^2\,T_{\mu\nu}T^{\mu\nu}, \tag{12.9}$$

where s is the canonically normalized scalar built from $S_{\rm spec}$ and $T_{\mu\nu}$ from $T^{\rm spec}_{\mu\nu}$; the linear couplings

$$s\,\mathcal{O}_S \ \propto\ \alpha_\Phi\,\varphi\,\mathrm{Tr}\,G^2, \qquad T^{\mu\nu}\,\mathcal{O}^{(2)}_{\mu\nu} \ \propto\ \gamma_\chi\,\nabla_{\langle\mu}\chi_{\nu\rangle}\ \mathrm{Tr}(G_\mu{}^\alpha G_{\nu\alpha}), \tag{12.10}$$

arise directly from (12.5). Parity–odd mixing involves

$$\mathcal{L}_{0^{-+}} = -\frac{1}{2}\,p\,\mathcal{O}_P + \frac{1}{2}(\partial p)^2 - \frac{1}{2}m_p^2 p^2 + \kappa_\theta\,p\,\partial_\mu K^\mu, \tag{12.11}$$

where p is the pseudoscalar built from $P_{\rm spec}$ and K^μ is the Chern–Simons current with $\partial_\mu K^\mu = \mathcal{O}_P$. The last term encodes the axial anomaly alignment.

Spin–2 projector and kinetic term. Working on the $g_{\rm eff}$ background, the transverse traceless (TT) projector for a symmetric tensor is

$$\mathcal{P}^{\rm TT}_{\mu\nu,\rho\sigma}(q) = \frac{1}{2}\,(\Pi_{\mu\rho}\Pi_{\nu\sigma} + \Pi_{\mu\sigma}\Pi_{\nu\rho}) - \frac{1}{3}\Pi_{\mu\nu}\Pi_{\rho\sigma}, \qquad \Pi_{\mu\nu} = g_{{\rm eff},\mu\nu} - \frac{q_\mu q_\nu}{q^2}. \tag{12.12}$$

The quadratic kinetic operator of $T_{\mu\nu}$ is chosen such that only TT modes propagate (no ghosts), consistent with the (G) gate.

3. Correlators, pole equations, and spectral decomposition

Matrix of two–point functions. Introduce the operator vectors

$$\mathbb{O}_{0^{++}} = (\mathcal{O}_S,\ s), \qquad \mathbb{O}_{0^{-+}} = (\mathcal{O}_P,\ p), \qquad \mathbb{O}_{2^{++}} = (\mathcal{O}^{(2)},\ T).$$

Define the momentum–space correlators $\Pi_{ab}(q^2) = i\int d^4x\, e^{iqx}\langle 0|T\{\mathbb{O}_a(x)\mathbb{O}_b(0)\}|0\rangle$. To one–particle–irreducible (1PI) order the inverse propagators take the form

$$\Delta^{-1}_{J^{PC}}(q^2) \ =\ \begin{pmatrix} \Pi_{gg}(q^2) & \lambda \\ \lambda & q^2 - m_X^2 - \Sigma_{XX}(q^2) \end{pmatrix}, \qquad X \in \{s, p, T\}, \tag{12.13}$$

where Π_{gg} is the pure–glue polarization in the chosen channel, Σ_{XX} is the self–energy of the spectral mode, and λ is the mixing vertex inherited from (12.10)–(12.11). Poles (glueball eigenmasses) solve

$$\det\Delta^{-1}_{J^{PC}}(q^2)\Big|_{q^2=m^2_{J^{PC}}} = 0\ . \tag{12.14}$$

Källén–Lehmann form and residues. The correlators admit

$$\Pi_{ab}(q^2) \;=\; \int_0^\infty ds\,\frac{\rho_{ab}(s)}{s-q^2-i0^+}\,, \qquad \rho_{ab}(s) \;=\; \sum_n 2\pi\,\delta(s-m_n^2)\,Z_a^{(n)}Z_b^{(n)}+\rho_{ab}^{\rm cont}(s), \tag{12.15}$$

with *positive* diagonal densities (TQ gate). Near a narrow pole,

$$\Pi_{gg}(q^2) \;\sim\; \frac{(f_g^{(n)})^2}{m_n^2-q^2}, \qquad f_g^{(n)} := \langle 0|\,\mathcal{O}_{\rm glue}\,|n\rangle, \tag{12.16}$$

and analogously for s,p,T. The mixing rotates the primaries into mass eigenstates with glue/spectral fractions fixed by the eigenvectors of (12.13).

4. Scalar and pseudoscalar channels: trace and axial anomalies

Scalar (0^{++}). The trace anomaly relates $\mathcal{O}_S$ to the stress tensor:

$$\Theta^\mu{}_\mu \;=\; \frac{\beta(g_s)}{2g_s}\,\mathcal{O}_S+\cdots \tag{12.17}$$

(see Sec. 10.3), implying a nonzero vacuum form factor $\langle 0|\mathcal{O}_S|0\rangle\neq 0$. The scalar glueball 0^{++} dominantly couples to $\mathcal{O}_S$ but can mix with $s\sim\varphi$ through α_Φ in (12.5). At the level of (12.13) this is (Π_{gg},λ,m_s) with typical level repulsion:

$$m_\pm^2 \;\simeq\; \frac{1}{2}\Big[m_g^2+m_s^2 \;\pm\; \sqrt{(m_g^2-m_s^2)^2+4\lambda^2}\,\Big], \tag{12.18}$$

where m_g is the unmixed gluonic mass scale extracted from Π_{gg}.

Pseudoscalar (0^{-+}) and topological susceptibility. The divergence of the Chern–Simons current yields

$$\chi_t \;:=\; \int d^4x\,\langle 0|\,T\{\mathcal{O}_P(x)\mathcal{O}_P(0)\}\,|0\rangle \;=\; \Pi_{PP}(0), \tag{12.19}$$

the topological susceptibility. Coupling to p in (12.11) allows a light pseudoscalar admixture only if the spectral sector carries nontrivial parity–odd elasticity. In full QCD, χ_t feeds the Witten–Veneziano relation; in the pure gauge sector it sets the scale of the 0^{-+} glueball and the θ–dependence of the vacuum energy.

5. Spin–2 channel and emergent tensor elasticity

Tensor glueball (2^{++}). The correlator of $\mathcal{O}^{(2)}_{\mu\nu}$ projects on TT modes via (12.12). A spectral tensor $T_{\mu\nu}$ arises from anisotropic phase elasticity (gradients of χ^μ) and couples linearly to $\mathcal{O}^{(2)}$ as in (12.10). Hyperbolicity requires the kinetic operator to propagate exactly five TT degrees of freedom with luminal/sub-luminal speed inside the $g_{\rm eff}$ cone.

6. Sum rules, moments, and Borel analysis

Moments and Borel sum rules. Define Euclidean moments $M_n(Q^2)=\frac{1}{n!}\big(\frac{d}{dQ^2}\big)^n\Pi(Q^2)$ and the Borel transform

$$\widehat{\Pi}(\mathcal{M}^2) \;:=\; \lim_{\substack{Q^2,n\to\infty\\ Q^2/n=\mathcal{M}^2}} \frac{(Q^2)^n}{(n-1)!}\,M_{n-1}(Q^2) \;=\; \int_0^\infty ds\,e^{-s/\mathcal{M}^2}\,\rho(s). \tag{12.20}$$

In a "pole + continuum" model,

$$\widehat{\Pi}(\mathcal{M}^2) \;\simeq\; f^2\, e^{-m^2/\mathcal{M}^2} + \int_{s_0}^{\infty} ds\, e^{-s/\mathcal{M}^2}\, \rho_{\rm pert}(s), \tag{12.21}$$

where $\rho_{\rm pert}$ is the perturbative cut and s_0 the continuum threshold. The *spectral* input enters through condensates and the correlator $\mathcal{D}_E$ controlling IR power corrections (Sec. 10.2).

7. Linking masses to the confining scale

Dual mass and correlation length. In Sec. 10.2 we found the dual 2-form mass $m_B = \xi_c^{-1}$ and $\sigma_f \sim g_s^2 \mathcal{D}_0 \xi_c^2$. Dimensional analysis and lattice–inspired spectra suggest

$$m_{0^{++}} \;=\; c_S\, m_B, \qquad m_{2^{++}} \;=\; c_T\, m_B, \qquad m_{0^{-+}} \;=\; c_P\, m_B, \tag{12.22}$$

with $c_{S,T,P} = \mathcal{O}(1)$ calculable from the eigenvalue problem (12.14). Equivalently, in units of $\sqrt{\sigma_f}$,

$$R_S := \frac{m_{0^{++}}}{\sqrt{\sigma_f}}\,, \quad R_T := \frac{m_{2^{++}}}{\sqrt{\sigma_f}}\,, \quad R_P := \frac{m_{0^{-+}}}{\sqrt{\sigma_f}}\,, \tag{12.23}$$

are *dimensionless benchmarks* that weakly depend on mixing. The spectral–lattice program of Sec. 12.3 targets these ratios directly.

8. Mixing with isoscalar mesons (unquenching)

Scalar mixing. With dynamical light quarks, the isosinglet scalar quarkonium operator $S_q = \bar{q}q$ mixes with $\mathcal{O}_S$ and s:

$$\mathbb{O}_{0^{++}} = (\mathcal{O}_S,\; s,\; S_q), \qquad \Delta^{-1} = \begin{pmatrix} \Pi_{gg} & \lambda_{gs} & \lambda_{gq} \\ \lambda_{gs} & q^2 - m_s^2 - \Sigma_{ss} & \lambda_{sq} \\ \lambda_{gq} & \lambda_{sq} & q^2 - m_q^2 - \Sigma_{qq} \end{pmatrix}. \tag{12.24}$$

Trace–anomaly dominance guides the glue fraction, while GMOR/PCAC (Sec. 11.2) controls the quarkonium components.

Pseudoscalar sector. Anomalous mixing relates 0^{-+} glue to η, η'; the topological susceptibility χ_t and the WZW term (Sec. 11.4) constrain the mixing angles.

9. Finite temperature and screening masses

Center symmetry and deconfinement. As $T \to T_c^-$, center symmetry restores and the correlation length changes: $\mathcal{D}_E(\zeta) \to$ longer range for spacelike correlators while timelike confinement melts. Screening masses in Euclidean spatial correlators remain finite above T_c; the spectral mapping predicts

$$m^{\rm scr}_{0^{++}}(T) \;\downarrow, \qquad m^{\rm scr}_{2^{++}}(T) \;\downarrow, \qquad \lim_{T\to T_c^-} \frac{m^{\rm scr}_{J^{PC}}(T)}{\sqrt{\sigma_s(T)}} \;\to\; \text{const}, \tag{12.25}$$

where σ_s is the spatial string tension (Sec. 10.2).

10. Positivity, hyperbolicity, and consistency gates

(TQ) positivity. Diagonal spectral densities are nonnegative, ensuring real masses and widths; the mixing matrix preserves positivity if $|\lambda|^2 \le \Pi'_{gg}(m^2)\,\Sigma'_{XX}(m^2)$.

(G) hyperbolicity. Kinetic terms of s, p, T and of gluonic composites must share the $g_{\rm eff}$ cone. Anisotropies from χ^μ are allowed only through higher–derivative, subluminal operators; the TT projector (12.12) guarantees causal tensor propagation.

(S)/(D) stability. Mass terms $m^2_{s,p,T} > 0$ and the absence of negative–norm states (no longitudinal tensors) ensure bounded energy; dissipative couplings may broaden poles but cannot generate tachyons.

11. Practical extraction strategy

Spectral–lattice pipeline.

1. Measure gauge–invariant correlators of $\mathcal{O}_S, \mathcal{O}_P, \mathcal{O}^{(2)}$ and of spectral composites $S_{\rm spec}, P_{\rm spec}, T^{\rm spec}$ on spectral–lattice ensembles with fixed σ_f.
2. Build the correlator matrices $\Pi_{ab}(\tau)$ in Euclidean time; perform a generalized eigenvalue analysis (GEVP) to extract $m_{J^{PC}}$ and overlaps $Z_a^{(n)}$.
3. Scan the mixing strength by varying background $(\bar\Phi, \bar\chi^\mu)$ to map (c_S, c_T, c_P) in (12.22); report the ratios $R_{S,T,P}$ in (12.23) with continuum/chiral extrapolations.

12. Summary

Glueballs in the spectral model are *hybrid* bound modes of gluonic primaries and spectral composites sourced by the constitutive dependence of the Yang–Mills kernel on (Φ, χ^μ). A gauge–invariant mixing Lagrangian leads to a coupled–channel pole problem whose solutions yield the $0^{++}, 2^{++}, 0^{-+}$ spectra and glue/spectral fractions. Masses scale with the dual mass $m_B = \xi_c^{-1}$ and with $\sqrt{\sigma_f}$, providing dimensionless benchmarks insensitive to scheme choices. Anomalies constrain scalar and pseudoscalar channels via trace and topological susceptibilities. All constructions satisfy positivity and hyperbolicity gates and are amenable to spectral–lattice validation.

12.2 Monopoles, vortices, and confinement mechanisms

Aim. We develop a gauge–invariant, spectral description of magnetic monopoles and center vortices and show how their condensates realize confinement via the dual Meissner effect. We (i) define non–Abelian monopoles through an invariant 't Hooft tensor built from a color–direction field derived from the internal fiber, (ii) write the dual effective theory whose Abrikosov–Nielsen–Olesen (ANO) vortices are the confining flux tubes, (iii) formalize center vortices and their algebra with Wilson loops, (iv) relate monopole worldlines to vortex worldsheets, and (v) connect all scales to the spectral parameters m_B, σ_f, and the correlators of Secs. 10.2–10.3. Consistency is checked against the S/G/TQ/D gates.

1. Topological charges and gauge–invariant monopoles

Color direction from the fiber. Let $\mathfrak{n}(x)$ be an adjoint, unit–norm color director extracted from the internal spectral fiber along the $SU(3)$ isometries (Sec. 10.1), e.g.

$$\mathfrak{n}(x) \;:=\; \frac{Y(x)}{\sqrt{2\,\mathrm{Tr}[Y(x)^2]}} \;\in\; \mathfrak{su}(3), \qquad Y(x)=y^I(x)\,T_I \;\;\text{(adjoint)}. \tag{12.26}$$

Under G–gauge transformations $\mathfrak{n}\mapsto U\mathfrak{n}U^\dagger$.

't Hooft tensor and magnetic current. Define the (gauge–invariant) Abelian 2–form ('t Hooft tensor)

$$\mathcal{F}^m_{\mu\nu}[\mathfrak{n}] \;:=\; \mathrm{Tr}(\mathfrak{n}\,G_{\mu\nu}) - \frac{i}{g_s}\,\mathrm{Tr}(\mathfrak{n}\,[D_\mu\mathfrak{n}, D_\nu\mathfrak{n}])\,, \tag{12.27}$$

and the magnetic current

$$k^\mu \;:=\; \partial_\nu\,\tilde{\mathcal{F}}^{m\,\mu\nu}, \qquad \tilde{\mathcal{F}}^{m\,\mu\nu}=\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^m_{\rho\sigma}. \tag{12.28}$$

Then $\partial_\mu k^\mu = 0$ identically; nonzero k^μ signals monopole worldlines. For $SU(N)$, magnetic charges take values in the coroot lattice,

$$\vec{g}_m \;=\; \frac{4\pi}{g_s}\,\vec{\alpha}^{\,\vee}, \qquad \vec{\alpha}^{\,\vee}=\frac{2\vec{\alpha}}{\vec{\alpha}^{\,2}}, \tag{12.29}$$

consistent with Dirac quantization against all electric weights. The construction is projection–independent because $\mathfrak{n}$ is a bona fide field tied to the fiber section $y(x)$.

2. Dual Meissner effect and the dual Abelian–Higgs theory

Dual effective Lagrangian. In the infrared (IR) the confining dynamics is captured by a dual $U(1)^{N-1}$ Abelian–Higgs system for magnetic condensates ϕ_m minimally coupled to dual gauge fields C^i_μ (Cartan basis) or, in a fully gauge–invariant 2–form language, by a massive Kalb–Ramond field $B_{\mu\nu}$ (Sec. 10.2). A convenient form (single $U(1)$ for clarity) is

$$\mathcal{L}_{\text{dual}} \;=\; \frac{1}{4}\,\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + |(\partial_\mu + ig_m C_\mu)\phi_m|^2 - \lambda_m\left(|\phi_m|^2 - v_m^2\right)^2, \qquad \tilde{F}_{\mu\nu}=\partial_\mu C_\nu - \partial_\nu C_\mu, \tag{12.30}$$

with g_m the magnetic coupling, $g_m g_s = 4\pi$ (Dirac). Spontaneous breaking $\langle\phi_m\rangle = v_m$ gives the dual photon a mass

$$m_B \;=\; g_m v_m, \qquad m_H^{(\text{dual})}=\sqrt{2\lambda_m}\,v_m, \qquad \kappa \;:=\; \frac{m_H^{(\text{dual})}}{\sqrt{2}\,m_B}, \tag{12.31}$$

where κ is the (dual) Ginzburg–Landau parameter (type–II if $\kappa > 1/\sqrt{2}$).

Flux tubes and string tension. The ANO vortex with n quanta of center flux carries electric (original) flux by duality. Its tension obeys

$$\sigma_n \;=\; 2\pi n\,v_m^2\,\mathcal{F}(\kappa), \qquad \mathcal{F}(\tfrac{1}{\sqrt{2}})=1 \;\;\text{(BPS)}, \quad \mathcal{F}(\kappa)\gtrsim 1. \tag{12.32}$$

Using $v_m = m_B/g_m$,

$$\boxed{\sigma_n \;=\; \frac{2\pi n}{g_m^2}\, m_B^2\, \mathcal{F}(\kappa) \;=\; \frac{n\, g_s^2}{8\pi}\, m_B^2\, \mathcal{F}(\kappa), \qquad g_m = \frac{4\pi}{g_s}.} \tag{12.33}$$

Identifying $m_B = \xi_c^{-1}$ with the inverse correlation length of the color–electric correlator $\mathcal{D}_E$ and recalling $\sigma_f \sim g_s^2 \mathcal{D}_0 \xi_c^2$ (Sec. 10.2), Eq. (12.33) matches the scaling $\sigma_f \propto g_s^2 m_B^{-2}$ up to a calculable constant absorbed in $\mathcal{F}(\kappa)$ and $\mathcal{D}_0$.

2–form formulation. Equivalently,

$$\mathcal{L}_B \;=\; \frac{1}{12 g_B^2} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{m_B^2}{4}\, B_{\mu\nu} B^{\mu\nu} + i\,\pi\, B_{\mu\nu}\, \Sigma^{\mu\nu}, \qquad H = dB, \quad \partial_\mu \Sigma^{\mu\nu} = 0, \tag{12.34}$$

where $\Sigma^{\mu\nu}$ is the conserved vortex worldsheet current. Integrating out B produces an area law for Wilson loops with tension $\sigma_f \propto m_B^2/g_B^2$ (cf. Sec. 10.2).

3. Center vortices, N–ality, and 't Hooft algebra

Center flux quantization. For $SU(N)$ the nontrivial topology $\pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N$ allows vortices whose holonomy lies in the center. If a Wilson loop $W_R(C)$ in representation R links a vortex of type $k \in \{1, \dots, N-1\}$, it acquires the center phase

$$W_R(C) \;\longrightarrow\; z_k^{\nu_R}\, W_R(C), \qquad z_k = e^{2\pi i k/N}, \quad \nu_R = N\text{–ality(R)}. \tag{12.35}$$

't Hooft loop. Define the disorder operator $V_k(C)$ creating a thin center vortex along C. It obeys the algebra

$$V_k(C)\, W_R(C') \;=\; z_k^{\nu_R\, L(C,C')}\, W_R(C')\, V_k(C), \tag{12.36}$$

with $L(C, C')$ the linking number. In the *confining* phase, $\langle W_R \rangle \sim e^{-\sigma_{\nu_R} \mathcal{A}}$ (area law) while $\langle V_k \rangle$ shows a perimeter law; in the Higgs phase the roles swap. Random percolation of vortices in 4D yields the area law with σ_{ν_R} depending only on N–ality (Casimir vs. sine–law patterns discussed below).

4. Monopole–vortex networks and non–Abelian flux

Chains and junctions. In Abelian projections $SU(N) \to U(1)^{N-1}$, center vortices decompose into segments of Cartan flux joined at monopoles carrying charges in the coroot lattice. Gauge–invariantly, this is encoded by the identity

$$\partial_\mu \tilde{\mathcal{F}}^{m\,\mu\nu} \;=\; k^\nu \;=\; \frac{2\pi}{g_s}\, \partial_\mu \Sigma^{\mu\nu} \quad\Longrightarrow\quad \partial \Sigma = \frac{g_s}{2\pi}\, k, \tag{12.37}$$

so monopole worldlines are boundaries of vortex worldsheets (or vice versa in the dual picture). For $SU(3)$, Y–junctions realize color flux conservation at monopole sites.

k–strings. For sources of N–ality k, flux tubes ("k–strings") have tensions

$$\sigma_k/\sigma_1 \;\approx\; \begin{cases} \dfrac{k(N-k)}{N-1} & \text{(Casimir scaling, intermediate distances),} \\ \dfrac{\sin(\pi k/N)}{\sin(\pi/N)} & \text{(sine law, asymptotic, thin vortices).} \end{cases} \tag{12.38}$$

In the spectral framework the crossover pattern is governed by κ and the mixing of Cartan components in (12.30); spectral–lattice measurements of σ_k/σ_1 at fixed m_B and κ provide incisive diagnostics.

5. Lattice/continuum dictionary and spectral diagnostics

Monopoles. On the lattice, DeGrand–Toussaint currents $k^\mu_{\rm lat}$ detect violations of Bianchi identities in Abelian–projected plaquettes; in the continuum, k^μ in (12.28) is gauge–invariant thanks to $\mathfrak{n}(x)$. The monopole condensate/disorder parameter is the expectation value of a monopole–creation operator $\langle\mu\rangle$, nonzero in the confining phase.

Vortices. Center projection identifies P–vortices whose density and percolation correlate with confinement. Gauge–invariantly, the 2–form coupling $i\pi B\wedge\Sigma$ (Eq. (12.34)) provides a direct handle on vortex worldsheets; the surface susceptibility of Σ diagnoses percolation.

String observables. From (12.34) one obtains the flux–tube width $w\sim\lambda=1/m_B$ and the Lüscher term $-\pi/(12\,L)$ in the long–string potential; subleading corrections are governed by κ.

6. Quantitative links to spectral parameters

Tension, width, and jet power corrections. Combining (12.33) with Sec. 10.4 gives

$$\sqrt{\sigma_f}\;\sim\;\frac{g_s}{\sqrt{8\pi}}\,m_B\,\sqrt{\mathcal{F}(\kappa)},\qquad w\;\sim\;\frac{1}{m_B},\qquad \Omega_1\;=\;\kappa_e\sqrt{\sigma_f}\;\text{ (Eq. (10.53)).} \tag{12.39}$$

Thus collider–extracted Ω_1 and lattice–extracted w jointly determine (m_B,κ), overconstraining the confining sector.

Positivity constraints. $\sigma_f>0$ requires $m_B^2>0$ and $\mathcal{F}(\kappa)>0$; spectral positivity of $\mathcal{D}_E$ enforces both (Sec. 10.2). Hyperbolicity demands the dual modes propagate within the $g_{\rm eff}$ cone, i.e. no superluminal B– or C_μ–excitations.

7. Consistency gates (monopole–vortex sector)

(S) stability. The dual mass m_B and Higgs mass $m_H^{(\rm dual)}$ are positive; the energy of vortex/monopole configurations is bounded from below (BPS bound at $\kappa=1/\sqrt{2}$).

(G) hyperbolicity/causality. Kinetic terms $(\tilde F)^2$ and H^2 share the $g_{\rm eff}$ principal symbol; the massive spectrum $(m_B,m_H^{(\rm dual)})$ ensures subluminal group velocities.

(TQ) unitarity/positivity. The disorder operators V_k are unitary; spectral densities of $\mathcal{F}^m$ and B–propagators are positive. The 't Hooft algebra (12.36) is anomaly–free.

(D) dissipation. Dissipative couplings of the medium cannot break the topological conservation $\partial_\mu k^\mu = 0$ nor $\partial_\mu \Sigma^{\mu\nu} = 0$; they at most broaden flux–tube spectral lines.

8. Summary

Non–Abelian monopoles arise as gauge–invariant defects defined by the 't Hooft tensor built from a fiber–induced color director $\mathfrak{n}$. Their condensation realizes a dual superconductor whose ANO vortices (or, equivalently, massive $B_{\mu\nu}$ strings) confine color, with string tension $\sigma_f \propto g_s^2 m_B^2$ up to the function $\mathcal{F}(\kappa)$. Center vortices implement N–ality via the 't Hooft algebra, and monopole worldlines act as boundaries of vortex worldsheets, assembling confining networks. All scales (σ_f, w, m_B, κ) are tied to spectral correlators, enabling quantitative, overconstrained tests through lattice and collider observables.

12.3 Spectral–lattice simulations and validation

Aim. We formulate a *spectral–lattice* program that couples non–Abelian gauge fields to the spectral medium $(\Phi, \chi^\mu, \ldots)$ in a gauge–invariant, reflection–positive Euclidean discretization. We (i) define lattice actions for the gauge links, the spectral scalars/vectors, and the dual two–form; (ii) construct observables for glueballs, flux tubes, monopoles/vortices, and topological susceptibilities; (iii) give step–scaling/continuum procedures and operator–basis variational methods; (iv) provide a dictionary to extract the spectral IR parameters $(\sigma_f, m_B, \xi_c, \kappa)$ and the dimensionless ratios introduced in Sec. 12.1; (v) state validation and reproducibility protocols consistent with the S/G/TQ/D gates.

1. Euclidean discretization and fields

Lattice and links. Work on a hypercubic lattice Λ with spacing a and volume $N_s^3 \times N_t$; unit vectors $\hat{\mu}$ label directions $\mu = 1, \ldots, 4$ with Euclidean metric $\delta_{\mu\nu}$. Gauge fields are links $U_\mu(x) \in SU(3)$ on (x, μ) and plaquettes $U_{\mu\nu}(x)$.

Spectral site fields. Introduce a real scalar fluctuation $\varphi(x) = \Phi(x) - \bar{\Phi}$ and a *director* $\chi_\mu(x)$ constrained by

$$\sum_{\mu=1}^{4} \chi_\mu(x)\chi_\mu(x) \;=\; 1, \qquad \chi_4(x) \geq 0, \tag{12.40}$$

enforced with a Lagrange multiplier or soft penalty (see below). An anisotropy parameter $\xi = a_s/a_t$ can be used to emulate the effective cone g_{eff} (Sec. 10.2).

2. Gauge action with spectral stiffness

Constitutive coupling. The continuum constitutive factor $Z_{1,c}(\Phi,\chi)$ in Eq. (12.4) becomes a local plaquette weight. With a tree–level Symanzik improvement,

$$S_g[U,\varphi,\chi] \;=\; \beta\sum_x\Bigg\{c_0\sum_{\mu<\nu} w_{\mu\nu}(x)\Big[1-\tfrac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,U_{\mu\nu}(x)\Big] \\ +\; c_1\sum_{\mu\neq\nu} w_{\mu\nu}(x)\Big[1-\tfrac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,U_{\mu,\mu\nu}(x)\Big]\Bigg\}, \tag{12.41}$$

with $c_0+8c_1=1$ ($c_1=-1/12$ for LW) and rectangle $U_{\mu,\mu\nu}$. The spectral weight expands as

$$w_{\mu\nu}(x) \;=\; 1+\alpha_\Phi\,\varphi(x)+\tfrac{1}{2}\beta_\Phi\,\varphi(x)^2+\gamma_\chi\Big[\chi_\mu(x)\chi_\mu(x)+\chi_\nu(x)\chi_\nu(x)-\tfrac{1}{2}\Big]+\cdots, \tag{12.42}$$

ensuring gauge invariance and reflection positivity for small α_Φ,γ_χ (positivity gate).

Spectral sector. Discretize φ with a nearest–neighbor kinetic term and stabilizing potential,

$$S_\varphi \;=\; \sum_x\left\{\frac{\kappa_\varphi}{2}\sum_\mu\Big[\varphi(x+\hat\mu)-\varphi(x)\Big]^2 \;+\; \frac{m_\varphi^2}{2}\,\varphi(x)^2+\frac{\lambda_\varphi}{4!}\,\varphi(x)^4\right\}, \tag{12.43}$$

and the director with an $O(4)$ sigma–model action plus constraint

$$S_\chi \;=\; \frac{\kappa_\chi}{2}\sum_{x,\mu}\Big[\chi_\nu(x+\hat\mu)-\chi_\nu(x)\Big]^2 \;+\; \frac{\zeta}{2}\sum_x\Big[\chi_\nu(x)\chi_\nu(x)-1\Big]^2. \tag{12.44}$$

The $\zeta\to\infty$ limit enforces (12.40); finite ζ implements a soft constraint.

3. Dual two–form on the lattice (Kalb–Ramond)

Cochain formulation. Place a compact two–form B on plaquettes of the *dual* lattice $\Lambda^\star$, with lattice exterior derivative d (coboundary) and Hodge star $\star$. Let $H=dB$ be the three–form on dual cubes. The gauge symmetry is $B\to B+d\Lambda$ (with Λ a dual 1–form).

Action and vortex coupling. A massive two–form discretization reads

$$S_B \;=\; \frac{1}{2g_B^2}\sum_{\text{dual cubes } c^\star} H(c^\star)^2 \;+\; \frac{m_B^2}{2}\sum_{\text{dual plaquettes } p^\star} B(p^\star)^2 \;+\; i\pi\sum_{p^\star} B(p^\star)\,\Sigma(p^\star), \tag{12.45}$$

where Σ is an *integer* dual 2–form marking center–vortex worldsheets (closed: $d\Sigma=0$). Integrating B yields an area law with $\sigma_f\propto m_B^2/g_B^2$, cf. Eq. (12.34).

4. Observables and measurement protocols

(i) String tension σ_f. Measure rectangular Wilson loops $W(R,T)$ and Creutz ratios

$$\chi(R,T) \;:=\; -\ln\frac{\langle W(R,T)\,W(R-1,T-1)\rangle}{\langle W(R,T-1)\rangle\langle W(R-1,T)\rangle} \;\xrightarrow[R,T\to\infty]{}\; a^2\sigma_f, \tag{12.46}$$

with multilevel variance reduction (Lüscher–Weisz). Polyakov–loop correlators give σ_f at finite T; gradient flow improves signal.

(ii) Flux–tube width and m_B. Measure the longitudinally averaged action–density profile $E_\perp^2(\mathbf{r}_\perp)$ in the plane transverse to a static $Q\bar{Q}$ pair; fit to a Bessel–K_0 profile $E_\perp(r) \propto K_0(m_B r)$ to extract $m_B = \xi_c^{-1}$ and width $w \simeq 1/m_B$.

(iii) Glueball spectra. Build a variational basis $\{O_i^{J^{PC}}\}$ of gauge–invariant spatial Wilson loops (with smearing/distillation). Form the correlator matrix $C_{ij}(t) = \langle O_i(t) O_j(0) \rangle$ and solve the GEVP

$$C(t)\, v_n(t,t_0) \;=\; \lambda_n(t,t_0)\, C(t_0)\, v_n(t,t_0), \qquad m_n^{\mathrm{eff}}(t) \;=\; -\ln \frac{\lambda_n(t+1,t_0)}{\lambda_n(t,t_0)}. \tag{12.47}$$

Augment the basis with *spectral composites* (built from φ, χ) to map mixing (Sec. 12.1).

(iv) Monopoles and center vortices. Construct the lattice 't Hooft tensor using a smooth adjoint director $\mathfrak{n}(x)$ extracted from the fiber (smoothed via stout/gradient flow), then define the magnetic current

$$k_\mu(x + \tfrac{\hat{\mu}}{2}) \;:=\; \sum_{\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma}\, \Delta_\nu\, \mathcal{F}^m_{\rho\sigma}(x), \tag{12.48}$$

where Δ_ν is the forward difference. Vortex worldsheets are sampled either by center–projection diagnostics or directly via the integer field Σ coupled in (12.45). The 't Hooft algebra is checked by measuring phase jumps of W_R upon linking with Σ.

(v) Topological susceptibility. Compute

$$\chi_t \;=\; \frac{1}{V} \Big\langle \Big(\sum_x q(x) \Big)^2 \Big\rangle, \qquad q(x) \;=\; \frac{1}{32\pi^2}\, \epsilon_{\mu\nu\rho\sigma}\, \mathrm{Tr}\big[G_{\mu\nu}(x) G_{\rho\sigma}(x) \big], \tag{12.49}$$

with clover $G_{\mu\nu}$ after gradient flow to remove UV noise; compare with the pseudoscalar channel analysis (Sec. 12.1).

5. Extraction of spectral parameters and dimensionless ratios

Reference scales. Define flow scales t_0 or w_0 via $t^2\langle E \rangle|_{t_0} = 0.3$ (or similar). Alternatively use r_0 from the static potential or simply work in units of $\sqrt{\sigma_f}$.

Primary map.

$$\boxed{ m_B \;=\; w^{-1} \quad \text{(from flux–tube profile)}, \qquad \sigma_f \;=\; \lim_{R,T\to\infty} \frac{\chi(R,T)}{a^2}, \qquad \xi_c \;=\; m_B^{-1}. } \tag{12.50}$$

Then form the dimensionless benchmarks (Sec. 12.1)

$$R_S = \frac{m_{0^{++}}}{\sqrt{\sigma_f}}, \quad R_T = \frac{m_{2^{++}}}{\sqrt{\sigma_f}}, \quad R_P = \frac{m_{0^{-+}}}{\sqrt{\sigma_f}}, \tag{12.51}$$

and the dual GL parameter κ by fitting the ANO profile (or the $(m_H^{\mathrm{dual}}, m_B)$ dependence of σ_f), cf. Eq. (12.31)–(12.32).

Running and step–scaling. Define an *effective* coupling from the force scheme,

$$\alpha_{q\bar{q}}(r) \;:=\; \frac{r^2}{C_F}\,\frac{dV(r)}{dr}, \qquad V(r) = -\lim_{T\to\infty}\frac{1}{T}\ln W(r,T), \tag{12.52}$$

and its step–scaling $\sigma(u,s) = \alpha(s\,r)$ with $u = \alpha(r)$; compare to the spectral RG predictions of Ch. 3 including threshold gates.

6. Algorithms and variance reduction

HMC and over–relaxation. Evolve (U,φ,χ) with HMC: the plaquette force includes $w_{\mu\nu}(x)$ and its derivatives $\partial w/\partial\varphi$, $\partial w/\partial\chi$. Interleave heatbath/over–relaxation for links with local updates of (φ,χ). For the two–form, update B with Gaussian heatbath in momentum space (diagonal quadratic form) or via local Metroplis respecting $B \to B + d\Lambda$.

Multilevel and multigrid. Apply multilevel factorization for Wilson loops and correlators, GEVP with distillation or APE/HYP smearing to improve overlap, and multigrid solvers for extended operators. Autocorrrelations are monitored via τ_{int} of Q, σ_f, and $m_{0^{++}}$.

7. Systematics: continuum, volume, and anisotropy

Continuum limit. Simulate at $a \in \{a_1, a_2, a_3\}$ (fixed physical volume), extrapolating *ratios* $R_{S,T,P}$ and $m_B/\sqrt{\sigma_f}$ linearly or quadratically in a^2 (Symanzik). Reflection positivity of the action pieces (12.41)–(12.44) controls $O(a)$ artifacts.

Finite–volume effects. Use $L\sqrt{\sigma_f} \gtrsim 4$ and $T\sqrt{\sigma_f} \gtrsim 6$ for spectroscopy; check stability under L variations. For flux tubes, ensure $R \ll L/2$ and account for universal Lüscher correction $-\pi/(12\,R)$ when extracting σ_f.

Anisotropy tuning. If employing $a_t \neq a_s$, nonperturbatively tune ξ from the relativistic dispersion $E^2(\mathbf{p}) = m^2 + \mathbf{p}^2/\xi^2$ for the lowest glueball.

8. Cross–checks and consistency gates

(TQ) positivity/unitarity. Euclidean reflection positivity is preserved by each quadratic sector; correlator eigenvalues $\lambda_n(t,t_0)$ are positive and yield monotone effective masses. Topological susceptibility $\chi_t \geq 0$ after flow.

(G) hyperbolicity/causality. The anisotropic discretization reproduces the g_{eff} cone in the continuum; no superluminal two–form or director modes arise when $m_B^2 > 0$ and $\kappa_\chi > 0$.

(S)/(D) stability. Boundedness requires $\beta > 0$, $\kappa_{\varphi,\chi} > 0$, $m_B^2 > 0$, and $\zeta \geq 0$. Dissipative effects are absent in Euclidean dynamics; widths appear only as finite–volume/finite–a artifacts.

9. Validation checklist and reproducibility

Benchmarks.

1. Reproduce the pure–gauge ratios R_S, R_T in the decoupling limit $\alpha_\Phi = \gamma_\chi = 0$ (baseline).
2. Switch on $(\alpha_\Phi, \gamma_\chi)$ and verify stability of the continuum extrapolation and controlled mixing in $0^{++}/2^{++}$ channels.
3. Extract (σ_f, m_B) both from Wilson loops and from flux–tube widths; check consistency with Eq. (12.33).
4. Validate the 't Hooft algebra by measuring phase factors $z_k^{\nu_R}$ upon linking W_R with Σ.
5. Compare $R_{S,T,P}$ and χ_t with the coupled–channel pole analysis of Sec. 12.1.

Open data. Release gauge/link, spectral, and two–form ensembles; provide measurement scripts, random seeds, flow times, and raw correlator matrices $C_{ij}(t)$ (HDF5/NPZ) with YAML metadata for actions (12.41)–(12.45).

10. Summary

The spectral–lattice framework augments $SU(3)$ link dynamics with local constitutive couplings to (φ, χ) and, in dual variables, with a massive two–form B that captures confining flux. It delivers a coherent measurement pipeline for σ_f, $m_B = \xi_c^{-1}$, glueball spectra and mixing, monopole currents, vortex worldsheets, and χ_t, with systematic control via Symanzik improvement, multilevel variance reduction, and continuum/volume extrapolations. Dimensionless ratios $m_{J^{PC}}/\sqrt{\sigma_f}$ and $m_B/\sqrt{\sigma_f}$ provide robust benchmarks that overconstrain the confining sector and test the spectral model against first–principles simulations while respecting the S/G/TQ/D consistency gates.

12.4 Phenomenological handles

Aim. We assemble a set of *experiment-facing* observables that directly probe the confining sector of the spectral model: (1) glueball production and decay channels sensitive to the gluonic overlap; (2) string–tension extractions from hadron spectroscopy and Regge slopes; (3) collider handles linked to flux–tube dynamics and power corrections; (4) signatures tied to monopole/vortex networks; (5) a compact parameter map and "kill–criteria" for global fits. Throughout we express rates and ratios in terms of the spectral scales $\{\sigma_f, m_B, \kappa\}$ and mixing overlaps introduced in Secs. 12.1–12.3.

1. Glueball content and production filters

Gluonic overlap. For a mass eigenstate $|G_n\rangle$ in a given J^{PC} channel define

$$Z_g^{(n)} := \frac{|\langle 0|\, \mathcal{O}_{\rm glue}\, |G_n\rangle|}{\sqrt{\sum_a |\langle 0|\, \mathbb{O}_a\, |G_n\rangle|^2}} \in [0,1], \tag{12.53}$$

where $\mathbb{O}_a$ spans the coupled operator basis (glue + spectral composites), cf. Eq. (12.15). Z_g is basis–independent up to orthogonal rotations and serves as the *glue fraction*.

Radiative quarkonium decays. For $J/\psi,\ \Upsilon \to \gamma + X$ the short–distance amplitude filters gg–dominated states. Parametrically

$$\mathcal{B}\Big(J/\psi \to \gamma\, G_{0^{++}}\Big) \ \propto\ \alpha\, \alpha_s^2(m_c)\, \frac{|f_g^{(n)}|^2}{m_c^4} \ \times\ Z_g^{(n)2}, \qquad f_g^{(n)} := \langle 0|\, \mathcal{O}_S\, |G_n\rangle, \tag{12.54}$$

up to known kinematic factors. Enhanced branching with suppressed $\gamma\gamma$ width (below) is a glueball discriminator.

Central exclusive production (CEP). In $pp \to p + G_{J^{PC}} + p$ via double–Pomeron exchange the $J_z = 0$ selection favors $0^{++}, 2^{++}$ with rates scaling as

$$\sigma_{\rm CEP}(G_{J^{PC}}) \ \propto\ \mathcal{L}_{gg}^{\rm excl}(M_G, y)\, \Big|\, \mathcal{M}(gg \to G)\Big|^2 \ \propto\ \mathcal{L}_{gg}^{\rm excl}\ Z_g^2, \tag{12.55}$$

where $\mathcal{L}_{gg}^{\rm excl}$ is the exclusive gluon luminosity. The absence of leading $q\bar{q}$ backgrounds further sharpens glue sensitivity.

Two–photon couplings. Glueballs couple to $\gamma\gamma$ only via quark loops or mixing; for a scalar state

$$\Gamma(G_{0^{++}} \to \gamma\gamma) \ \simeq\ \sin^2\theta\, \Gamma(q\bar{q} \to \gamma\gamma) \ +\ \mathcal{O}(\alpha^2 \alpha_s^2), \qquad \sin^2\theta = 1 - Z_g^2, \tag{12.56}$$

so a small $\gamma\gamma$ width at fixed mass points to large Z_g.

Stickiness parameter. Define the dimensionless "stickiness"

$$S_G \ :=\ \frac{\Gamma(J/\psi \to \gamma G)\, {\rm Br}(G \to {\rm hadrons})}{\Gamma(G \to \gamma\gamma)} \ \propto\ \frac{Z_g^2}{\sin^2\theta}, \tag{12.57}$$

large for glue–dominated states. Eqs. (12.54)–(12.57) offer orthogonal handles on (Z_g, θ).

Trace–anomaly coupling and scalar decays. If the leading scalar couples as $\mathcal{L} \supset \frac{G}{F_\Theta}\, \Theta^\mu{}_\mu$, with $\Theta^\mu{}_\mu$ given by Eq. (12.17), then near chiral limit

$$\mathcal{A}(G \to \pi\pi) \ \propto\ \frac{m_\pi^2}{F_\Theta}, \qquad \Gamma(G \to \pi\pi) \ \sim\ \frac{3}{32\pi}\, \frac{m_G^3}{F_\Theta^2} \left(1 - \frac{4m_\pi^2}{m_G^2}\right)^{1/2}, \tag{12.58}$$

modulo SU(3) breaking and mixing. Fitting $\{\Gamma_{\pi\pi}, \Gamma_{KK}, \Gamma_{\eta\eta}\}$ constrains F_Θ and θ.

2. String tension from spectroscopy and Regge slopes

Regge trajectories. For mesons on leading trajectories

$$J \ \simeq\ \alpha_0 \ +\ \alpha'\, m^2, \qquad \boxed{\alpha' \ \simeq\ \frac{1}{2\pi\, \sigma_f}}, \tag{12.59}$$

so empirical α' determines σ_f model–independently. Baryon trajectories probe the Y–string energy $V_{3Q} \simeq \sigma_f L_Y +$ perimeter terms.

Heavy–quarkonia. The static potential $V(r) = \sigma_f r - \frac{\pi}{12r} + \cdots$ (including the universal Lüscher term) fixes level spacings. Using spin–averaged $1P$–$1S$ and $2S$–$1S$ splittings,

$$\sigma_f \;\simeq\; \frac{\Delta E_{n\ell}}{\langle r \rangle_{n\ell}} \quad \text{(Schrödinger fit)}, \tag{12.60}$$

with relativistic corrections controlled by short–distance Wilson coefficients; consistency with Eq. (12.59) is a stringent cross–check.

3. Flux tubes at colliders: power corrections and event shapes

Nonperturbative power correction. Event–shape distributions (e.g. thrust $\tau = 1 - T$) in e^+e^- admit leading power shifts

$$\langle \tau \rangle \;=\; \langle \tau \rangle_{\text{pQCD}} \;+\; \frac{C_\tau}{Q}\,\Omega_1 \;+\; \mathcal{O}\!\left(\frac{1}{Q^2}\right), \qquad \boxed{\Omega_1 \;=\; \kappa_e \sqrt{\sigma_f}} \;\text{(Sec. 10.4)}, \tag{12.61}$$

where Q is the hard scale and C_τ is calculable. A global fit to thrust, heavy–jet mass, and angularities maps $\Omega_1 \leftrightarrow \sqrt{\sigma_f}$, complementing spectroscopy.

Energy–energy correlators (EEC). At small angles $\chi \to 0$ the EEC probes Sudakov + NP physics; the first power correction scales with Ω_1 and thus with $\sqrt{\sigma_f}$. Joint fits of event shapes and EEC reduce hadronization scheme dependence.

Underlying event and rope hadronization. Multiplicity–dependent strangeness/baryon enhancement in high–multiplicity pp final states correlates with an *effective* string–bundle tension $\sigma_{\text{eff}} > \sigma_f$. The spectral picture predicts

$$\frac{\sigma_{\text{eff}}}{\sigma_f} \;\simeq\; 1 + \mathcal{O}\!\left(\rho_{\text{flux}}\,\kappa\right), \qquad \rho_{\text{flux}}:\ \text{overlap density of color flux tubes}, \tag{12.62}$$

linking soft–QCD systematics to (m_B, κ) inferred from flux–tube widths (Sec. 12.3).

4. Monopoles, vortices, and heavy–ion links

Screening and spatial tension at $T > 0$. In the deconfined plasma, the *spatial* string tension $\sigma_s(T)$ remains nonzero; the spectral mapping (Sec. 12.1) predicts

$$\frac{m^{\text{scr}}_{0^{++}}(T)}{\sqrt{\sigma_s(T)}} \;\approx\; \text{const. near } T_c^-, \tag{12.63}$$

testable via spatial correlators and Euclidean screening masses.

Monopole transport. Monopole worldline density controls the dual photon mass m_B and hence the flux–tube width $w \sim 1/m_B$. In heavy–ion collisions, the late–time hadronization radius distribution inherits w, offering an interferometric handle (two–particle correlations).

5. Compact parameter map and fit strategy

Primary parameters and benchmarks.

$$\mathcal{P}_{\rm conf} := \Big\{ \sigma_f,\ m_B,\ \kappa,\ Z_g^{(n)},\ \theta^{(n)} \Big\}, \qquad \mathcal{R} := \Big\{ R_{S,T,P},\ m_B/\sqrt{\sigma_f},\ \alpha',\ \Omega_1/\sqrt{\sigma_f} \Big\}, \tag{12.64}$$

with $R_{S,T,P}$ from Eq. (12.51). A minimal, overconstrained fit proceeds as:

1. Extract σ_f from Regge slope and/or heavy–quarkonia ((12.59), (12.60));
2. Determine m_B from flux–tube widths (Sec. 12.3); infer κ from ANO profiles;
3. Fit Z_g, θ to $\{\mathcal{B}(J/\psi \to \gamma X),\ \Gamma(\gamma\gamma),\ \text{CEP rates}\}$ ((12.54)–(12.57));
4. Cross–validate with $R_{S,T,P}$ and Ω_1 ((12.61)).

Predictive relations (no new dials). Given (σ_f, m_B, κ) fixed by *soft* data, the suite

$$\Big\{ \text{CEP cross sections},\ \text{radiative quarkonium rates},\ \text{two–photon widths},\ R_{S,T,P} \Big\} \tag{12.65}$$

is determined up to the glue fraction Z_g and mixing angle θ in each channel. This is a strict test of internal consistency.

6. Consistency gates and kill criteria (phenomenology)

(S) stability. Fits must yield $\sigma_f > 0$, $m_B > 0$, and $\kappa > 0$; negative or tachyonic inferences invalidate the sector.

(G) hyperbolicity. Inferred m_B and κ cannot imply superluminal dual or director modes; EFT cones must nest inside $g_{\rm eff}$.

(TQ) unitarity/positivity. Glueball spectral densities extracted from lineshapes are positive; anomalous channels obey WZW normalizations fixed by N_c (Part II).

(D) dissipation. Effective widths in soft observables must be explainable by hadronic rescattering or IR radiation, not by explicit breaking of topological conservation ($\partial_\mu k^\mu = 0$, $\partial_\mu \Sigma^{\mu\nu} = 0$).

Kill criteria. Any *simultaneous* violation of two or more relations among

$$\alpha' \simeq (2\pi\sigma_f)^{-1}, \quad m_B \simeq w^{-1}, \quad \Omega_1 \propto \sqrt{\sigma_f}, \quad \text{CEP} \ \propto Z_g^2, \quad \Gamma_{\gamma\gamma} \ \propto \sin^2\theta$$

within stated uncertainties constitutes a falsification of the minimal spectral confining sector.

7. Summary

Glue–sensitive production (radiative quarkonia, CEP) and suppressed $\gamma\gamma$ widths isolate the glue fraction Z_g and mixing θ of scalar/tensor candidates. String tension σ_f is fixed by Regge slopes and heavy–quarkonia, while flux–tube profiles determine m_B and κ. Collider event–shape power corrections provide an *independent* measurement of $\sqrt{\sigma_f}$ via Ω_1. Heavy–ion screening and interferometry add complementary thermal handles. A compact set of relations then overconstrains the parameter space, providing clear, quantitative tests of the spectral model's confining dynamics.

Part V

Flavor, Neutrinos, and CP Structure

Chapter 13

Generations from Topological Indices

13.1 Winding/Hopf indices as flavor labels

Aim. We construct *topological* labels for fermion generations from smooth spectral textures supported by the internal fiber fields introduced in Part I. Concretely, we define two integer-valued invariants on a compactified spatial slice $\Sigma \simeq S^3$: the *SU(2) winding number* $w[U] \in \mathbb{Z}$ of the phason map $U : \Sigma \to SU(2) \simeq S^3$, and the *Hopf invariant* $H[\hat{n}] \in \mathbb{Z}$ of a director map $\hat{n} : \Sigma \to S^2$ built from the aether director χ^μ (Vol. I) or a CP^1 submanifold of the fiber. We then show that the *mod-3* class of a specific linear combination,

$$\boxed{\mathfrak{f} \;\equiv\; H[\hat{n}] \;-\; w[U] \pmod 3\,, \qquad \mathfrak{f} \in \{0,1,2\},} \tag{13.1}$$

is gauge-invariant and conserved under admissible deformations (S/G/TQ/D gates), thus providing three superselection sectors naturally identified with the three observed generations. This mod-3 reduction is enforced by color center ($\mathbb{Z}_3$) structure and the quantized WZW inflow (Part II).

1. Compactification and spectral fields on S^3

Setup. Choose a constant aether-time hypersurface Σ in (M, g_{eff}) with the boundary condition that spectral fields approach a uniform vacuum at spatial infinity, allowing the one-point compactification $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$. On Σ we consider:

$$U :\ \Sigma \to SU(2) \simeq S^3, \qquad L_i := U^{-1}\partial_i U\ , \tag{13.2}$$

$$\hat{n} :\ \Sigma \to S^2, \qquad \hat{n}\cdot\hat{n} = 1, \quad \hat{n} = \hat{n}(\chi^\mu, \Phi; \mathcal{S}), \tag{13.3}$$

where $\hat{n}$ is a unit CP^1 director extracted from the fiber (e.g. normalized projection of χ^i onto the spatial slice or a bilinear of U via Hopf map, see below).

2. Topological invariants: degree and Hopf number

SU(2) winding (degree). The degree of U is

$$w[U] \;:=\; \frac{1}{24\pi^2}\int_{S^3} d^3x\, \epsilon^{ijk}\,\mathrm{Tr}\Big(L_i L_j L_k\Big) \;\in\; \mathbb{Z}. \tag{13.4}$$

Hopf invariant. Let ω_{S^2} be the normalized area 2-form on S^2, $\int_{S^2}\omega_{S^2}=4\pi$. The pullback $F:=\hat{n}^*\omega_{S^2}$ is a closed 2-form on S^3 and hence exact, $F=dA$ for some globally defined 1-form A on S^3. The Hopf invariant is

$$H[\hat{n}] \;:=\; \frac{1}{4\pi^2}\int_{S^3} A\wedge dA \;\in\; \mathbb{Z}, \tag{13.5}$$

equal to the linking number of preimages $\hat{n}^{-1}(p)$ and $\hat{n}^{-1}(q)$ for generic $p,q\in S^2$.

Hopf from phason via the Hopf map. Given $U:S^3\to S^3\simeq SU(2)$, the standard Hopf map $\mathcal{H}:S^3\to S^2$ yields $\hat{n}=\mathcal{H}\circ U$, for which

$$H[\hat{n}] \;=\; w[U]^2, \tag{13.6}$$

a useful check but *not* the most general situation (independent $\hat{n}$-textures are allowed and will be important below).

3. A conserved family current and the mod-3 index

Topological 3-form. Define the 3-form density on S^3

$$\Omega_{\rm fam} \;:=\; \underbrace{\frac{1}{4\pi^2}\,A\wedge dA}_{\text{Hopf density}} \;-\; \underbrace{\frac{1}{24\pi^2}\,\mathrm{Tr}\,(U^{-1}dU)^3}_{SU(2)\text{ Chern–Simons}}, \tag{13.7}$$

whose integral equals $H-w$ by (13.4)–(13.5). Promoting to spacetime with a time foliation, the associated *family current*

$$J^\mu_{\rm fam} \;:=\; \frac{1}{8\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,A_\nu\,\partial_\rho A_\sigma \;-\; \frac{1}{24\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(L_\nu L_\rho L_\sigma) \tag{13.8}$$

is identically conserved, $\partial_\mu J^\mu_{\rm fam}=0$, provided A is the CP^1 Berry connection of $\hat{n}$ and $L_\mu=U^{-1}\partial_\mu U$ (no equations of motion are used).

Mod-3 reduction from color center/WZW. Coupling to dynamical color with $N_c=3$ induces a quantized WZW functional $\Gamma_{\rm WZW}[U_{\rm chiral}]$ (Part II) and a mixed *topological* contact between the CP^1 Berry connection and the chiral SU(2) phason at the flavor scale. The net effect is that the rephasing of the path integral by

$$\exp\Big\{\,2\pi i\,N_c\int_\Sigma \Omega_{\rm fam}\Big\} \;=\; \exp\{\,2\pi i\,N_c\,(H-w)\} \;=\; 1 \tag{13.9}$$

is physically invisible, because it can be absorbed by attaching three center ($\mathbb{Z}_3$) vortices (Sec. 12.2) to the texture without changing any gauge-invariant observable. Hence only the residue of $H-w$ *modulo* 3 is an invariant label measurable by gauge-invariant probes:

$$\mathfrak{f} \;\equiv\; \int_{S^3}\Omega_{\rm fam} \pmod 3 \;\in\{0,1,2\}. \tag{13.10}$$

We call $\mathfrak{f}$ the *spectral family index*.

Invariance. Local, finite-energy deformations permitted by the S/G/TQ/D gates can change H and w only through events that shift both by multiples of 3 (creation/annihilation of color-centered Hopf rings or large phason maps screened by three center vortices). Therefore $\mathfrak{f}$ is conserved.

4. Dirac zero modes and index matching

Spectral Dirac operator. Consider a chiral composite fermion coupled to $(U, \hat{n})$ through a Yukawa-like mass functional $M[U, \hat{n}]$ (built from the fiber map $y(x)$), and define the Euclidean Dirac operator on $\Sigma \times \mathbb{R}$,

$$\mathcal{D} \;=\; i\not{D}_{g_{\rm eff}} + \mathcal{Y}[U, \hat{n}] \;, \qquad \mathcal{Y} :=\; y_1\, U^{\gamma_5} + y_2\, \hat{n}\cdot\vec{\sigma}, \tag{13.11}$$

with appropriate chiral/projector structure. Under standard regularity assumptions, the spectral flow of $\mathcal{D}$ across time equals the integral of a local index density, giving

$$\operatorname{ind} \mathcal{D} \;=\; \Delta N_{\rm L} - \Delta N_{\rm R} \;=\; H[\hat{n}] \;-\; w[U]. \tag{13.12}$$

Thus the number of protected chiral zero modes bound to a *single* flavor texture equals $H - w$. Modulo 3, this reproduces the family index $\mathfrak{f}$, i.e. each texture injects one member of a generation multiplet labeled by $\mathfrak{f}$.

5. Constructive examples yielding $\mathfrak{f} = 0, 1, 2$

Type A (pure Hopf knot). Take $w = 0$ and a minimal Hopf texture with $H = 1$ (the standard Hopf fibration image of a unit-charge map $S^3 \to S^3$ followed by a local relaxation that unwinds w but preserves H). Then $\mathfrak{f} = 1$.

Type B (pure phason winding). Take $H = 0$ and $w = 1$ (a hedgehog-like $SU(2)$ phason). Then $\mathfrak{f} = -1 \equiv 2 \pmod 3$.

Type C (trivial). Take $H = w = 0$, giving $\mathfrak{f} = 0$. These three minimal constructions realize the three distinct family classes without invoking large charges.

6. Gauge/anomaly consistency and additivity across fields

Additivity. For multiple textures (e.g. different Weyl species) the indices add: $\mathfrak{f}_{\rm tot} = \sum_i \mathfrak{f}_i \pmod 3$. The 't Hooft anomaly constraints of Ch. 6 are unaffected because they depend on gauge representations, while $\mathfrak{f}$ is a *spectral* superselection label tied to the texture hosting a zero mode.

Electroweak couplings. Gauging $SU(2)_L \times U(1)_Y$ (Part III) does not alter the mod-3 classification: the Bardeen counterterm rearranges vector/axial currents but leaves the integer $H - w$ invariant and the center-induced mod-3 reduction intact.

7. Positivity, hyperbolicity, and stability gates

(TQ) positivity/unitarity. The family current $J^{\mu}_{\rm fam}$ is identically conserved and does not enter the quadratic kinetic operator; it cannot induce negative-norm states.

(G) hyperbolicity/causality. The construction involves only topological densities on spacelike slices; propagation cones of dynamical fields remain those of $g_{\rm eff}$.

(S) stability. Finite energy of textures requires $U \to \mathbf{1}$ and $\hat{n} \to \hat{n}_0$ at infinity; the corresponding energy functionals are bounded below by gradient terms inherited from spectral stiffness.

(D) dissipation. Dissipative couplings cannot change $\mathfrak{f}$: creation/annihilation of texture charge proceeds via events that shift H and w by equal multiples of 3, leaving $\mathfrak{f}$ invariant.

8. Summary

The pair of integer invariants (H, w) associated with CP^1 and $SU(2)$ spectral textures on S^3 defines a conserved mod-3 class $\mathfrak{f} = H - w \pmod 3$ fixed by color-center structure and WZW quantization. This *spectral family index* yields precisely three robust superselection sectors, providing natural, gauge-invariant labels for generations. In the next section we quantize bound internal modes on each texture to derive mass hierarchies and mixing patterns consistent with the spectral RG and anomaly constraints.

13.2 Hierarchy from mode quantization

Aim. We derive fermion mass hierarchies from the *quantization of bound modes* living on the family textures introduced in Sec. 13.1. The key elements are: (i) chiral zero modes localized on smooth S^3 textures of $(U, \hat{n})$; (ii) a *fiber* (Hopf) angle ψ with a Berry/Aharonov–Bohm phase fixed by the family index $\mathfrak{f} = (H - w) \pmod 3$; (iii) mode quantization along ψ that yields a discrete label $m \in \mathbb{Z}$ shifted by a topological phase $\alpha_{\mathrm{fam}} \in [0, 1)$; (iv) Higgs-induced overlaps between left/right texture sheets producing Dirac masses $m_f = y_f v/\sqrt{2}$ with exponentially ordered Yukawas y_f. The outcome is a *geometric* hierarchy controlled by two spectral lengths: the texture radius R_{tex} and the Higgs coherence length ℓ_H.

1. Chiral bound modes on family textures

Local geometry (Hopf fibration). Near a smooth texture, introduce tubular coordinates adapted to the Hopf fibration of S^3:

$$ds^2 \;=\; R_H^2\Big[\, d\theta^2 + \sin^2\theta\, d\phi^2 + \Big(d\psi + \cos\theta\, d\phi\Big)^2 \,\Big], \qquad \psi \sim \psi + 2\pi, \tag{13.13}$$

where R_H is the local Hopf radius of the texture. The CP^1 director $\hat{n}$ furnishes the Berry one-form

$$a \;:=\; \tfrac{1}{2}\Big(d\psi + \cos\theta\, d\phi\Big), \qquad f := da = \tfrac{1}{2}\sin\theta\, d\theta \wedge d\phi, \tag{13.14}$$

whose Chern–Simons integral on S^3 gives the Hopf number (Sec. 13.1).

Topological boundary condition along the fiber. Color-center/WZW dressing implies that spinors transported around the fiber acquire a phase tied to the family index,

$$\Psi(\psi + 2\pi) \;=\; e^{2\pi i \alpha_{\mathrm{fam}}}\, \Psi(\psi), \qquad \alpha_{\mathrm{fam}} \;\equiv\; \frac{H - w}{3} \pmod 1, \tag{13.15}$$

i.e. an Aharonov–Bohm shift by α_{fam}.

Spectral Dirac operator and zero modes. For a composite chiral fermion coupled to the texture, the local Dirac operator (schematically) reads

$$\mathcal{D} \;\simeq\; i\gamma^{\psi}\frac{1}{R_H}\Big(\partial_\psi - i\alpha_{\rm fam}\Big) + i\gamma^{\perp}\!\cdot\!\nabla_\perp \;+\; M(r)\,\gamma^5, \tag{13.16}$$

with $M(r)$ a domain-wall mass changing sign across the tubular core of radius $R_{\rm tex}$. Solving $\mathcal{D}\Psi = 0$ yields exponentially localized chiral channels propagating along ψ with quantized momenta

$$k_\psi R_H \;=\; m - \alpha_{\rm fam}, \qquad m \in \mathbb{Z}, \tag{13.17}$$

and a transverse profile $\Psi_\perp(r) \propto \exp[-\int_0^r M(r')dr'/v_\perp]$ with $v_\perp$ a spectral "Fermi" speed.

2. Effective 1D spectrum and selection of three light channels

Linearized dispersion (pre-EWSB). Projecting onto the bound transverse zero mode, the 1D Hamiltonian along ψ is

$$H_{\rm 1D} \;=\; v_\psi\,\sigma^3\frac{k_\psi}{R_H}\,, \qquad E_m \;=\; \pm\, v_\psi\,\frac{|m-\alpha_{\rm fam}|}{R_H}. \tag{13.18}$$

Modes with the *smallest* $|m-\alpha_{\rm fam}|$ are the lowest-energy channels surviving at long distances. Because $\alpha_{\rm fam} = (H-w)/3$, the three integers $m \in \{0,1,2\}$ provide the three nearest representatives modulo 3; all others lie at $|m-\alpha_{\rm fam}| \geq 3/2$ and are heavy/decoupled at the electroweak scale. This selects *three* light chiral wires per flavor species.

3. Higgs-induced Dirac masses from overlap integrals

Setup. Let left/right chiral wires live on nearby, parallel texture sheets separated by d (along a normal geodesic). Coupling to the Higgs order parameter (Part III) gives the effective matrix element

$$(m_f)_{mn} \;=\; \frac{v}{\sqrt{2}}\; y_\star \underbrace{\int d\psi\, \Psi_m^{(L)*}(\psi)\,\Psi_n^{(R)}(\psi)}_{\delta_{mn}} \underbrace{\int d^2x_\perp\, \Psi_\perp^{(L)*}(x_\perp)\,h(x_\perp)\,\Psi_\perp^{(R)}(x_\perp)}_{\mathcal{O}_\perp}, \tag{13.19}$$

where h is the Higgs profile in the tube and $y_\star$ a spectral Yukawa density. Fiber orthogonality enforces *diagonal* $m = n$ at leading order.

Transverse overlap and hierarchy kernel. For a smooth domain wall ($M(r) \simeq M_0\, r/R_{\rm tex}$) and a Higgs coherence length ℓ_H, the transverse factor is well approximated by a Gaussian kernel in k_ψ:

$$\mathcal{O}_\perp(m) \;\simeq\; \exp\!\Big[-\;\frac{R_{\rm tex}^2}{\Lambda_H^2}\,\big(m-\alpha_{\rm fam}\big)^2\Big]\;\exp\!\Big[-\;\frac{d}{\ell_H}\Big], \qquad \Lambda_H \;:=\; \sqrt{2}\,\frac{R_H}{\ell_H}\,\frac{v_\perp}{v_\psi}. \tag{13.20}$$

Thus the *effective Yukawa* for the m-th channel is

$$y_m \;=\; y_\star\,\exp\!\Big[-\,\kappa_H\,(m-\alpha_{\rm fam})^2\Big], \qquad \kappa_H \;:=\; \frac{R_{\rm tex}^2}{\Lambda_H^2}, \tag{13.21}$$

up to a common suppression e^{-d/ℓ_H} (flavor-universal if the sheet spacing is common).

Resulting masses. The physical Dirac masses are

$$m_{(m)} \;=\; \frac{v}{\sqrt{2}}\, y_\star\, e^{-d/\ell_H}\;\exp\Big[-\kappa_H\,(m-\alpha_{\rm fam})^2\Big], \qquad m\in\{0,1,2\}. \tag{13.22}$$

Ordering follows the distances $|m-\alpha_{\rm fam}|$: the channel nearest to $\alpha_{\rm fam}$ is heaviest.

4. Geometric patterns and parameter–economy

Near-rational phases. Take $\alpha_{\rm fam}=\frac{1}{3}$ (a canonical choice when $H-w\equiv 1 \pmod 3$). Then

$$(m-\alpha_{\rm fam})^2 \;\in\; \Big\{\tfrac{1}{9},\;\tfrac{4}{9},\;\tfrac{25}{9}\Big\} \quad \text{for } m=0,1,2, \tag{13.23}$$

yielding the *log–quadratic* hierarchy

$$\ln\frac{m_{(1)}}{m_{(0)}} \;=\; -\tfrac{1}{3}\kappa_H, \qquad \ln\frac{m_{(2)}}{m_{(1)}} \;=\; -\tfrac{7}{3}\kappa_H. \tag{13.24}$$

A *single* dimensionless stiffness κ_H thus fixes two adjacent mass ratios in each sector; small sector-dependent shifts of $\alpha_{\rm fam}$ (from weak hypercharge couplings to the fiber) fine-tune the pattern without new dials.

Universal slope with sector prefactors. Allowing modest sector prefactors $y_\star^{(u,d,\ell,\nu)}$ (spectral charges) but *common* (κ_H,ℓ_H) across sectors, one obtains

$$\ln m^{(s)}_{(m)} \;=\; \ln\Big(\tfrac{v}{\sqrt{2}}\,y^{(s)}_\star\Big) \;-\; \frac{d}{\ell_H} \;-\; \kappa_H\,(m-\alpha^{(s)}_{\rm fam})^2, \qquad s\in\{u,d,\ell,\nu\}, \tag{13.25}$$

i.e. a set of *parallel parabolas* in $(m-\alpha)^2$ with small sector-dependent horizontal offsets. This realizes quark–lepton correlated hierarchies from geometry.

5. WKB justification of the Gaussian kernel

Radial reduction. Separating variables $\Psi(\psi,r,\Omega_\perp)=\frac{e^{i(m-\alpha)\psi}}{\sqrt{2\pi}}\,\psi_m(r,\Omega_\perp)$ and projecting onto the lowest transverse channel yields the 2D Schrödinger problem

$$\Big[-\nabla_\perp^2 + M(r)^2 - v_\perp\,\sigma^\perp\!\cdot\nabla_\perp M(r) + \frac{(m-\alpha)^2}{R_H^2}\Big]\psi_m \;=\; 0. \tag{13.26}$$

Around the core, $M(r)\simeq M_0 r/R_{\rm tex}$ makes the potential quadratic, giving a ground-state width $\Delta r\simeq\min\{\ell_H,R_{\rm tex}\}$ and a suppression $\psi_m(0)\propto\exp[-(m-\alpha)^2R_{\rm tex}^2/\Lambda_H^2]$, which, when squared in the overlap, reproduces Eq. (13.20).

6. Matching to spectral scales and predictive windows

Spectral lengths. The texture radius is set by chiral stiffness and the Skyrme/BPS parameters (Sec. 11.3): $R_{\rm tex}\sim 1/(eF)$. The coherence scale ℓ_H follows from the Higgs curvature at the minimum, $\ell_H=(2\lambda_H v^2)^{-1/2}$ with λ_H the *spectral* quartic (Sec. 8.3). Hence

$$\kappa_H \;\sim\; \Big(\frac{R_{\rm tex}}{\ell_H}\Big)^2\,\frac{v_\perp^2}{2v_\psi^2}, \tag{13.27}$$

is calculable from stiffness ratios and mode velocities (fixed by the medium's response functions).

Predictivity. Fix $(\kappa_H, \alpha_{\rm fam}^{(s)})$ by two ratios in one sector; the third ratio and the absolute normalization $y_\star^{(s)} e^{-d/\ell_H}$ are then predictions. Cross-sector universality of κ_H (modulo small RG drift) provides nontrivial consistency checks.

7. Consistency gates (hierarchy sector)

(S) stability. Bound modes exist for monotone $M(r)$ with $M_0 > 0$; the Gaussian kernel is normalizable for $\kappa_H > 0$ and any $\alpha_{\rm fam}$.

(G) hyperbolicity/causality. Mode velocities $v_{\psi,\perp}$ are bounded by the $g_{\rm eff}$ cone; no superluminal propagation arises from the Berry shift.

(TQ) positivity/unitarity. Orthogonality along the fiber ensures a positive-definite mass matrix at leading order; loop corrections respect forward-limit positivity (Part VI).

(D) dissipation. Dissipative couplings may renormalize d/ℓ_H but cannot change $\alpha_{\rm fam}$ nor destroy the exponential ordering, which is geometric.

8. Summary

Family textures impose a *twisted* quantization $k_\psi R_H = m - \alpha_{\rm fam}$ on chiral wires running along Hopf fibers. Electroweak masses arise from Higgs overlaps between parallel left/right wires, yielding Yukawas $y_m \propto \exp[-\kappa_H (m - \alpha_{\rm fam})^2]$ with $\kappa_H \propto (R_{\rm tex}/\ell_H)^2$. Exactly three light channels ($m = 0, 1, 2$ modulo 3) survive at low energy, and their squared distances from $\alpha_{\rm fam}$ organize the observed mass hierarchies with few geometric inputs. The next section develops *stability gates and inter-generation mixing*, where small misalignments of texture frames generate CKM/PMNS angles while preserving the exponential mass ordering.

13.3 Stability gates and inter-generation mixing

Aim. We analyze how the three light fiber modes selected in Sec. 13.2 remain *stable* under the S/G/TQ/D gates and how small *misalignments* of family textures generate inter-generation mixing matrices (CKM/PMNS). We construct the effective 3×3 Yukawa/mass operators from geometric overlaps, identify the dominant sources of off-diagonal entries, derive compact analytic expressions for mixing angles and the CP phase in terms of a few geometric control parameters, and state quantitative stability criteria ensuring decoupling from heavy tower modes.

1. Sources of mixing: geometry of the three-mode subspace

Family triads and frame misalignment. For each fermion sector $s \in \{u, d, \ell, \nu\}$, the three light fiber modes ("wires") are labeled by $m \in \{0, 1, 2\}$ with phases $e^{i(m - \alpha_{\rm fam}^{(s)})\psi}$ along the Hopf fiber (Sec. 13.2). Let $\{|m\rangle_L^{(s)}\}$ and $\{|n\rangle_R^{(s)}\}$ denote orthonormal left/right bases on their respective texture sheets. If the left frames of two sectors (e.g. u_L vs. d_L)

differ by a small unitary rotation $R_L^{(ud)} \in U(3)$, then

$$|m\rangle_L^{(d)} \;=\; \sum_{n=0}^{2} (R_L^{(ud)})_{nm}\; |n\rangle_L^{(u)}\,, \qquad R_L^{(ud)} \;=\; \exp\!\Big(i\,\epsilon_a^{(ud)}\,T^a\Big), \quad |\epsilon_a^{(ud)}| \ll 1, \tag{13.28}$$

with $\{T^a\}$ a basis of traceless Hermitian generators acting in the *family* space.

Adiabatic mixing among fiber harmonics. Slow spatial variations of $(U, \hat{n})$ induce a non-Abelian Berry connection $\mathcal{A}_\psi^{(s)}$ within the three-mode subspace,

$$\Big(\mathcal{A}_\psi^{(s)}\Big)_{mn} \;:=\; i_L^{(s)} \langle m|\partial_\psi|n|m|\partial_\psi|n\rangle_L^{(s)}\;, \qquad \mathcal{W}^{(s)} \;:=\; \mathcal{P}\exp\!\Big(i\int_0^{2\pi} d\psi\,\mathcal{A}_\psi^{(s)}\Big), \tag{13.29}$$

whose *holonomy* $\mathcal{W}^{(s)} \in U(3)$ multiplies the fiber phases $e^{2\pi i(m-\alpha^{(s)})}$ and can seed complex off-diagonal overlaps.

Heavy-mode decoupling. Couplings to higher $|m-\alpha| \geq 3/2$ states are suppressed by the spectral gap

$$\Delta_{\rm gap} \;=\; v_\psi\,\frac{3}{2R_H}\;, \tag{13.30}$$

so adiabatic variations ($\|\partial\mathcal{S}\| \ll \Delta_{\rm gap}$) keep dynamics inside the protected three-mode manifold (G gate).

2. Geometric Yukawas and the emergence of off-diagonal entries

Aligned limit (no mixing). In the perfectly aligned case

$$R_L^{(ud)} = R_R^{(ud)} = \mathbf{1}, \qquad \mathcal{A}_\psi^{(s)} = \mathrm{diag}\Big(\alpha_{\rm fam}^{(s)}\Big), \tag{13.31}$$

the overlap $\int_0^{2\pi} d\psi\, e^{i(m-\alpha_L)\psi}\, e^{-i(n-\alpha_R)\psi} = 2\pi\,\delta_{mn}$ and the Yukawa matrix is diagonal,

$$\Big(Y_s\Big)_{mn} \;=\; y_\star^{(s)}\, e^{-d/\ell_H}\;\exp\Big[-\kappa_H\,(m-\alpha_{\rm fam}^{(s)})^2\Big]\;\delta_{mn}, \tag{13.32}$$

reproducing the hierarchical pattern of Eq. (13.22) with *no* inter-generation mixing.

Leading deformations. Small misalignment and Berry holonomy generate off-diagonal entries,

$$Y_s \;\simeq\; y_\star^{(s)}\, e^{-d/\ell_H}\,\Big[\,\mathsf{D}\Big(\alpha^{(s)}\Big) \;+\; \varepsilon_L^{(s)}\,\mathsf{D}^{1/2}\Big(\alpha^{(s)}\Big)\,\Xi\,\mathsf{D}^{1/2}\Big(\alpha^{(s)}\Big) \;+\; \varepsilon_R^{(s)}\,(\cdots)\,\Big], \tag{13.33}$$

where $\mathsf{D}_{mn}(\alpha) := \delta_{mn}\exp[-\kappa_H(m-\alpha)^2]$, Ξ is an $\mathcal{O}(1)$ Hermitian matrix built from the geometric overlaps induced by $\mathcal{W}^{(s)}$ and transverse profile misalignments, and $|\varepsilon_{L,R}^{(s)}| \ll 1$ are small, dimensionless *alignment parameters* proportional to frame angles ϵ_a and to the norm of the non-Abelian Berry curvature in the three-mode sector. The factorized $\mathsf{D}^{1/2}\Xi\mathsf{D}^{1/2}$ structure expresses that off-diagonals inherit the *geometric suppression* of both participating modes.

3. Mixing angles from $Y_s Y_s^\dagger$: closed forms

Left-unitary diagonalization. Physical mixing in charged currents depends on the left-unitary matrices $U_L^{(s)}$ that diagonalize $M_s := Y_s Y_s^\dagger$. To leading order in $\varepsilon_L^{(s)}$ one finds, for $i \neq j$,

$$\big(M_s\big)_{ij} \;\simeq\; y_\star^{(s)2}\, e^{-2d/\ell_H}\, \varepsilon_L^{(s)}\, e^{-\kappa_H\left[(i-\alpha)^2+(j-\alpha)^2\right]}\, \Xi_{ij}, \qquad \big(M_s\big)_{ii}-\big(M_s\big)_{jj} \;\simeq\; y_\star^{(s)2}\, e^{-2d/\ell_H}\, \Delta_{ij}, \tag{13.34}$$

with $\Delta_{ij} := e^{-2\kappa_H(i-\alpha)^2} - e^{-2\kappa_H(j-\alpha)^2}$. Hence the small mixing angle in the (ij) subspace is

$$\boxed{\theta_{ij}^{(s)} \;\simeq\; \frac{\big|(M_s)_{ij}\big|}{\big|(M_s)_{ii}-(M_s)_{jj}\big|} \;\simeq\; \frac{|\varepsilon_L^{(s)}\,\Xi_{ij}|}{2\sinh\!\Big(\kappa_H\,[\,(j-\alpha)^2-(i-\alpha)^2\,]\Big)}\,.} \tag{13.35}$$

Equation (13.35) is the central result: *geometric* hyperbolic suppressions determined by κ_H and the distances $d_i = (i-\alpha)^2$ multiply the microscopic alignment parameter.

CKM and PMNS matrices. For quarks, the CKM matrix is $V_{\rm CKM} = U_L^{(u)\dagger} U_L^{(d)}$; for leptons, $U_{\rm PMNS} = U_L^{(\ell)\dagger} U_L^{(\nu)}$. If the three-mode geometry is similar in u and d sectors, the small net angles follow from (13.35) with $\varepsilon_L^{(u)}$ and $\varepsilon_L^{(d)}$ partially cancelling. Conversely, if the neutrino three-mode manifold is *near-degenerate* (e.g. smaller κ_H or $\alpha^{(\nu)}$ close to half-integers so that $(j-\alpha)^2-(i-\alpha)^2 \approx 0$), the denominator in (13.35) shrinks and large leptonic mixing naturally emerges.

4. Geometric CP violation

Phases from Berry holonomy. Complex phases in Y_s arise from Wilson lines $\mathcal{W}^{(s)}$ and relative frame twists between sectors; write $\Xi_{ij} = |\Xi_{ij}|e^{i\phi_{ij}}$. The Jarlskog invariant (quark sector) is, to cubic order in small angles,

$$J_{\rm CKM} \;\simeq\; \frac{|\varepsilon_L^{(u)}\Xi_{12}^{(u)}|\,|\varepsilon_L^{(u)}\Xi_{23}^{(u)}|\,|\varepsilon_L^{(u)}\Xi_{13}^{(u)}|}{8\,\prod_{(ij)}\sinh\!\Big(\kappa_H[\,(j-\alpha^{(u)})^2-(i-\alpha^{(u)})^2\,]\Big)}\,\sin\Phi_{\rm geom} \;+\; (u\leftrightarrow d), \tag{13.36}$$

where $\Phi_{\rm geom}$ is a gauge-invariant *geometric* phase built from the sum of holonomy phases around the triangle of family indices. A nonzero J requires non-commuting frame twists or nontrivial $\mathcal{W}^{(s)}$.

5. Stability gates and quantitative bounds

(S) Stability: bounded spectra and decoupling. The three-mode EFT is stable if (i) masses are positive and hierarchical (already ensured by (13.32)), and (ii) mixings with heavy modes are perturbative:

$$\frac{\|\delta Y\|}{\Delta_{\rm gap}} \;\ll\; 1\,, \qquad \Delta_{\rm gap} = \frac{3v_\psi}{2R_H}, \quad \|\delta Y\| \;\sim\; |\varepsilon_{L,R}|\, y_\star\, e^{-d/\ell_H}. \tag{13.37}$$

This guarantees that integrating out the tower does not induce tachyonic instabilities nor invert the mass ordering.

(G) Hyperbolicity: cone compatibility. Mixing arises in *mass* matrices and does not alter the principal symbol of kinetic terms. Requiring all mode velocities to lie within the g_{eff} cone enforces

$$0 < v_\psi,\ v_\perp \le 1, \qquad \|\partial \mathcal{S}\| \ll \Delta_{\text{gap}} \ \Rightarrow\ \text{adiabatic decoupling of heavy modes.} \tag{13.38}$$

(TQ) Positivity/unitarity: forward gates. Tree-level four-fermion operators induced by integrating out heavy modes satisfy forward-limit positivity provided $\varepsilon_{L,R}$ are real to leading order; complex phases only enter through unitary rotations (Berry holonomy), preserving unitarity. Moreover $M_s = Y_s Y_s^\dagger$ is positive definite by construction.

(D) Dissipation: robustness of textures. Dissipative medium couplings may renormalize $\varepsilon_{L,R}$ and κ_H, but cannot change the *topological* family index $\mathfrak{f}$ nor collapse the spectral gap as long as the texture remains smooth (finite stiffness).

6. Quark–lepton contrast as a geometric corollary

Small quark vs. large lepton mixing. Equations (13.35)–(13.36) predict:

$$\text{Quarks:}\quad \kappa_H^{(u,d)} \text{ moderate/large}, \ \Rightarrow\ \theta_{ij}^{\text{CKM}} \sim \frac{|\varepsilon_L^{(d)} - \varepsilon_L^{(u)}|}{2\sinh(\kappa_H \Delta d_{ij})} \ll 1, \tag{13.39}$$

$$\text{Leptons:}\quad \kappa_H^{(\nu)} \text{ smaller or } \alpha^{(\nu)} \text{ near half-integers } \Rightarrow\ \sinh(\kappa_H \Delta d_{ij}) \text{ small } \Rightarrow\ \text{large PMNS angles.} \tag{13.40}$$

Here $\Delta d_{ij} = (j-\alpha)^2 - (i-\alpha)^2$. Thus the observed pattern (small CKM, large PMNS) arises from purely geometric data of the textures.

7. Worked two-state reduction (analytic check)

Two-level formula. For a 2×2 sub-block with diagonal entries $m_i^2 \propto e^{-2\kappa_H d_i}$, $m_j^2 \propto e^{-2\kappa_H d_j}$ and off-diagonal $\delta \propto \varepsilon_L\, e^{-\kappa_H(d_i+d_j)}$, the exact mixing angle satisfies

$$\tan 2\theta_{ij} \ = \ \frac{2|\delta|}{|m_j^2 - m_i^2|} \ = \ \frac{|\varepsilon_L\, \Xi_{ij}|}{\sinh\!\big(\kappa_H (d_j - d_i)\big)}\ , \tag{13.41}$$

confirming (13.35). The eigenvalue shifts are $\mathcal{O}(\delta^2/|m_j^2 - m_i^2|)$ and respect the hierarchy for small ε_L (S gate).

8. Summary

Inter-generation mixing in the spectral framework is a *purely geometric* effect of (i) small frame misalignments and Berry holonomies within the protected three-mode family manifold and (ii) the exponential mode suppressions controlled by κ_H and the fiber offsets $(m-\alpha)$. Closed-form expressions (13.35)–(13.36) relate CKM/PMNS observables to a handful of alignment parameters and texture geometry; quark–lepton contrast follows from different geometric denominators. Stability, hyperbolicity, and positivity gates hold provided the spectral gap Δ_{gap} is large against alignment-induced deformations, ensuring that mixing emerges without jeopardizing the hierarchical mass spectrum.

13.4 Links to anomaly constraints

Aim. We show that the *family index* $\mathfrak{f} = H - w \pmod 3$ introduced in Sec. 13.1 and the three–mode quantization/hierarchy of Sec. 13.2–13.3 are *compatible* with gauge and gravitational anomaly constraints, and we extract sharpened, model–independent selection rules. The analysis proceeds by (i) organizing the Standard–Model (SM) anomaly conditions in a compact spectral form; (ii) gauging the *discrete* $\mathbb{Z}_3$ family label as a background probe and checking all mixed 't Hooft anomalies; (iii) tracking anomaly inflow during texture rearrangements $\Delta(H - w) \in 3\mathbb{Z}$; and (iv) stating constraints on mixing and CP phases that are *anomaly–insensitive* vs. those that are not.

1. Anomaly bookkeeping in the spectral basis

Conventions. Let ψ_i be left–handed Weyl fermions, irreps R_i of $G_{\rm SM} = SU(3)_c \times SU(2)_L \times U(1)_Y$ with hypercharge Y_i and Dynkin index $T(R_i)$ normalized by $T(\mathbf{3}) = \frac{1}{2}$, $T(\mathbf{2}) = \frac{1}{2}$. Denote

$$\mathcal{A}[G^2\!-\!U(1)_Y] = \sum_i Y_i\, T_G(R_i), \qquad G \in \{SU(3), SU(2)\}, \tag{13.42}$$

$$\mathcal{A}[U(1)_Y^3] = \sum_i Y_i^3, \qquad \mathcal{A}[\mathrm{grav}^2\!-\!U(1)_Y] = \sum_i Y_i. \tag{13.43}$$

For the SM *per generation*, these sums vanish:

$$\mathcal{A}[SU(3)^2\!-\!Y] = 0, \quad \mathcal{A}[SU(2)^2\!-\!Y] = 0, \quad \mathcal{A}[Y^3] = 0, \quad \mathcal{A}[\mathrm{grav}^2\!-\!Y] = 0, \tag{13.44}$$

and the global Witten anomaly for $SU(2)_L$ is absent because the number of $SU(2)$ doublets is even (three colored quark doublets + one lepton doublet $\Rightarrow$ 4 per generation).

Spectral replication. In the spectral model each family texture carries the *same* gauge representation content as one SM generation; the family label $\mathfrak{f}$ is topological and does not alter R_i nor Y_i. Therefore Eq. (13.44) holds *family–wise*, and the total SM anomalies vanish generation by generation, independently of hierarchy/mixing.

2. Discrete $\mathbb{Z}_3$ family label and mixed 't Hooft anomalies

Background probe for $\mathbb{Z}_3^{\rm fam}$. Treat $\mathfrak{f} \in \{0, 1, 2\}$ as a *discrete, global* label and couple it to a non–dynamical $\mathbb{Z}_3$ background one–form field $A^{(0)}$ (or, equivalently, to a flat $\mathbb{Z}_3$ bundle) to test possible mixed anomalies with $G_{\rm SM}$. Assign each left–handed Weyl field a family charge $q^{\rm fam} \equiv \mathfrak{f} \pmod 3$.

Necessary $\mathbb{Z}_3$ anomaly–cancellation conditions. Possible mixed anomalies are defined modulo 3:

$$\mathcal{A}\Big[\mathbb{Z}_3^{\rm fam}\!-\!SU(3)_c^2\Big] \equiv \sum_i q_i^{\rm fam}\, T_{SU(3)}(R_i) \pmod 3, \tag{13.45}$$

$$\mathcal{A}\Big[\mathbb{Z}_3^{\rm fam}\!-\!SU(2)_L^2\Big] \equiv \sum_i q_i^{\rm fam}\, T_{SU(2)}(R_i) \pmod 3, \tag{13.46}$$

$$\mathcal{A}\Big[\mathbb{Z}_3^{\rm fam}\!-\!U(1)_Y^2\Big] \equiv \sum_i q_i^{\rm fam}\, Y_i^2 \pmod 3, \tag{13.47}$$

$$\mathcal{A}\Big[\mathbb{Z}_3^{\rm fam}\!-\!\mathrm{grav}^2\Big] \equiv \sum_i q_i^{\rm fam} \pmod 3. \tag{13.48}$$

Because each generation carries identical $\{R_i, Y_i\}$, assigning the three families charges $\{0,1,2\}$ implies

$$\sum_{\text{families}} q^{\text{fam}} \;=\; 0+1+2 \;\equiv\; 0 \;(\text{mod } 3) \quad\Longrightarrow\quad \mathcal{A}\Big[\mathbb{Z}_3^{\text{fam}} - G^2\Big] \equiv 0, \quad \forall\, G, \tag{13.49}$$

and similarly for $U(1)_Y^2$ and gravity. Hence the *canonical* spectral assignment $\mathfrak{f} = \{0,1,2\}$ is automatically free of mixed $\mathbb{Z}_3^{\text{fam}}$ anomalies.

Relation to color center. Physically, $\mathbb{Z}_3^{\text{fam}}$ descends from the color center (Sec. 12.2) via the mod–3 reduction of $H - w$; attaching a center vortex to a texture shifts $H - w$ by 1 but leaves all gauge–invariant observables unchanged. This explains why Eq. (13.49) is enforced by center–symmetry triviality.

3. Witten anomaly and texture–wise parity

Per–texture parity of doublets. Since each family texture supports a *full* SM generation, it contains an even number (4) of $SU(2)_L$ doublets. Therefore the *global* $SU(2)$ anomaly (*Witten*) cancels *texture by texture* and hence in any superposition of textures, independently of how families are distributed among $\mathfrak{f} = 0, 1, 2$.

4. ABJ anomalies, inflow, and texture rearrangements

ABJ under smooth deformations. Let $\mathscr{A}_\mu$ be background gauge fields for G_{SM}. The five–dimensional inflow functional reproducing ABJ anomalies is

$$S_{\text{inflow}} \;=\; 2\pi i \int_{\mathcal{M}_5} I_5(\mathscr{A}), \qquad dI_5 = I_6 \propto \operatorname{Tr} F^3, \quad \operatorname{Tr} F^2\, d\mathcal{A}_Y, \;\ldots \tag{13.50}$$

(Part II). A change of texture with $\Delta(H - w) = 3k$ modifies the family current by an exact form on $\mathcal{M}_4$ and the path–integral phase by $2\pi i\, N_c\, \Delta(H - w) = 2\pi i \cdot 3k$, which is trivial. Hence *all* ABJ coefficients (including mixed gauge–gravitational) are invariant under allowed spectral moves; anomaly matching across scales is automatic.

5. Hypercharge quantization and mixed constraints

Per–generation constraints. The fiber embedding of $U(1)_Y$ in Part III fixes a geometric normalization for Y. Per generation one has the vanishing sums in (13.44) and, in addition, quantization conditions ensuring integrality of Berry phases around non–contractible loops of the fiber:

$$\sum_i Y_i \in \mathbb{Z}, \qquad \sum_i Y_i^3 \in \mathbb{Z}, \qquad \sum_i Y_i\, T_{SU(2)}(R_i) \in \mathbb{Z}, \quad \sum_i Y_i\, T_{SU(3)}(R_i) \in \mathbb{Z}, \tag{13.51}$$

(all satisfied by SM charges). Since each family texture carries identical $\{Y_i, R_i\}$, Eqs. (13.44)–(13.51) are preserved by replication and by mixing.

6. Sphaleron selection rules and family index

Instanton/sphaleron. An $SU(2)_L$ instanton of winding Q violates B and L according to

$$\Delta(B+L) \;=\; 3\,Q\,N_{\rm gen}, \qquad \Delta(B-L)=0, \tag{13.52}$$

with $N_{\rm gen}=3$. In the spectral picture, the instanton event induces spectral flow of chiral zero modes bound to textures; the net change of the family charge is

$$\Delta\sum_{\rm fam}\mathfrak{f} \;\equiv\; 0 \;(\mathrm{mod}\;3), \tag{13.53}$$

because the flow can be compensated by attaching/removing three center vortices, i.e. by $\Delta(H-w)=3k$. Thus sphaleron processes preserve the *set* $\{\mathfrak{f}\}$ modulo permutations while reproducing Eq. (13.52).

7. Mixing, CP phases, and anomaly insensitivity

Unitary rotations. Inter–generation mixing corresponds to unitary transformations in the three–mode subspaces (Sec. 13.3). Such rotations leave all gauge/gravity anomalies unchanged: Eqs. (13.42)–(13.43) are basis–independent.

Strong CP. Anomaly constraints do *not* solve the strong–CP problem by themselves: $\bar\theta = \theta_{\rm QCD} + \arg\det(Y_u Y_d)$ is physical unless an additional symmetry/mechanism (axion/relaxation or spectral alignment enforcing $\arg\det(Y_uY_d)=0$) is present. The geometric WZW structures of Part II guarantee consistent phases but do not force $\bar\theta=0$; this remains a separate model ingredient.

Jarlskog invariant. The CKM/PMNS CP phases arise from Berry holonomies and frame twists (Sec. 13.3) and are unconstrained by gauge anomalies beyond the requirement that Yukawa rephasings respect the anomaly–free combinations.

8. Summary: anomaly gate for families

Anomaly gate (family sector). The family construction is consistent iff the following hold:

(i) Per–texture SM anomalies vanish: $\mathcal{A}[SU(3)^2{-}Y]=\mathcal{A}[SU(2)^2{-}Y]=\mathcal{A}[Y^3]=\mathcal{A}[\mathrm{grav}^2{-}Y]=0$; (13.54)

(ii) Witten anomaly canceled *per texture*: even # of $SU(2)$ doublets; (13.55)

(iii) Mixed $\mathbb{Z}_3^{\rm fam}$ anomalies vanish: families carry charges $\{0,1,2\} \Rightarrow \sum q^{\rm fam} \equiv 0 \;(\mathrm{mod}\;3)$; (13.56)

(iv) Texture moves obey $\Delta(H-w)\in 3\mathbb{Z} \;\Rightarrow$ trivial inflow phase and preserved ABJ data. (13.57)

When (13.54)–(13.57) are met, anomaly matching is automatic, mixing/CP are geometric and anomaly–insensitive, and sphaleron selection rules remain the SM ones with $N_{\rm gen}=3$.

9. Outlook

The next chapter constructs *Yukawa textures and mixing matrices* explicitly from overlap integrals (Part 14), now constrained by the anomaly gate above and by the positivity/causality bounds of Part VI. There we quantify the allowed parameter domains for $(\kappa_H, \alpha_{\mathrm{fam}}^{(s)})$, Berry holonomies, and frame misalignments that reproduce CKM/PMNS data while maintaining anomaly–safe family assignments $\mathfrak{f} \in \{0, 1, 2\}$.

Chapter 14

Yukawa Textures and CKM/PMNS from Overlaps

14.1 Spectral overlap integrals and texture zeros

Aim. We derive a *master formula* for fermion Yukawa matrices from first–principles spectral overlaps of the three light family modes constructed in Secs. 13.1–13.3. The resulting structure makes transparent (i) why diagonal dominance is generic, (ii) how controlled off–diagonals arise from fiber holonomies and weak longitudinal modulations of the Higgs profile, and (iii) which matrix entries are *exactly* forbidden ("structural zeros") by residual spectral symmetries versus those that are merely *exponentially* suppressed ("effective zeros").

1. Geometric setup and light-mode basis

Light family modes. For a given sector $s \in \{u, d, \ell, \nu\}$ we denote by $\{|m\rangle_L^{(s)}\}_{m=0,1,2}$ and $\{|n\rangle_R^{(s)}\}_{n=0,1,2}$ the orthonormal left/right bound modes localized on nearby texture sheets (Sec. 13.2). In adapted Hopf–fiber coordinates $(\psi, x_\perp)$,

$$\langle \psi, x_\perp | m | \psi, x_\perp | m \rangle_L^{(s)} \;=\; \frac{e^{\,i(m-\alpha_L^{(s)})\psi}}{\sqrt{2\pi}}\, \Psi_{L,\perp}^{(s)}(x_\perp; m)\,, \qquad \langle \psi, x_\perp | n | \psi, x_\perp | n \rangle_R^{(s)} \;=\; \frac{e^{\,i(n-\alpha_R^{(s)})\psi}}{\sqrt{2\pi}}\, \Psi_{R,\perp}^{(s)}(x_\perp; n), \tag{14.1}$$

where $\alpha_{L,R}^{(s)}$ encode the family (Berry) phases of Sec. 13.2. Normalization is $\int d^2x_\perp\, \Psi_{A,\perp}^\dagger \Psi_{A,\perp} = 1$ for $A = L, R$.

Wilson–line dressing. Slow adiabatic twists of the three–mode subbundles along the fiber generate a non–Abelian Berry connection $\mathcal{A}_{\psi;A}^{(s)}$ in family space (Sec. 13.3). The left–right relative Wilson line entering Yukawa overlaps is

$$\mathcal{W}^{(s)}(\psi) \;:=\; \mathcal{P}\exp\!\left\{ i\int_0^\psi d\psi' \left(\mathcal{A}_{\psi';L}^{(s)} - \mathcal{A}_{\psi';R}^{(s)} \right) \right\} \;\in U(3). \tag{14.2}$$

2. Master overlap formula

Definition. Let $h^{(s)}(\psi, x_\perp)$ denote the (possibly weakly modulated) Higgs order–parameter profile seen by sector s within the tube. The *Yukawa matrix* in the light basis is

$$\big(Y_s\big)_{mn} \;=\; y_\star^{(s)} \int_0^{2\pi} \frac{d\psi}{2\pi} \int d^2x_\perp \,\Big[\Psi_{L,\perp}^{(s)}(x_\perp; m)\Big]^\dagger \, h^{(s)}(\psi, x_\perp) \,\Big(\mathcal{W}^{(s)}(\psi)\Big)_{mn} \,\Psi_{R,\perp}^{(s)}(x_\perp; n) \\ \times \exp\Big\{ i\Big[(m-\alpha_L^{(s)}) - (n-\alpha_R^{(s)})\Big]\psi \Big\}. \tag{14.3}$$

Factorization at leading order. In the weak–twist, weak–modulation regime (precisely quantified below), the transverse and fiber parts factorize:

$$\big(Y_s\big)_{mn} \;\simeq\; y_\star^{(s)} \underbrace{\mathcal{O}_\perp^{(s)}(m,n)}_{\text{transverse}} \underbrace{\mathcal{I}_{mn}^{(s)}}_{\text{fiber}}, \qquad \mathcal{O}_\perp^{(s)}(m,n) \;=\; \delta_{mn}\, e^{-d/\ell_H^{(s)}} \,\exp\!\Big[-\frac{\kappa_H^{(s)}}{2}\Big((m-\alpha_L)^2+(n-\alpha_R)^2\Big)\Big], \tag{14.4}$$

where d is the sheet separation, $\ell_H^{(s)}$ the coherence length, and $\kappa_H^{(s)}$ the hierarchy stiffness (Sec. 13.2). The fiber integral

$$\mathcal{I}_{mn}^{(s)} \;:=\; \int_0^{2\pi} \frac{d\psi}{2\pi}\, \mathrm{e}^{\,i[(m-\alpha_L)-(n-\alpha_R)]\psi}\; H^{(s)}(\psi)\,\Big(\mathcal{W}^{(s)}(\psi)\Big)_{mn}, \qquad H^{(s)}(\psi) := \frac{\int d^2x_\perp\, \Psi_{L,\perp}^\dagger\, h^{(s)}\, \Psi_{R,\perp}}{\int d^2x_\perp\, \Psi_{L,\perp}^\dagger \Psi_{R,\perp}}, \tag{14.5}$$

encodes selection rules and phases. Equation (14.4) already explains diagonal dominance: the Gaussian factors suppress entries away from the family phases.

3. Fourier/Bessel representation and controlled off–diagonals

Fourier content of the Higgs profile. Expand the smooth, 2π–periodic $H^{(s)}(\psi)$ in Fourier modes

$$H^{(s)}(\psi) \;=\; \sum_{k\in\mathbb{Z}} h_k^{(s)}\, e^{ik(\psi-\psi_0)}\,, \qquad h_{-k}^{(s)} = h_k^{(s)\,*}. \tag{14.6}$$

If the relative Wilson line is near–Abelian in the three–mode subspace we may parametrize

$$\Big(\mathcal{W}^{(s)}(\psi)\Big)_{mn} \;\simeq\; \exp\Big\{ i\beta_{mn}^{(s)}\, \sin(\psi-\psi_0)\Big\} \;=\; \sum_{\ell\in\mathbb{Z}} J_\ell\Big(\beta_{mn}^{(s)}\Big)\, e^{i\ell(\psi-\psi_0)}, \tag{14.7}$$

with J_ℓ the Bessel function. Inserting (14.6)–(14.7) into (14.5) gives the *Bessel–Fourier* master form

$$\boxed{\;\mathcal{I}_{mn}^{(s)} \;=\; \sum_{k,\ell\in\mathbb{Z}} h_k^{(s)}\, J_\ell\Big(\beta_{mn}^{(s)}\Big)\, \delta_{\,m-n,\;k+\ell}\; e^{-i(k+\ell)\psi_0}\;.\;} \tag{14.8}$$

Therefore Y_{mn} is nonzero only if $m-n$ equals a *Fourier–Wilson* harmonic $k+\ell$.

Small–twist limit. For $|\beta_{mn}^{(s)}| \ll 1$, $J_0 = 1 + \mathcal{O}(\beta^2)$, $J_{\pm1} = \pm\,\beta/2 + \mathcal{O}(\beta^3)$, and $J_{|\ell|\ge 2} = \mathcal{O}(\beta^{|\ell|})$. Keeping $k \in \{0, \pm1\}$ and $\ell \in \{0, \pm1\}$ one finds

$$\mathcal{I}_{mn}^{(s)} \;\simeq\; h_0^{(s)}\,\delta_{mn} \;+\; \frac{\beta_{mn}^{(s)}}{2}\Big(h_1^{(s)}\,\delta_{m,n+1} - h_{-1}^{(s)}\,\delta_{m,n-1}\Big) \;+\; \mathcal{O}(\beta^2, |h_{\pm2}|). \tag{14.9}$$

Hence the *nearest–neighbor* off–diagonals are linearly controlled by the modulation amplitude $|h_{\pm1}|$ and the holonomy parameter $\beta_{mn}^{(s)}$, while further off–diagonals are parametrically smaller.

4. Texture zeros: structural vs. effective

Structural (exact) zeros. Equation (14.8) implies the **selection rule**

$$\mathcal{I}^{(s)}_{mn} = 0 \quad \text{if} \quad m-n \notin \mathcal{K}^{(s)} \;:=\; \{\, k+\ell \mid h^{(s)}_k \neq 0,\; J_\ell(\beta^{(s)}_{mn}) \neq 0 \,\}. \tag{14.10}$$

Thus:

- If $H^{(s)}$ preserves a *fiber shift* $U(1)_\psi$ ($h_k = 0$ for $k \neq 0$) and $\beta^{(s)}_{mn} = 0$, then $\mathcal{I}^{(s)}_{mn} \propto \delta_{mn}$ and *all* off–diagonals vanish *exactly*.
- If a residual $\mathbb{Z}_3$ fiber symmetry holds ($h_k = 0$ unless $k \in 3\mathbb{Z}$, and $\beta = 0$), then Y_s is *circulant*: $Y_{mn} = F_{m-n \bmod 3}$ and is diagonalized by the discrete Fourier transform in family space.

Effective (exponentially small) zeros. Even when the selection rule allows a given entry, (14.4) yields

$$|(Y_s)_{mn}| \;\lesssim\; y^{(s)}_\star\, e^{-d/\ell^{(s)}_H}\, \exp\!\Big[-\frac{\kappa^{(s)}_H}{2}\Big((m-\alpha_L)^2 + (n-\alpha_R)^2\Big)\Big]\; \sum_{k,\ell} |h_k|\, |J_\ell(\beta_{mn})|, \tag{14.11}$$

so entries involving modes far from the family phases are *Gaussian suppressed*, furnishing "texture zeros" in practice.

5. Canonical textures from spectral data

Quasi–diagonal texture. If $H^{(s)}(\psi) = h^{(s)}_0$ and $\beta^{(s)} = 0$, then

$$Y_s \;=\; y^{(s)}_\star\, h^{(s)}_0\, e^{-d/\ell^{(s)}_H}\, \mathrm{diag}\Big(e^{-\kappa^{(s)}_H(0-\alpha^{(s)})^2},\; e^{-\kappa^{(s)}_H(1-\alpha^{(s)})^2},\; e^{-\kappa^{(s)}_H(2-\alpha^{(s)})^2}\Big), \tag{14.12}$$

which reproduces the Gaussian hierarchy of Sec. 13.2.

Nearest–neighbor (tridiagonal) texture. For a single harmonic $k = \pm 1$ and small twist, Eq. (14.9) gives

$$Y_s \;\simeq\; \begin{pmatrix} A_0 & B_{01} & 0 \\ B_{10} & A_1 & B_{12} \\ 0 & B_{21} & A_2 \end{pmatrix},\quad A_m \equiv \mathcal{N}_s\, e^{-\kappa^{(s)}_H(m-\alpha^{(s)})^2},\quad B_{m,m\pm1} \equiv \mathcal{N}_s\, \frac{\beta^{(s)}_{m,m\pm1}}{2}\, h^{(s)}_{\pm1}\, e^{-\frac{\kappa^{(s)}_H}{2}\left[(m-\alpha_L)^2+(m\pm1-\alpha_R)^2\right]}, \tag{14.13}$$

with $\mathcal{N}_s := y^{(s)}_\star e^{-d/\ell^{(s)}_H}$. This realizes a Fritzsch–like nearest–neighbor texture with *controlled* off–diagonals.

Circulant texture. If a residual $\mathbb{Z}_3$ symmetry is approximately preserved (e.g. $H^{(s)}(\psi) \approx h_0 + h_3 e^{3i\psi} + \text{c.c.}$, $\beta \approx 0$) then

$$Y^{(s)}_{mn} \;\approx\; F^{(s)}_{m-n \bmod 3}, \qquad F^{(s)}_0 \propto h_0, \quad F^{(s)}_{\pm1} \propto h_{\pm3}, \tag{14.14}$$

diagonalized by the 3×3 discrete Fourier matrix; small symmetry breaking then perturbs away to realistic mixing.

6. Normalization, rank, and invariants

Rank and determinant. Generically $\mathrm{rank}(Y_s) = 3$ once at least one $h_{\pm 1}$ or nonzero β is present (otherwise Y_s is diagonal). To leading order in small off–diagonals,

$$\det Y_s \;\simeq\; \prod_{m=0}^{2} (Y_s)_{mm} \left[1 + \mathcal{O}\left(\sum_{m \neq n} \frac{|(Y_s)_{mn}|^2}{|(Y_s)_{mm}(Y_s)_{nn}|} \right) \right], \tag{14.15}$$

consistent with the hierarchical structure.

Rephasing invariants. Global family–space rephasings $Y_s \to e^{i\phi_L} Y_s e^{-i\phi_R}$ leave the overlap physics unchanged; physical CP phases reside in closed *loops* of phases from products $(Y_s)_{ij}(Y_s)_{jk}(Y_s)_{ki}$, ultimately descending from ψ–holonomy in (14.2).

7. Validity regime and consistency gates

Small parameters and control. The factorization (14.4) holds when

$$\epsilon_{\mathrm{twist}} := \max_{mn} |\beta^{(s)}_{mn}| \ll 1, \qquad \epsilon_{\mathrm{mod}} := \frac{\sqrt{\sum_{k\neq 0} |h_k|^2}}{|h_0|} \ll 1, \qquad \epsilon_{\mathrm{gap}} := \frac{\|\partial \mathcal{S}\|}{\Delta_{\mathrm{gap}}} \ll 1, \tag{14.16}$$

with Δ_{gap} given by Eq. (13.30). Corrections are systematically organized by powers of $(\epsilon_{\mathrm{twist}}, \epsilon_{\mathrm{mod}}, \epsilon_{\mathrm{gap}})$.

S/G/TQ/D gates.

- **(S) Stability**: Gaussian suppressions ensure bounded spectra; small off–diagonals cannot induce tachyonic directions.
- **(G) Hyperbolicity**: Overlaps do not alter kinetic principal symbols; adiabaticity ($\epsilon_{\mathrm{gap}} \ll 1$) guarantees cone compatibility.
- **(TQ) Positivity**: Two–to–two forward limits for contact operators induced by integrating out heavy fiber harmonics are positive (sum of squares of overlaps).
- **(D) Dissipation**: Medium dissipation renormalizes h_k and κ_H but cannot violate the exact selection rule (14.10).

8. Summary

Yukawa matrices are controlled products of a universal Gaussian *hierarchy kernel* (transverse physics) and a fiber overlap obeying sharp harmonic selection rules. Exact texture zeros arise when residual fiber symmetries restrict the Fourier support of $H^{(s)}$ and suppress holonomy; controlled nearest–neighbor textures follow from small $k = \pm 1$ components and weak twists. This provides a minimal, *geometric* origin for realistic textures with few continuous dials, preparing the ground for CKM/PMNS phenomenology (Sec. 14.2) and for neutrino mass generation via spectral see–saw (Sec. 14.3).

14.2 CKM phenomenology and CP violation

Aim. We connect the spectral overlap construction of Sec. 14.1 to the observed pattern of quark mixing and CP violation. Starting from hierarchical, nearly diagonal Yukawas with controlled nearest–neighbor deformations, we derive: (i) analytic formulae for the CKM angles in terms of geometric parameters $(\kappa_H, \alpha^{(u,d)}, \beta^{(u,d)}, h_k^{(u,d)})$; (ii) a Wolfenstein–like parameterization with spectral content; (iii) expressions for the Jarlskog invariant $J_{\rm CKM}$ and unitarity–triangle phases in terms of *geometric* holonomies; (iv) stability/positivity gates and phenomenological *kill criteria*.

1. From spectral Yukawas to CKM

Yukawa inputs. For $s \in \{u, d\}$, Eqs. (14.4)–(14.13) give

$$Y_s \simeq \mathcal{N}_s \begin{pmatrix} A_0^{(s)} & B_{01}^{(s)} & \varepsilon_{013}^{(s)} \\ B_{10}^{(s)} & A_1^{(s)} & B_{12}^{(s)} \\ \varepsilon_{310}^{(s)} & B_{21}^{(s)} & A_2^{(s)} \end{pmatrix}, \qquad A_m^{(s)} = e^{-\kappa_H^{(s)}(m-\alpha^{(s)})^2}, \quad B_{m,m\pm1}^{(s)} = \frac{\beta_{m,m\pm1}^{(s)}}{2}\, h_{\pm1}^{(s)}\, e^{-\frac{\kappa_H^{(s)}}{2}\left[(m-\alpha_L)^2+(m\pm1-\alpha_R\right.} \tag{14.17}$$

with $\mathcal{N}_s = y_\star^{(s)} e^{-d/\ell_H^{(s)}}$ and small next–nearest entries $\varepsilon_{013}^{(s)}, \varepsilon_{310}^{(s)} = \mathcal{O}\big(\beta^2, |h_{\pm2}|\big)$. Define $M_s := Y_s Y_s^\dagger$ and diagonalize $M_s = U_L^{(s)}\, \mathrm{diag}(m_u^2, m_c^2, m_t^2)\, (U_L^{(s)})^\dagger$ (and analogously for d). The CKM matrix is

$$V_{\rm CKM} \;=\; U_L^{(u)\,\dagger}\, U_L^{(d)}. \tag{14.18}$$

Small–angle expansion. Using Eq. (13.35),

$$\theta_{ij}^{(s)} \;\simeq\; \frac{\left|\varepsilon_L^{(s)}\, \Xi_{ij}^{(s)}\right|}{2\sinh\!\Big(\kappa_H^{(s)}[(j-\alpha^{(s)})^2 - (i-\alpha^{(s)})^2]\Big)}\,, \qquad i<j, \tag{14.19}$$

and, to leading order,

$$|V_{us}| \;\simeq\; |\theta_{12}^{(d)} - \theta_{12}^{(u)}|\,, \quad |V_{cb}| \;\simeq\; |\theta_{23}^{(d)} - \theta_{23}^{(u)}|\,, \quad |V_{ub}| \;\simeq\; |\theta_{13}^{(d)} - \theta_{13}^{(u)}\, e^{i\Delta\phi_{13}}|\,, \tag{14.20}$$

with $\Delta\phi_{13}$ a *geometric* phase difference defined below.

2. Wolfenstein map with spectral content

Definition. Introduce the Wolfenstein parameters $\{\lambda, A, \rho, \eta\}$ via

$$\lambda \;:=\; |V_{us}|\,, \qquad A\lambda^2 \;:=\; |V_{cb}|\,, \qquad A\lambda^3\sqrt{\rho^2+\eta^2} \;:=\; |V_{ub}|\,, \qquad \delta \;:=\arg\!\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right). \tag{14.21}$$

Equations (14.19)–(14.20) give the *spectral* map

$$\lambda \;\simeq\; \left|\frac{\varepsilon_L^{(d)}\Xi_{12}^{(d)}}{2\sinh\left(\kappa_H^{(d)}\Delta d_{12}^{(d)}\right)} - \frac{\varepsilon_L^{(u)}\Xi_{12}^{(u)}}{2\sinh\left(\kappa_H^{(u)}\Delta d_{12}^{(u)}\right)}\right|, \tag{14.22}$$

$$A\lambda^2 \;\simeq\; \left|\frac{\varepsilon_L^{(d)}\Xi_{23}^{(d)}}{2\sinh\left(\kappa_H^{(d)}\Delta d_{23}^{(d)}\right)} - \frac{\varepsilon_L^{(u)}\Xi_{23}^{(u)}}{2\sinh\left(\kappa_H^{(u)}\Delta d_{23}^{(u)}\right)}\right|, \tag{14.23}$$

$$A\lambda^3\sqrt{\rho^2+\eta^2} \;\simeq\; \left|\frac{\varepsilon_L^{(d)}\Xi_{13}^{(d)}}{2\sinh\left(\kappa_H^{(d)}\Delta d_{13}^{(d)}\right)} - \frac{\varepsilon_L^{(u)}\Xi_{13}^{(u)}\, e^{i\Delta\phi_{13}}}{2\sinh\left(\kappa_H^{(u)}\Delta d_{13}^{(u)}\right)}\right|, \tag{14.24}$$

with $\Delta d_{ij}^{(s)} := (j-\alpha^{(s)})^2 - (i-\alpha^{(s)})^2$ and $\Delta\phi_{13}$ a Berry/holonomy phase difference between u and d three–mode bundles.

3. Geometric CP phase and Jarlskog invariant

Holonomy–induced phase. Let $\mathcal{W}^{(s)}$ be the relative Wilson line (14.2), and write nearest–neighbor complex entries as

$$B_{m,m+1}^{(s)} \;=\; |B_{m,m+1}^{(s)}|\, e^{i\phi_{m,m+1}^{(s)}}, \qquad \phi_{m,m+1}^{(s)} = \arg\Big[\beta_{m,m+1}^{(s)}\, h_{+1}^{(s)} \left(\mathcal{W}^{(s)}\right)_{m,m+1}\Big]. \tag{14.25}$$

The *geometric* CKM phase is then controlled by the gauge–invariant loop

$$\Phi_{\rm geom} \;:=\; \left(\phi_{12}^{(d)} + \phi_{23}^{(d)} - \phi_{13}^{(d)}\right) \;-\; \left(\phi_{12}^{(u)} + \phi_{23}^{(u)} - \phi_{13}^{(u)}\right), \tag{14.26}$$

equal to a combination of Berry curvatures integrated over a minimal 2–simplex in the family bundle.

Jarlskog invariant. For small angles,

$$J_{\rm CKM} \;= \mathrm{Im}\left(V_{us}V_{cb}V_{ub}^{*}V_{cs}^{*}\right) \;\simeq\; \lambda\, A^2\lambda^4\,\sqrt{\rho^2+\eta^2}\,\sin\delta \;\simeq\; \left|\theta_{12}\theta_{23}\theta_{13}\right|\,\sin\Phi_{\rm geom}, \tag{14.27}$$

which, using (14.19)–(14.24), becomes a *pure* function of $(\kappa_H^{(u,d)}, \alpha^{(u,d)}, \varepsilon_L^{(u,d)}\Xi_{ij}^{(u,d)}, \Phi_{\rm geom})$.

4. GST–type relations from tridiagonal textures

Mass–mixing links. For the tridiagonal form (14.13) with hierarchical diagonals and $|B_{ij}| \ll A_i$, the diagonalization gives (Fritzsch/Gatto–Sartori–Tonin–type)

$$\theta_{12}^{(s)} \;\simeq\; \left|\sqrt{\frac{m_1^{(s)}}{m_2^{(s)}}}\, e^{i\phi_{12}^{(s)}} \;-\; \sqrt{\frac{m_1^{(s')}}{m_2^{(s')}}}\, e^{i\phi_{12}^{(s')}}\right| \quad \text{(for } s \text{ vs. } s' \text{ sectors)}, \tag{14.28}$$

$$\theta_{23}^{(s)} \;\simeq\; \left|\sqrt{\frac{m_2^{(s)}}{m_3^{(s)}}}\, e^{i\phi_{23}^{(s)}} \;-\; \sqrt{\frac{m_2^{(s')}}{m_3^{(s')}}}\, e^{i\phi_{23}^{(s')}}\right|, \qquad \theta_{13}^{(s)} \;\simeq\; \left|\sqrt{\frac{m_1^{(s)}}{m_3^{(s)}}}\, e^{i\phi_{13}^{(s)}} \;-\; \sqrt{\frac{m_1^{(s')}}{m_3^{(s')}}}\, e^{i\phi_{13}^{(s')}}\right|, \tag{14.29}$$

where $m_i^{(s)}$ denote the physical masses of sector s. In the spectral model the *mass ratios* are controlled by Gaussian distances, Eq. (13.22), e.g.

$$\frac{m_1^{(s)}}{m_2^{(s)}} \;\simeq\; \exp\Big[-\kappa_H^{(s)}\Big((m_1-\alpha^{(s)})^2 - (m_2-\alpha^{(s)})^2\Big)\Big], \tag{14.30}$$

so the GST relations become *geometric consequences* of the hierarchy kernel.

5. Unitarity triangle and spectral phases

Angles. Define

$$\alpha \;:= \arg\left(-\frac{V_{td}V_{tb}^{*}}{V_{ud}V_{ub}^{*}}\right), \qquad \beta \;:= \arg\left(-\frac{V_{cd}V_{cb}^{*}}{V_{td}V_{tb}^{*}}\right), \qquad \gamma \;:= \arg\left(-\frac{V_{ud}V_{ub}^{*}}{V_{cd}V_{cb}^{*}}\right) \equiv \delta. \tag{14.31}$$

In our framework, γ coincides with the geometric phase $\delta \simeq \Phi_{\rm geom}$ of (14.26); α and β follow from the same phase data plus the moduli fixed by (14.22)–(14.24). Consistency requires $\alpha+\beta+\gamma=\pi$ (unitarity), providing a sharp check of numerical implementations.

6. RG robustness and MFV structure

Running. Between the spectral matching scale $\mu_\star$ and the electroweak scale, $Y_{u,d}$ obey standard one–loop RGEs. Because off–diagonals are already small and aligned with $Y_{u,d}$, the evolution preserves the *Minimal Flavor Violation* (MFV) pattern: all flavor/CP violation remains spurionic in $Y_{u,d}$ with no extra sources from the spectral medium at leading order. Hence the CKM structure derived above is RG–stable up to calculable $\mathcal{O}(\alpha_{s,2,1}/4\pi)$ effects.

7. Phenomenological handles and kill criteria

Handles. The following correlations are *predictions* of the spectral geometry:

$$\text{(H1)}: \quad \frac{|V_{ub}|/|V_{cb}|}{|V_{td}|/|V_{ts}|} \approx \frac{\sinh\left(\kappa_H^{(u)}\Delta d_{23}^{(u)}\right)}{\sinh\left(\kappa_H^{(u)}\Delta d_{13}^{(u)}\right)} \cdot \frac{\sinh\left(\kappa_H^{(d)}\Delta d_{13}^{(d)}\right)}{\sinh\left(\kappa_H^{(d)}\Delta d_{23}^{(d)}\right)}\,, \tag{14.32}$$

$$\text{(H2)}: \quad \frac{|V_{us}|}{\sqrt{m_d/m_s}} \sim \left|1 - \sqrt{\frac{m_u/m_c}{m_d/m_s}}\, e^{i(\phi_{12}^{(u)} - \phi_{12}^{(d)})}\right|, \tag{14.33}$$

$$\text{(H3)}: \quad J_{\rm CKM} \simeq |V_{us}||V_{cb}||V_{ub}| \sin\Phi_{\rm geom}\,, \tag{14.34}$$

with mass ratios given by Eq. (14.30).

Kill criteria. The minimal spectral CKM sector is falsified if any of the following *mutually consistent* inequalities fail by more than stated uncertainties:

1. **K1 (hierarchy gate)**: the observed ordering $|V_{us}| \gg |V_{cb}| \gg |V_{ub}|$ cannot be reproduced for *any* real $\kappa_H^{(u,d)} > 0$ and phases $\alpha^{(u,d)} \in [0, 1)$;
2. **K2 (Jarlskog bound)**: the measured $J_{\rm CKM}$ exceeds the maximal geometric value $J_{\max} = |V_{us}||V_{cb}||V_{ub}|$ (i.e. $\sin\Phi_{\rm geom} > 1$);
3. **K3 (GST tension)**: the GST–type relation (14.33) fails grossly for *all* allowed phases $\phi_{12}^{(u)} - \phi_{12}^{(d)}$;
4. **K4 (triangle closure)**: $\alpha + \beta + \gamma \neq \pi$ beyond fit/systematic errors when $V_{\rm CKM}$ is reconstructed from the spectral map (14.22)–(14.24).

8. Consistency gates (S/G/TQ/D)

(S) Stability. Small off–diagonals and positive–definite M_s ensure a bounded mass spectrum; integrating out heavy fiber harmonics yields higher–dimensional operators suppressed by the spectral gap (no tachyons).

(G) Hyperbolicity. Mixing matrices act in internal family space and leave kinetic principal symbols unchanged; adiabatic conditions maintain cone compatibility.

(TQ) Positivity/unitarity. Forward–limit bounds are satisfied because four–fermion Wilson coefficients generated by heavy modes are sums of *squares* of overlaps. $V_{\rm CKM}$ is unitary by construction from $U(3)$ rotations in the three–mode manifolds.

(D) Dissipation. Dissipative medium effects renormalize the small parameters $(\varepsilon_L, \beta, h_{\pm 1})$ but cannot generate new flavor structures beyond those already present in the spectral overlaps (MFV).

9. Summary

The CKM matrix emerges from controlled geometric misalignments between the u and d three–mode family bundles. Its magnitudes are governed by hyperbolic suppressions set by (κ_H, α), while its unique CP phase is the *Berry–holonomy* Φ_{geom}. Wolfenstein parameters admit a direct spectral map (14.22)–(14.24), yielding sharp correlations (14.32)–(14.34) and falsifiable *kill criteria*. This completes the quark–sector phenomenology; Sec. 14.3 turns to neutrino masses and the spectral see–saw.

14.3 Neutrino masses: spectral see-saw

Aim. We derive Majorana neutrino masses from a *spectral see–saw* in which heavy, gauge–singlet fermions arise as bound modes on family textures and acquire lepton–number–violating (LNV) masses from CP^1 director condensates and/or spectral instantons. The light neutrino mass matrix follows the universal form

$$\boxed{m_\nu \;=\; -\,m_D\,M_R^{-1}\,m_D^{\mathsf{T}} \;=\; -\,\frac{v^2}{2}\,Y_\nu\,M_R^{-1}\,Y_\nu^{\mathsf{T}}\,,} \tag{14.35}$$

with the Dirac Yukawa Y_ν given by the overlap integrals of Sec. 14.1 and the Majorana mass M_R determined by spectral LNV sources specified below. We provide: (i) a geometric construction of M_R; (ii) matching to the Weinberg operator; (iii) rank/texture implications; (iv) RG evolution and leptogenesis handles; (v) consistency gates and falsifiability.

1. Spectral origin of lepton number violation

Emergent $U(1)_L$ and its breaking. Lepton number $U(1)_L$ is an accidental symmetry of the leading spectral action in the leptonic sector. LNV arises from two, not mutually exclusive, spectral mechanisms:

1. **Director–condensate (DC) mechanism**: a CP^1 bilinear built from the director $\hat{n}$ (or its time–like component tied to the internal clock T) condenses at a high scale Λ_{LNV}, generating a gauge–singlet bilinear $\langle \mathcal{O}_{\text{LNV}} \rangle \neq 0$ that sources a right–handed Majorana mass.
2. **Instanton/monopole (IM) mechanism**: rare spectral tunnelling events (Hopf/center–vortex reconnections, Sec. 12.2) produce an LNV 't Hooft–like vertex with action S_{IM}; the induced M_R is exponentially suppressed, $M_R \sim \Lambda_\star e^{-S_{\text{IM}}}$.

In both cases, the heavy states are *spectral singlets* N_R—bound zero modes localized on family textures but neutral under $SU(2)_L \times U(1)_Y$.

Right–handed sector and Majorana kernel. Let $\{|p\rangle_R\}_{p=0,1,2}$ denote the three lightest singlet modes along the Hopf fiber on a companion texture sheet (not necessarily the same as the left leptonic sheet). The LNV kernel is the overlap

$$(M_R)_{pq} \;=\; \Lambda_{\rm LNV} \int \frac{d\psi}{2\pi} \int d^2x_\perp \, \Big[\Psi_{R,\perp}(x_\perp;p)\Big]^{\mathsf T} \, \mathcal{C}\, \mathcal{K}_{\rm LNV}(\psi,x_\perp)\, \Psi_{R,\perp}(x_\perp;q)\, e^{\,i(p-\tilde\alpha_R)\psi}\, e^{\,i(q-\tilde\alpha_R)\psi}, \tag{14.36}$$

where $\mathcal{C}$ is charge–conjugation, $\tilde\alpha_R$ the singlet Berry phase, and $\mathcal{K}_{\rm LNV}$ encodes either DC ($\propto \langle \hat n \hat n \rangle$) or IM contributions (multi–fermion vertex contracted to a bilinear). Selection rules analogous to Sec. 14.1 apply: only harmonics $p+q = k+\ell$ with k from the Fourier support of $\mathcal{K}_{\rm LNV}$ and ℓ from its Wilson–line holonomy contribute.

2. Dirac overlaps and smallness from geometry

Dirac matrix. From the master overlap formula,

$$(m_D)_{mn} \;=\; \frac{v}{\sqrt{2}}\,(Y_\nu)_{mn} \;\simeq\; \frac{v}{\sqrt{2}}\, y_\star^{(\nu)}\, e^{-d_\nu/\ell_H^{(\nu)}}\, \exp\!\Big[-\frac{\kappa_H^{(\nu)}}{2}\Big((m-\alpha_L^{(\nu)})^2 + (n-\alpha_R^{(\nu)})^2\Big)\Big]\, \mathcal{I}^{(\nu)}_{mn}, \tag{14.37}$$

with $\mathcal{I}^{(\nu)}_{mn}$ the fiber factor (§14.1). Even if M_R sits well below the UV cutoff, the double Gaussian in (14.37) yields naturally tiny entries, providing an *intrinsic* suppression besides the M_R^{-1} in (14.35).

3. Matching to the Weinberg operator

Tree matching. Integrating out N_R at $\mu = \mu_R$ gives

$$\mathcal{L}_{\rm eff} \;\supset\; \frac{1}{2}\, c_5^{ij}\, (\overline{L_i}\tilde H)\, (\tilde H^{\mathsf T} L_j^{\mathsf c}) \;+\; \text{h.c.}, \qquad \boxed{\,c_5 \;=\; Y_\nu\, M_R^{-1}\, Y_\nu^{\mathsf T}\,} \tag{14.38}$$

so $m_\nu = c_5\, v^2/2$ reproduces (14.35). The *rank* of m_ν equals the number of heavy singlets with finite mass: with two N_R one predicts $\mathrm{rank}(m_\nu) = 2$ and one massless neutrino.

Spectral scaling. Writing $\|m_D\| \sim (v/\sqrt{2})\, y_\star^{(\nu)}\, e^{-d_\nu/\ell_H^{(\nu)}}$ and $\|M_R\| \sim \Lambda_{\rm LNV}\, \xi_R$ (dimensionless overlap), a norm estimate yields

$$\|m_\nu\| \;\sim\; \frac{v^2}{2\,\Lambda_{\rm LNV}}\, \frac{\Big(y_\star^{(\nu)}\, e^{-d_\nu/\ell_H^{(\nu)}}\Big)^2}{\xi_R} \;\lesssim\; \mathcal{O}(0.1\,\text{eV}) \quad\Rightarrow\quad \Lambda_{\rm LNV} \;\gtrsim\; 10^{14}\,\text{GeV} \times \frac{\Big(y_\star^{(\nu)} e^{-d_\nu/\ell_H^{(\nu)}}\Big)^2}{\xi_R}, \tag{14.39}$$

consistent with a grand–unified–like scale or with exponentially suppressed IM contributions.

4. Canonical textures and rank statements

Diagonal M_R. If $\mathcal{K}_{\rm LNV}$ preserves a fiber shift (no off–diagonal harmonics), then

$$M_R \;\simeq\; \mathrm{diag}(M_{R,0},\, M_{R,1},\, M_{R,2})\,, \qquad M_{R,n} \propto \exp\!\Big[-2\kappa_R\,(n-\tilde\alpha_R)^2\Big], \tag{14.40}$$

with a *right–handed* stiffness κ_R. Combined with a near–diagonal m_D, this yields m_ν diagonal at leading order; large PMNS angles then demand either (i) misalignment between left and right phases ($\alpha_L^{(\nu)} \neq \alpha_R^{(\nu)}$), (ii) non–zero holonomies in $\mathcal{I}^{(\nu)}$, or (iii) off–diagonal harmonics in M_R.

Two–singlet (minimal) scenario. If only two N_R are light (third decouples), $\det m_\nu = 0$ and one neutrino is massless. This robust prediction is stable under RG and small geometric deformations.

Circulant M_R. A residual $\mathbb{Z}_3$ symmetry in $\mathcal{K}_{\rm LNV}$ produces a circulant M_R, diagonalized by the discrete Fourier matrix. Combining with the non–circulant m_D from Sec. 14.1 yields naturally large mixings.

5. Majorana phases and observables

Diagonalization. Let $m_\nu = U_\nu^* \,\mathrm{diag}(m_1, m_2, m_3)\, U_\nu^\dagger$. The PMNS matrix is $U_{\rm PMNS} = U_L^{(\ell)\dagger} U_\nu$, containing one Dirac phase δ (Sec. 14.4) and two Majorana phases α_{21}, α_{31}.

Neutrinoless double beta decay. The effective mass is

$$m_{\beta\beta} \;=\; \Big| (U_{\rm PMNS})_{e1}^2 m_1 + (U_{\rm PMNS})_{e2}^2 m_2 e^{i\alpha_{21}} + (U_{\rm PMNS})_{e3}^2 m_3 e^{i\alpha_{31}} \Big| \,. \tag{14.41}$$

In the spectral picture the Majorana phases are Wilson–loop phases inherited from $\mathcal{W}^{(\nu)}$ and from the phase structure of M_R (Eq. (14.36)); thus $m_{\beta\beta}$ probes *fiber holonomy*.

6. RG evolution and threshold matching

Below and above M_R. Between $\mu = M_R$ and $\mu = m_Z$, the Weinberg operator runs as

$$16\pi^2 \frac{d\, c_5}{d \ln \mu} \;=\; \Big[-3g_2^2 + 2\lambda_H + 6\,\mathrm{Tr}(Y_u Y_u^\dagger) \Big] c_5 \;-\; \frac{1}{2}\Big[(Y_e Y_e^\dagger)\, c_5 + c_5\, (Y_e Y_e^\dagger)^{\mathsf{T}} \Big]. \tag{14.42}$$

Above M_R the coupled (Y_ν, M_R) RGEs apply; spectral holonomies are scale–independent to leading order (topological). Thresholds split if the three $M_{R,n}$ are hierarchical, with matching performed sequentially.

7. Leptogenesis handle (qualitative)

CP asymmetry from spectral phases. Decays $N_{R,p} \to \ell H$ carry CP asymmetry

$$\varepsilon_p \;\propto\; \frac{1}{8\pi} \sum_{q \neq p} \frac{\mathrm{Im}\Big[(Y_\nu^\dagger Y_\nu)_{pq}^2 \Big]}{(Y_\nu^\dagger Y_\nu)_{pp}} \; f\!\left(\frac{M_{R,q}^2}{M_{R,p}^2} \right), \tag{14.43}$$

with the imaginary parts inherited from the same holonomies producing PMNS phases. Successful thermal leptogenesis thus correlates with PMNS CP violation, providing a spectral consistency check once washout is included.

8. Consistency gates and kill criteria

(S) Stability. m_ν is positive semi–definite; heavy–light mixing $\Theta \sim m_D M_R^{-1}$ must satisfy $\|\Theta\| \ll 1$ to keep kinetic hyperbolicity intact and to evade non–unitarity bounds.

(G) Hyperbolicity/causality. The see–saw modifies mass terms only; principal symbols (light–cone structure of $g_{\rm eff}$) are unchanged.

(TQ) Positivity/unitarity. Forward–limit bounds are satisfied since $c_5 = Y_\nu M_R^{-1} Y_\nu^\mathsf{T}$ is a *Gram*-type matrix (sum of squares weighted by positive M_R^{-1}).

(D) Dissipation. Dissipative renormalizations shift $(y_\star^{(\nu)}, \kappa_H^{(\nu)}, \ell_H^{(\nu)})$ but cannot generate or remove topology (holonomies); LNV requires the DC or IM sources explicitly.

Kill criteria. The minimal spectral see–saw is falsified if any hold:

1. **K1**: m_ν cannot reach the observed mass splittings for *any* $(\Lambda_{\rm LNV}, y_\star^{(\nu)}, \kappa_H^{(\nu)}, \alpha_{L,R}^{(\nu)})$ within the S/G/TQ/D domain;
2. **K2**: $\mathrm{rank}(m_\nu) \leq 2$ is excluded experimentally while the model employs only two N_R;
3. **K3**: predicted correlation between $m_{\beta\beta}$ and PMNS phases implied by the holonomy structure conflicts with $0\nu\beta\beta$ limits and oscillation data simultaneously;
4. **K4**: non–unitarity from $\Theta\Theta^\dagger$ exceeds precision bounds when fitting masses.

9. Summary

Neutrino masses emerge naturally from a spectral see–saw: small Dirac overlaps from fiber geometry combine with a high–scale, spectrally generated M_R to yield $m_\nu = -m_D M_R^{-1} m_D^\mathsf{T}$. Selection rules and holonomies control textures and phases; rank statements (e.g. one massless neutrino with two N_R) and correlations with $m_{\beta\beta}$ provide sharp tests. RG evolution preserves the geometric structure, while leptogenesis links the same spectral phases to the cosmic baryon asymmetry. The next section turns these ingredients into quantitative PMNS predictions against oscillation data.

14.4 PMNS predictions and oscillation data

Aim. We turn the spectral see–saw construction of Sec. 14.3 into quantitative predictions for the leptonic mixing matrix (PMNS), mass ordering, absolute scales, and neutrinoless double–beta decay. The derivation is anchored in two inputs: (i) the Dirac overlaps Y_ν from the *three–mode* family geometry (Sec. 14.1); and (ii) the Majorana kernel M_R from spectral LNV sources (Sec. 14.3). We provide analytic control of angles/phases, identify robust correlations, and state falsifiable gates.

1. From spectral inputs to PMNS: Takagi map and parameterization

Light–neutrino mass matrix. At the matching scale μ_R (where heavy singlets are integrated out), the Majorana mass matrix reads

$$m_\nu \;=\; -\,m_D\, M_R^{-1}\, m_D^\mathsf{T} \;=\; -\,\frac{v^2}{2}\, Y_\nu\, M_R^{-1}\, Y_\nu^\mathsf{T}\,, \qquad m_\nu^\mathsf{T} = m_\nu. \tag{14.44}$$

In the charged–lepton mass basis (i.e. after left–unitary rotation $U_L^{(\ell)}$ that diagonalizes Y_ℓ), the PMNS matrix is the Takagi factor of m_ν:

$$U_{\rm PMNS} \;:=\; U_L^{(\ell)\,\dagger}\, U_\nu, \qquad U_\nu^\mathsf{T}\, m_\nu\, U_\nu \;=\; \mathrm{diag}(m_1, m_2, m_3), \qquad m_i \geq 0. \tag{14.45}$$

We adopt the PDG parameterization

$$U_{\rm PMNS} \;=\; R_{23}(\theta_{23})\,\Gamma_\delta\,R_{13}(\theta_{13})\,\Gamma_\delta^\dagger\,R_{12}(\theta_{12})\,\mathrm{diag}\Big(1,e^{i\alpha_{21}/2},e^{i\alpha_{31}/2}\Big)\,, \tag{14.46}$$

with one Dirac phase δ and two Majorana phases α_{21},α_{31}.

2. Analytic control of mixing angles from m_ν

Flavor basis and small–off–diagonal expansion. Work in the charged–lepton mass basis; denote $m_{\alpha\beta}\equiv(m_\nu)_{\alpha\beta}$, $\alpha,\beta=e,\mu,\tau$. When $|m_{\alpha\beta}|\ll|m_{\beta\beta}-m_{\alpha\alpha}|$ one has the textbook relations

$$\tan 2\theta_{12}^{(\nu)} \simeq \frac{2|m_{e\mu}c_{23}+m_{e\tau}s_{23}|}{|m_{\mu\mu}c_{23}^2+2m_{\mu\tau}c_{23}s_{23}+m_{\tau\tau}s_{23}^2-m_{ee}|}, \tag{14.47}$$

$$\tan 2\theta_{13}^{(\nu)} \simeq \frac{2|m_{e\tau}c_{23}-m_{e\mu}s_{23}|}{|m_{\tau\tau}c_{23}^2-2m_{\mu\tau}c_{23}s_{23}+m_{\mu\mu}s_{23}^2-m_{ee}|}, \tag{14.48}$$

$$\tan 2\theta_{23}^{(\nu)} \simeq \frac{2|m_{\mu\tau}|}{|m_{\tau\tau}-m_{\mu\mu}|}, \tag{14.49}$$

with phases of $m_{\alpha\beta}$ feeding the Dirac phase δ through rephasing–invariant triangles. The physical angles are $\theta_{ij}\simeq\theta_{ij}^{(\nu)}-\theta_{ij}^{(\ell)}$; in our framework $\theta_{ij}^{(\ell)}$ are small (Sec. 13.3), so U_ν dominantly sets PMNS.

Seesaw composition rule. Using $m_\nu=-\sum_k m_D^{(\cdot k)}m_D^{(\cdot k)\,\mathsf{T}}/M_{R,k}$, the off–diagonals are controlled by *pairwise* geometric overlaps:

$$m_{\alpha\beta} \;=\; -\sum_{k=0}^{2}\frac{(m_D)_{\alpha k}\,(m_D)_{\beta k}}{M_{R,k}} \;\propto\; -\sum_k\Big[\mathcal{I}_{\alpha k}^{(\nu)}\,\mathcal{I}_{\beta k}^{(\nu)}\Big]\;e^{-\frac{\kappa_H^{(\nu)}}{2}\left[(m_\alpha-\alpha_L^{(\nu)})^2+(m_\beta-\alpha_L^{(\nu)})^2+2(n_k-\alpha_R^{(\nu)})^2\right]}, \tag{14.50}$$

where $m_\alpha,n_k\in\{0,1,2\}$ label the family harmonics of the left/right modes. Large mixing angles emerge whenever (i) two diagonal entries are *quasi–degenerate* and (ii) a single k–tower dominates the sum in (14.50), setting the phase of $m_{\alpha\beta}$.

3. Ordering, absolute scale, and spectral regimes

Normal vs. inverted ordering. Define eigenvalues $m_1\le m_2\le m_3$. Two robust tendencies follow from geometry:

1. **Diagonal M_R & hierarchical m_D ⇒ normal ordering.** If $M_R=\mathrm{diag}(M_{R,0},M_{R,1},M_{R,2})$ with mild hierarchy and m_D exhibits Gaussian ordering along fiber (Sec. 13.2), then the light spectrum inherits a normal pattern $m_3\gg m_{1,2}$ unless phases enforce tuned cancellations in (14.50).
2. **Off–diagonal M_R or near–circulant patterns ⇒ flexible ordering.** A residual $\mathbb{Z}_3$ symmetry in M_R or a dominant nearest–neighbor harmonic can invert the hierarchy by aligning m_{ee} with $m_{\mu\mu}\approx m_{\tau\tau}$ while enhancing $m_{e\mu},m_{e\tau}$.

Absolute scale and cosmology. Let $\Sigma_\nu := m_1 + m_2 + m_3$ be the cosmological sum. From (14.44) and the norm estimate (14.39),

$$\Sigma_\nu \;\sim\; \frac{v^2}{2\Lambda_{\rm LNV}}\,\frac{\|Y_\nu\|^2}{\xi_R} \;\lesssim\; \Sigma_\nu^{\max}, \qquad \Rightarrow \quad \Lambda_{\rm LNV} \;\gtrsim\; \frac{v^2}{2\Sigma_\nu^{\max}}\,\frac{\|Y_\nu\|^2}{\xi_R}, \tag{14.51}$$

where $\Sigma_\nu^{\max}$ is the empirical bound from cosmology. Thus the spectral see–saw translates cosmological limits into lower bounds on the LNV scale and/or on the geometric suppression $\|Y_\nu\|$.

4. Neutrinoless double–beta decay and Majorana phases

Effective mass and holonomies. The $0\nu\beta\beta$ amplitude probes

$$m_{\beta\beta} \;= \Big|\,(U_{\rm PMNS})_{e1}^2 m_1 + (U_{\rm PMNS})_{e2}^2 m_2 e^{i\alpha_{21}} + (U_{\rm PMNS})_{e3}^2 m_3 e^{i\alpha_{31}}\,\Big|. \tag{14.52}$$

In the spectral model, α_{21}, α_{31} are *Wilson–loop* phases inherited from the relative holonomy $\mathcal{W}^{(\nu)}$ and from complex structure in M_R; hence $m_{\beta\beta}$ becomes a direct probe of fiber–bundle phases. Parametrically,

$$m_{\beta\beta} \;\approx\; \left|\sum_{k=0}^{2} \frac{(m_D)_{ek}^2}{M_{R,k}}\right| \;\sim\; \frac{v^2}{2}\left|\sum_{k} \frac{(Y_\nu)_{ek}^2}{M_{R,k}}\right|, \tag{14.53}$$

so cancellations ("chimney" region) correspond to destructive interference among the k–towers controlled by holonomies.

5. Benchmark spectral scenarios and predictions

(A) Minimal rank–2 see–saw. With two heavy singlets effectively contributing, $\operatorname{rank}(m_\nu) = 2$ and $m_{\rm lightest} = 0$. Then

$$\Sigma_\nu \;= \begin{cases} \sqrt{\Delta m^2_{21}} + \sqrt{\Delta m^2_{31}} & \text{(normal)},\\ \sqrt{|\Delta m^2_{31}|} + \sqrt{|\Delta m^2_{31}| + \Delta m^2_{21}} & \text{(inverted)}, \end{cases} \qquad m_{\beta\beta} \text{ confined to a narrow band set by } \alpha_{21}. \tag{14.54}$$

Large atmospheric/solar angles arise naturally if a single k–tower dominates $m_{e\mu}, m_{e\tau}, m_{\mu\tau}$.

(B) Near–circulant M_R (residual $\mathbb{Z}_3$). If M_R is approximately circulant while m_D is hierarchical Gaussian, m_ν inherits a *trimaximal* column; then

$$|U_{e2}| \;\approx\; |U_{\mu 2}| \;\approx\; |U_{\tau 2}| \;\approx\; \frac{1}{\sqrt{3}}, \qquad \theta_{12} \text{ large},\ \theta_{13} \text{ generated by weak symmetry breaking in } M_R \text{ or } Y_\nu. \tag{14.55}$$

(C) Single–harmonic deformation. If one off–diagonal harmonic dominates M_R (say, $p \leftrightarrow p{+}1$), then from (14.48)–(14.49)

$$\theta_{23} \;\simeq\; \frac{|(m_\nu)_{\mu\tau}|}{|m_{\tau\tau} - m_{\mu\mu}|}, \qquad \theta_{13} \;\simeq\; \frac{|(m_\nu)_{e\tau} c_{23} - (m_\nu)_{e\mu} s_{23}|}{|m_{\tau\tau} c_{23}^2 - 2 m_{\mu\tau} c_{23} s_{23} + m_{\mu\mu} s_{23}^2 - m_{ee}|}, \tag{14.56}$$

yielding a controlled path to nonzero θ_{13} with correlated δ from the single complex harmonic.

6. Fit protocol and spectral parameter economy

Inputs. The neutrino sector at the matching scale is controlled by a small set of *geometric* parameters:

$$\Big\{\kappa_H^{(\nu)},\ \alpha_{L,R}^{(\nu)},\ \varepsilon_L^{(\nu)},\ \beta^{(\nu)},\ \text{Fourier amplitudes/phases of } H^{(\nu)}(\psi),\ \text{and of } \mathcal{K}_{\rm LNV}\Big\}$$

plus the three (or two) heavy masses $M_{R,k}$. Cross–sector universality ties (κ_H, ℓ_H) to the charged leptons/quarks (Parts 14, **??**), raising predictivity.

Procedure. (i) Build Y_ν from the master overlap (14.3); (ii) specify M_R from DC/IM sources (Sec. 14.3); (iii) compute m_ν and Takagi–diagonalize to obtain $\{m_i, \theta_{ij}, \delta, \alpha_{21}, \alpha_{31}\}$; (iv) run down with (14.42); (v) confront $\{\Delta m^2_{21}, \Delta m^2_{31}, \theta_{ij}\}$, Σ_ν, and $m_{\beta\beta}$.

7. Consistency gates and falsifiable targets

(S) Stability. m_ν is positive semi–definite by construction; heavy–light mixing $\Theta = m_D M_R^{-1}$ must satisfy $\|\Theta\| \ll 1$ to preserve perturbative unitarity in weak decays.

(G) Hyperbolicity. Mass generation does not modify kinetic principal symbols; adiabaticity in the three–mode manifold ensures cone compatibility.

(TQ) Positivity/unitarity. The Weinberg operator coefficient $c_5 = Y_\nu M_R^{-1} Y_\nu^{\mathsf{T}}$ is a Gram matrix and obeys forward–limit positivity; RG running preserves this property.

(D) Dissipation. Dissipative medium effects renormalize (κ_H, ℓ_H) and Fourier weights but cannot alter topological holonomies that set CP phases.

Kill criteria. The minimal spectral neutrino sector is falsified if any hold:

1. **K1 (ordering rigidity)**: all spectral choices with diagonal M_R force normal ordering while data require inverted (or vice versa);
2. **K2 (mixing gate)**: simultaneous fit of $(\theta_{12}, \theta_{13}, \theta_{23})$ cannot be achieved for any $(\alpha_{L,R}^{(\nu)}, \beta^{(\nu)})$ within the small–twist regime;
3. **K3 (cosmology)**: Σ_ν implied by (14.51) exceeds the cosmological limit for *all* allowed $(\Lambda_{\rm LNV}, \|Y_\nu\|, \xi_R)$;
4. **K4 ($0\nu\beta\beta$)**: the predicted band for $m_{\beta\beta}$ vs. the lightest mass from (14.52)–(14.53) is entirely excluded.

8. Summary

Starting from geometric overlaps and spectral LNV, we obtain m_ν via the Takagi map, with PMNS angles controlled by pairwise seesaw sums (14.50). Normal ordering is generic with diagonal M_R and Gaussian m_D, while residual symmetries in M_R allow large mixings and flexible ordering. Absolute–scale information from Σ_ν and phase sensitivity in $m_{\beta\beta}$ turn cosmology and $0\nu\beta\beta$ into direct probes of fiber holonomy. The framework remains predictive thanks to cross–sector geometric universality and admits sharp kill criteria, completing the spectral account of flavor and CP in the lepton sector.

Chapter 15

Rare Processes and EDMs

15.1 Lepton flavor violation and $\mu \to e\gamma$

Aim. We develop the effective–field–theory (EFT) description of lepton–flavor violation (LFV) with emphasis on $\mu \to e\gamma$, identify the *spectral* origins of the dipole operators, and map the predicted rates to the geometric parameters of the three–mode family sector and of the spectral see–saw. We provide master formulae, controlled loop estimates (seesaw–W, Barr–Zee, heavy spectral vectors/scalars), RG evolution, correlated channels ($\mu \to 3e$, $\mu N \to eN$), and S/G/TQ/D gates with falsifiable criteria.

1. EFT: dipole operators and branching ratio

SMEFT basis at dimension six. In the Warsaw basis the LFV dipoles read

$$\mathcal{L}_{\rm dip} \;=\; \frac{1}{\Lambda^2}\Big[C^{ij}_{eB}\,(\bar{L}_i\sigma^{\mu\nu}e_j)\,H\,B_{\mu\nu} \;+\; C^{ij}_{eW}\,(\bar{L}_i\sigma^{\mu\nu}\tau^a e_j)\,H\,W^a_{\mu\nu}\Big] \;+\; \text{h.c.}\,, \tag{15.1}$$

with $i \neq j$ flavor indices. After EWSB,

$$C^{ij}_{e\gamma} \;:=\; c_W\,C^{ij}_{eB} - s_W\,C^{ij}_{eW}, \qquad \mathcal{L} \;\supset\; \frac{v}{\sqrt{2}\Lambda^2}\,C^{ij}_{e\gamma}\,(\bar{\ell}_{Li}\sigma^{\mu\nu}\ell_{Rj})\,F_{\mu\nu} + \text{h.c.}\,. \tag{15.2}$$

Amplitude and rate. Equivalently one may write

$$\mathcal{L} \;\supset\; \frac{e\,m_\mu}{2}\,\bar{e}\,\sigma^{\mu\nu}\Big(A_L P_L + A_R P_R\Big)\,\mu\,F_{\mu\nu} + \text{h.c.}, \qquad A_{R(L)} \;=\; \frac{\sqrt{2}\,v}{e\,m_\mu\,\Lambda^2}\,C^{\mu e\,(e\mu)}_{e\gamma}\,. \tag{15.3}$$

The partial width is

$$\Gamma(\mu\to e\gamma) \;=\; \frac{m_\mu^3}{4\pi}\Big(|A_L|^2+|A_R|^2\Big), \qquad \text{Br}(\mu\to e\gamma) \;=\; \frac{48\pi^3\,\alpha}{G_F^2}\Big(|A_L|^2+|A_R|^2\Big), \tag{15.4}$$

where we used $\Gamma_\mu \simeq G_F^2 m_\mu^5/(192\pi^3)$. In terms of SMEFT Wilsons,

$$\boxed{\text{Br}(\mu\to e\gamma) \;=\; \frac{24\pi^2\,\alpha\,v^2}{G_F^2\,m_\mu^2\,\Lambda^4}\left(\,|C^{\mu e}_{e\gamma}|^2+|C^{e\mu}_{e\gamma}|^2\right).} \tag{15.5}$$

SM reference. In the SM with light Majorana neutrinos only, $\text{Br}(\mu \to e\gamma)$ is suppressed by $(\Delta m_\nu^2/m_W^2)^2$ (GIM) and is negligibly small. Any observable signal probes new dynamics—here, spectral in origin.

2. Spectral sources of $C_{e\gamma}$

(a) Seesaw–W penguins (heavy N_R). In the spectral see–saw (Sec. 14.3) with heavy singlets N_{Rk} one finds at one loop

$$C_{e\gamma}^{ij}\Big|_{N_R\text{–}W} \;\simeq\; \frac{g_2^2}{16\pi^2}\sum_{k=0}^{2}\frac{(Y_\nu)_{ik}\,(Y_\nu^\dagger)_{kj}}{M_{R,k}^2}\,\mathcal{G}\!\left(\frac{M_{R,k}^2}{m_W^2}\right), \qquad \mathcal{G}(x)=\mathcal{O}(1)\ \text{smooth},\ \mathcal{G}(x\to\infty)\to\frac{1}{2}. \tag{15.6}$$

Using $m_D = vY_\nu/\sqrt{2}$,

$$C_{e\gamma}^{ij}\Big|_{N_R\text{–}W} \;\sim\; \frac{g_2^2}{16\pi^2}\left(m_D M_R^{-2} m_D^\dagger\right)_{ij}\frac{2}{v^2}\,. \tag{15.7}$$

This contribution carries the *same* geometric phases that enter PMNS and leptogenesis.

(b) Barr–Zee two–loop diagrams (Higgs misalignment). Misalignment between charged–lepton Yukawas and mass eigenstates (Sec. 14.1, 13.3) induces off–diagonal couplings δY_ℓ^{ij}. Two–loop Barr–Zee graphs with a heavy fermion f (top) or a W loop yield

$$C_{e\gamma}^{ij}\Big|_{\rm BZ} \;\simeq\; \frac{\alpha}{8\pi^3}\left[\sum_f N_c^f Q_f^2\, y_f^2\,\frac{\delta Y_\ell^{ij}}{m_h^2}\,\mathcal{F}_f\!\left(\frac{m_f^2}{m_h^2}\right) \;+\; \frac{g_2^2\,\delta Y_\ell^{ij}}{m_h^2}\,\mathcal{F}_W\!\left(\frac{m_W^2}{m_h^2}\right)\right], \tag{15.8}$$

with loop functions $\mathcal{F}_{f,W}=\mathcal{O}(1)$.

(c) Heavy spectral vectors/scalars. If the fiber geometry supports heavy vector resonances V or scalar modes S coupling as $\bar\ell_i (g_{L,R}^{(V)})_{ij}\gamma^\mu \ell_j V_\mu$ or $\bar\ell_i (y^{(S)})_{ij}\ell_j S$, then (schematically)

$$C_{e\gamma}^{ij}\Big|_{V} \;\simeq\; \frac{1}{16\pi^2}\,\frac{e\,m_\mu}{M_V^2}\left(g_L^{(V)} g_R^{(V)\,\dagger}\right)_{ij}\mathcal{H}_V, \qquad C_{e\gamma}^{ij}\Big|_{S} \;\simeq\; \frac{1}{16\pi^2}\,\frac{e\,m_\mu}{M_S^2}\left(y^{(S)} y^{(S)\,\dagger}\right)_{ij}\mathcal{H}_S, \tag{15.9}$$

with $\mathcal{H}_{V,S}=\mathcal{O}(1)$. Couplings inherit the *Gaussian* suppressions and holonomies of the three–mode overlaps.

(d) Medium anisotropy (χ^μ) portals. Tiny Lorentz–violating tensors $\delta\kappa_{IJ}^{\mu\nu}(\chi)$ (Sec. 1.1) can mix vector eigenstates; any induced LFV is model–dependent but bounded by the same overlap suppressions and by $|\delta\kappa| \ll 1$ from positivity/causality gates.

3. From overlaps to Wilson coefficients: controlled estimates

Charged–lepton misalignment. From Eq. (14.13) (with $s=\ell$),

$$(\delta Y_\ell)_{01} \;\simeq\; \mathcal{N}_\ell\,\frac{\beta_{01}^{(\ell)}}{2}\,h_{+1}^{(\ell)}\,\exp\!\Big[-\frac{\kappa_H^{(\ell)}}{2}\Big((0-\alpha_L)^2+(1-\alpha_R)^2\Big)\Big], \quad \text{etc.} \tag{15.10}$$

Hence Barr–Zee contributions scale as

$$\Big|C_{e\gamma}^{\mu e}\Big|_{\rm BZ} \;\propto\; \frac{\alpha}{8\pi^3}\,\frac{|\beta^{(\ell)}h_{+1}^{(\ell)}|}{m_h^2}\,\exp\!\Big[-\frac{\kappa_H^{(\ell)}}{2}\Big((0-\alpha_L)^2+(1-\alpha_R)^2\Big)\Big]. \tag{15.11}$$

Seesaw scaling. Using Eq. (14.37), the seesaw penguin behaves as

$$\left|C_{e\gamma}^{\mu e}\right|_{N_R\text{–}W} \;\propto\; \frac{g_2^2}{16\pi^2}\,\sum_k \frac{e^{-\,\kappa_H^{(\nu)}[(1-\alpha_L^{(\nu)})^2+(n_k-\alpha_R^{(\nu)})^2]}}{M_{R,k}^2}\;. \tag{15.12}$$

Hierarchy in $M_{R,k}$ or in the Gaussian factors selects the dominant tower.

4. Correlated channels and dipole dominance

Predicted ratios (dipole dominance). If dipoles dominate LFV, then (neglecting small logs)

$$\mathrm{Br}(\mu\to 3e) \;\simeq\; \frac{\alpha}{3\pi}\left(\ln\frac{m_\mu^2}{m_e^2}-\frac{11}{4}\right)\mathrm{Br}(\mu\to e\gamma), \qquad \mathrm{CR}(\mu N\to eN) \;\simeq\; \kappa_N\,\mathrm{Br}(\mu\to e\gamma), \tag{15.13}$$

with nucleus–dependent $\kappa_N = \mathcal{O}(10^{-3}-10^{-2})$. Significant deviations signal contact operators (e.g. from heavy vectors) beyond the pure–dipole regime.

5. RG evolution and matching

Running to m_μ. Between Λ and m_W, C_{eB}, C_{eW} mix with four–fermion/leptoquark–type operators via gauge and Yukawa interactions. Below m_W, the photon dipole obeys (schematically)

$$\mu\,\frac{d}{d\mu}C_{e\gamma}^{ij} \;=\; \frac{\alpha}{\pi}\,C_{e\gamma}^{ij} \;+\; \frac{y_\ell^2}{16\pi^2}\,C_{e\gamma}^{ij} \;+\; \cdots, \tag{15.14}$$

which is a mild multiplicative renormalization; our leading–order overlap estimates are thus RG–stable up to $\mathcal{O}(\%)$ factors.

6. Consistency gates and falsifiable criteria

(S) Stability. LFV arises in higher–dimensional operators or loops; mass matrices remain positive–definite. Heavy–light mixing $\Theta = m_D M_R^{-1}$ must satisfy $\|\Theta\| \ll 1$ to keep kinetic normalization perturbative.

(G) Hyperbolicity/causality. Dipoles and contact operators do not alter the principal symbol of kinetic terms; cone compatibility from g_{eff} remains intact.

(TQ) Positivity/unitarity. Forward–limit positivity holds for dipoles induced by integrating out heavy, causal UV modes; spectral anisotropies $\delta\kappa$ are constrained to $|\delta\kappa| \ll 1$ by same bounds.

(D) Dissipation. Dissipative renormalizations may rescale κ_H, ℓ_H, β but cannot lift Gaussian/holonomy selection rules; LFV zeros protected by symmetry remain exact.

Kill criteria. The minimal spectral LFV sector is falsified if any hold:

1. **K1 (dipole scaling failure)**: the set $\{\mathrm{Br}(\mu\to e\gamma), \mathrm{Br}(\mu\to 3e), \mathrm{CR}(\mu N\to eN)\}$ violates the dipole–dominance ratios (15.13) *for all* allowed $(\kappa_H, \alpha, \beta, \Lambda_{\mathrm{LNV}}, M_{R,k})$.
2. **K2 (PMNS–LFV inconsistency)**: using Y_ν and M_R that fit oscillation data, Eq. (15.6) irreducibly predicts $\mathrm{Br}(\mu\to e\gamma)$ above experimental limits, *independent* of Barr–Zee and heavy–resonance uncertainties.
3. **K3 (symmetry–protected zeros broken)**: a residual fiber symmetry implies exact texture zeros (Sec. 14.1), yet data require nonzero entries in the corresponding LFV amplitudes.

7. Summary

$\mu\to e\gamma$ is controlled by photon dipoles whose Wilson coefficients inherit the *geometric* suppressions and phases of the spectral framework: Gaussian overlaps in family space, Berry holonomies, and the seesaw structure of Y_ν and M_R. Master expressions (15.5), (15.6) and (15.8) make quantitative predictions once the few spectral parameters $(\kappa_H, \alpha, \beta, \Lambda_{\mathrm{LNV}}, M_{R,k})$ are fixed by masses and PMNS. Correlated ratios in (15.13) provide sharp cross–checks across LFV channels, while S/G/TQ/D gates constrain allowed deformations and ensure UV–consistent interpretations.

15.2 Electric dipole moments as spectral probes

Aim. We develop the CP–odd effective operators that generate electric dipole moments (EDMs), show how their Wilson coefficients arise from *spectral* phases (Berry holonomies, Higgs–sector CP mixing, WZW–type functionals), and provide master formulae for leptonic, hadronic, and atomic EDMs. We organize the calculation in three layers: (i) **SMEFT** at $\mu\simeq m_W$; (ii) **low–energy EFT** below m_W with QED⊗QCD running; (iii) **hadronic/atomic** matching to observables. We close with S/G/TQ/D consistency gates and falsifiable criteria.

1. EDM operator basis and matching conventions

Leptons. At dimension six in SMEFT (Warsaw basis),

$$\mathcal{L}^{\ell}_{\mathrm{EDM}} \;=\; \frac{1}{\Lambda^2}\Big[C^{ii}_{eB}\,(\bar L_i\sigma^{\mu\nu}e_i)\,H\,B_{\mu\nu} \;+\; C^{ii}_{eW}\,(\bar L_i\sigma^{\mu\nu}\tau^a e_i)\,H\,W^a_{\mu\nu}\Big] \;+\; \mathrm{h.c.} \tag{15.15}$$

After EWSB ($H\to v/\sqrt{2}$) and field rotations,

$$\mathcal{L} \;\supset\; -\frac{i}{2}\,d_{\ell_i}\,\bar\ell_i\sigma^{\mu\nu}\gamma_5\ell_i\,F_{\mu\nu}, \qquad d_{\ell_i} \;=\; \frac{\sqrt{2}\,v}{\Lambda^2}\,\mathrm{Im}\big(C^{ii}_{e\gamma}\big), \qquad C^{ii}_{e\gamma} := c_W C^{ii}_{eB} - s_W C^{ii}_{eW}. \tag{15.16}$$

Quarks and gluons. Below m_W the relevant CP–odd operators are

$$\mathcal{L}^{q,g}_{\mathrm{EDM}} \;=\; -\frac{i}{2}\sum_{q=u,d,s} d_q\,\bar q\,\sigma^{\mu\nu}\gamma_5 q\,F_{\mu\nu} \;-\; \frac{i}{2}\sum_{q=u,d,s}\tilde d_q\,g_s\,\bar q\,\sigma^{\mu\nu}\gamma_5 T^a q\,G^a_{\mu\nu} \;+\; w\,\frac{1}{6}\,f^{abc}\,G^a_{\mu\rho}G^{b\,\rho}_{\ \ \nu}\,\tilde G^{c\,\mu\nu} \\ +\;\sum_i C^{(6)}_i\,\mathcal{O}^{(6)}_i \;+\;\cdots, \tag{15.17}$$

with the Weinberg operator coefficient w and possible semileptonic CP–odd four–fermion operators in the ellipsis (important for diamagnetic atoms). Matching from SMEFT at $\mu \simeq m_W$ fixes $\{d_q, \tilde{d}_q, w\}$ in terms of high–scale Wilsons and loop effects.

2. Spectral sources of CP violation

(i) Geometric (Berry) phases in Yukawas. As derived in Secs. 14.1–14.2, complex phases in $Y_{u,d,\ell,\nu}$ originate from fiber holonomies $\mathcal{W}^{(s)}$; in the SM these drive EDMs only at very high loop order (tiny). In the spectral framework, the *leading* EDMs are instead controlled by CP–odd structures in the Higgs/gauge sectors induced by the medium.

(ii) Higgs CP mixing from spectral textures. The Higgs–like resonance h can couple as

$$\mathcal{L}_{hff,VV} \supset -\frac{m_f}{v}\,\bar{f}\,(\kappa_f^S + i\kappa_f^P\gamma_5)\,f\,h + \frac{\alpha}{4\pi}\,\frac{h}{v}\left(c_\gamma F_{\mu\nu}F^{\mu\nu} + \tilde{c}_\gamma F_{\mu\nu}\tilde{F}^{\mu\nu}\right) + \frac{\alpha_s}{4\pi}\,\frac{h}{v}\left(c_g G^2 + \tilde{c}_g G\tilde{G}\right), \tag{15.18}$$

with $\kappa_f^P, \tilde{c}_{\gamma,g}$ generated by CP–odd spectral form factors (e.g. phason admixture, Sec. 8.3). These induce electron/quark EDMs dominantly via Barr–Zee two–loop diagrams.

(iii) WZW/CS emergent functionals. CP–odd Wess–Zumino–Witten terms (Part 5) and parity–odd Chern–Simons mixings (Sec. 5.2) induce $\tilde{c}_{\gamma,g} \neq 0$ at low energies, feeding EDMs without requiring large κ_f^P.

(iv) Tiny anisotropies. Medium anisotropy tensors $\delta\kappa_{IJ}^{\mu\nu}(\chi)$ can generate CPT–odd structures; S/G gates enforce $|\delta\kappa| \ll 1$, making such contributions subleading to (ii)–(iii).

3. Master formulae: Barr–Zee and dipole matching

Electron EDM (Barr–Zee). With the interaction (15.18), the dominant two–loop contributions read

$$\left.\frac{d_e}{e}\right|_{h-f\ \text{loop}} = -\frac{\alpha}{8\pi^3}\,\frac{m_e}{v^2}\sum_{f=t,b,\tau} N_c^f Q_f^2\left[\kappa_e^S\,\kappa_f^P\,F\!\left(\frac{m_f^2}{m_h^2}\right) + \kappa_e^P\,\kappa_f^S\,G\!\left(\frac{m_f^2}{m_h^2}\right)\right], \tag{15.19}$$

$$\left.\frac{d_e}{e}\right|_{h-W\ \text{loop}} = -\frac{\alpha}{8\pi^3}\,\frac{m_e}{v^2}\left[\kappa_e^P\,\mathcal{F}_W\!\left(\frac{m_W^2}{m_h^2}\right) + \tilde{c}_\gamma\,\mathcal{F}_\gamma\right], \tag{15.20}$$

with $F, G, \mathcal{F}_W, \mathcal{F}_\gamma = \mathcal{O}(1)$ loop functions.[1]

Quark (C)EDMs and Weinberg w. Analogous expressions yield

$$d_q \sim \frac{\alpha}{8\pi^3}\,\frac{m_q}{v^2}\left[\kappa_q^P\,\Xi_q^{(1)} + \kappa_q^S\,\Xi_q^{(2)}\right], \qquad \tilde{d}_q \sim \frac{\alpha_s}{8\pi^3}\,\frac{m_q}{v^2}\left[\kappa_q^P\,\tilde{\Xi}_q^{(1)} + \kappa_q^S\,\tilde{\Xi}_q^{(2)}\right], \qquad w \sim \frac{\alpha_s}{(4\pi)^3}\,\tilde{c}_g\,, \tag{15.21}$$

where Ξ's encode loop functions and group factors (top, W, heavy spectral resonances).

[1] Closed forms are standard: e.g. $F(z) = z\int_0^1 dx\,\frac{1-2x(1-x)}{x(1-x)-z}\ln\frac{x(1-x)}{z}$, etc.

Direct dipole matching. Equivalently, one may integrate out heavy modes at $\mu \simeq m_h$ to SMEFT and use

$$d_e \;=\; \frac{\sqrt{2}\,v}{\Lambda^2}\,\mathrm{Im}\,C^{ee}_{e\gamma}, \qquad d_q \;=\; \frac{\sqrt{2}\,v}{\Lambda^2}\,\mathrm{Im}\,C^{qq}_{u\gamma,\,d\gamma}, \qquad \tilde{d}_q \;=\; \frac{\sqrt{2}\,v}{\Lambda^2}\,\mathrm{Im}\,C^{qq}_{uG,\,dG}. \tag{15.22}$$

In the spectral model the Wilsons inherit *holonomy* phases and Gaussian suppressions per Sec. 14.1.

4. QED/QCD running to hadronic scale

Anomalous dimensions (schematic). Between $\mu \simeq m_W$ and $\mu \simeq 2$ GeV,

$$d_q(\mu) \;=\; \eta_E\, d_q(m_W) \;+\; \eta_{EG}\, \tilde{d}_q(m_W) \;+\; \eta_{EW}\, e\, w(m_W), \tag{15.23}$$

$$\tilde{d}_q(\mu) \;=\; \eta_C\, \tilde{d}_q(m_W) \;+\; \eta_{CW}\, w(m_W), \qquad w(\mu) \;=\; \eta_W\, w(m_W), \tag{15.24}$$

with η's the known QCD/QED running factors (order–unity multiplicative numbers). Leptonic d_e runs multiplicatively under QED and is nearly scale independent at percent level.

5. Hadronic/atomic matching kernels

Nucleon EDMs. At $\mu \simeq 2$ GeV one may write

$$d_n \;=\; \sum_{q=u,d,s} \Big(a_q\, d_q + \bar{a}_q\, e\, \tilde{d}_q \Big) + a_W\, e\, w, \qquad d_p \;=\; \sum_{q=u,d,s} \Big(b_q\, d_q + \bar{b}_q\, e\, \tilde{d}_q \Big) + b_W\, e\, w, \tag{15.25}$$

where $\{a_q, \bar{a}_q, a_W\}$ and $\{b_q, \bar{b}_q, b_W\}$ are hadronic matrix elements (from lattice/QCD sum rules); their uncertainties can be profiled in the global fit (Sec. 15.4).

Paramagnetic systems (ThO, HfF$^+$, YbF). These are dominated by d_e and a scalar electron–nucleon coupling C_S:

$$d_{\mathrm{para}} \;\simeq\; \mathcal{K}_e\, d_e \;+\; \mathcal{K}_S\, C_S\ , \tag{15.26}$$

with atomic coefficients $\mathcal{K}_{e,S}$ known from many–body calculations. In the spectral model, C_S arises from CP–odd $h\bar{q}q$ and semileptonic operators via Higgs exchange.

Diamagnetic systems (Hg, Ra, Xe). These are controlled by the nuclear Schiff moment $\mathcal{S}$, which depends on $\{\tilde{d}_q, w\}$ and CP–odd pion–nucleon couplings $\bar{g}_{0,1}$, plus subleading $d_e, C_{S,T}$:

$$d_{\mathrm{dia}} \;\simeq\; \mathcal{K}_{\mathcal{S}}\, \mathcal{S}(\tilde{d}_q, w, \ldots) \;+\; \mathcal{K}_e^{\mathrm{dia}}\, d_e \;+\; \mathcal{K}_{S,T}\, C_{S,T}. \tag{15.27}$$

6. Geometric control and predictions

Spectral origin of $\kappa_f^{S,P}$ and $\tilde{c}_{\gamma,g}$. From Secs. 8.1–8.3, CP–odd couplings follow from (i) a small CP–odd admixture of the Higgs with a phason φ: $\kappa_f^P \simeq \sin\alpha_{h\varphi}\, \hat{y}_f^{(\varphi)}$, $\tilde{c}_{g,\gamma} \simeq \sin\alpha_{h\varphi}\, \hat{c}_{g,\gamma}^{(\varphi)}$; (ii) WZW–induced contact terms $\tilde{c}_{g,\gamma} \neq 0$ even if $\kappa_f^P = 0$; (iii) holonomy–induced phases in Y_f contributing to $\kappa_f^{S,P}$ via the overlap integrals. All are *topologically* protected small parameters, naturally explaining tiny EDMs.

Correlation handles.

$$\text{(H1)}: \quad d_e \propto \kappa_e^S \kappa_t^P + \kappa_e^P \kappa_t^S + \tilde{c}_\gamma \;\Rightarrow\; \text{tight link with CP in } h\bar{t}t \text{ and } h\gamma\gamma, \tag{15.28}$$

$$\text{(H2)}: \quad d_n \text{ dominated by } \tilde{d}_q, w \;\Rightarrow\; \text{link with CP–odd } hGG \text{ and WZW gluonic phases}, \tag{15.29}$$

$$\text{(H3)}: \quad \text{Paramagnetic vs. diamagnetic systems separate } d_e \text{ from } \{\tilde{d}_q, w, C_{S,T}\}. \tag{15.30}$$

7. Consistency gates (S/G/TQ/D) and kill criteria

(S) Stability. EDM operators are higher dimensional; they do not destabilize the vacuum. CP–odd mixings $\sin\alpha_{h\varphi}$ and WZW coefficients must remain perturbative: $|\kappa_f^P|,\ |\tilde{c}_{\gamma,g}| \ll 1$.

(G) Hyperbolicity/causality. No modification of the kinetic principal symbol; causality cones of g_{eff} are respected. Tiny $\delta\kappa^{\mu\nu}$ are bounded by positivity and by wavefront–set arguments.

(TQ) Positivity/unitarity. EDM Wilsons generated by integrating out causal UV modes (Higgs, W, heavy spectral resonances) respect forward–limit positivity; unitarity bounds constrain $|\tilde{c}_g|, |\tilde{c}_\gamma|$ relative to CP–even c_g, c_γ.

(D) Dissipation. Dissipative medium effects renormalize small parameters but cannot generate large CP–odd couplings or alter holonomy quantization.

Kill criteria. The minimal spectral EDM sector is falsified if *any* of the following hold:

1. **K1 (paramagnetic tension)**: the set of paramagnetic measurements requires $|d_e|$ exceeding the maximal Barr–Zee value implied by (15.19)–(15.20) for *any* $(\kappa_e^{S,P}, \kappa_t^{S,P}, \tilde{c}_\gamma)$ consistent with Higgs data.
2. **K2 (hadronic inconsistency)**: nucleon/diamagnetic EDMs imply $\{d_q, \tilde{d}_q, w\}$ incompatible with any spectral $(\kappa_q^{S,P}, \tilde{c}_g)$ under QCD running (15.23)–(15.24).
3. **K3 (phase alignment failure)**: the geometric phases that fit CKM/PMNS (Secs. 14.2, 14.4) *necessarily* overproduce EDMs even when $\sin\alpha_{h\varphi}, \tilde{c}_{\gamma,g} \to 0$.

8. Summary

EDMs are *spectroscopes* of CP in the spectral medium. Two–loop Barr–Zee diagrams translate Higgs–sector CP mixing and WZW contact terms into d_e; gluonic WZW phases and quark (C)EDMs govern d_n and diamagnetic atoms. Holonomy phases set the *available* CP while Gaussian overlaps keep it naturally small, yielding predictive, correlated targets across paramagnetic/diamagnetic systems. The formalism here, combined with Sec. 15.4, enables global fits that confront spectral CP with current and future EDM frontiers.

15.3 Baryon/lepton number violation windows

Aim. We classify the leading baryon- and lepton-number violating (BNV/LNV) operators in the spectral framework, identify their *geometric* origins (director condensates, instantons/monopoles, WZW functionals, fiber–isometry mixings), and derive parametric *windows* where (i) experimental bounds are satisfied, (ii) baryogenesis is not washed out, and (iii) discovery prospects exist. We provide master formulae for $\Delta L = 2$ (*Weinberg*), $\Delta B = 1$ (proton decay), and $\Delta B = 2$ (n–$\bar{n}$) channels, with consistency gates (S/G/TQ/D) and falsifiability criteria.

1. Operator taxonomy and symmetry structure

Charges. We organize operators by $(\Delta B, \Delta L)$ and by $B-L$. Electroweak sphalerons violate $B+L$ but preserve $B-L$; spectral sources may violate $B-L$ (e.g. Majorana masses) and thus can seed/leak the asymmetry.

Leading operators. Using SMEFT notation (schematic flavor/color indices suppressed):

$$\Delta L = 2,\ \Delta B = 0\ \textbf{(Weinberg)}: \quad \mathcal{O}_5 = (\overline{L}\tilde{H})(\tilde{H}^{\mathsf{T}} L^{\mathsf{c}}), \qquad c_5 = \frac{1}{\Lambda_L}\,\hat{c}_5, \tag{15.31}$$

$$\Delta B = 1,\ \Delta L = 1\ \textbf{(dim-6)}: \quad \mathcal{O}_{QQQL} = (QQQL), \quad \mathcal{O}_{uude} = (u^c u^c d^c e^c), \quad C_B^{(6)} = \frac{1}{\Lambda_B^2}\,\hat{C}_B, \tag{15.32}$$

$$\Delta B = 2,\ \Delta L = 0\ \textbf{(dim-9)}: \quad \mathcal{O}^{(9)}_{\Delta B=2} \sim (udd)^2, \qquad C^{(9)}_{\Delta B=2} = \frac{1}{\Lambda^5_{\Delta B=2}}\,\hat{C}_{\Delta B=2} \tag{15.33}$$

where hatted coefficients $\hat{c}_5, \hat{C}_B, \hat{C}_{\Delta B=2}$ are $\mathcal{O}(1)$ tensors in flavor space determined by spectral overlaps.

2. Spectral origins of BNV/LNV

(i) Director–condensate (DC) sources. CP^1 director bilinears (Vol. I; Sec. 14.3) at scale Λ_{LNV} produce LNV Majorana masses and, through quark–lepton fiber mixings, can induce suppressed BNV currents. The effective vertices are spectral overlap integrals weighted by Gaussian factors and holonomies (Sec. 14.1).

(ii) Instanton/monopole (IM) sources. Topological events in the spectral medium (Hopf reconnections, center vortices; Sec. 12.2) generate 't Hooft–like vertices with action S_{IM}:

$$\mathcal{L}_{\mathrm{IM}} \supset \Lambda_\star^4 e^{-S_{\mathrm{IM}}} \left[\det_f(\bar{\psi}_L \psi_R) \right] + \text{h.c.} \ \Rightarrow\ \left\{ c_5,\ C_B^{(6)},\ C^{(9)}_{\Delta B=2} \right\} \propto e^{-S_{\mathrm{IM}}} \times (\text{overlaps}). \tag{15.34}$$

(iii) WZW/CS functionals. Anomaly inflow and emergent WZW terms (Part 5) can mediate baryo-/leptogenesis–relevant interactions and induce $B-L$ violation when coupled to the Higgs/phason sector.

(iv) Fiber–isometry unification nodes. When quark and lepton doublets share a common sub–isometry of the spectral fiber (Part **??**), heavy vector/scalar resonances at Λ_B couple across sectors, generating the dim-6 BNV operators in (15.32) after integration.

3. $\Delta L = 2$ rates and the sphaleron washout window

Weinberg scattering rate. The operator $\mathcal{O}_5$ induces $LL \leftrightarrow HH$ scatterings with thermally averaged rate

$$\Gamma_{\Delta L=2}(T) \;\simeq\; \frac{T^3}{\pi^3}\,\|c_5\|^2 \;\approx\; \frac{T^3}{\pi^3}\,\frac{\|m_\nu\|^2}{v^4}, \tag{15.35}$$

using $m_\nu = \frac{v^2}{2} c_5$.

Out-of-equilibrium criterion. Comparing with the Hubble rate $H(T) = 1.66\sqrt{g_*}\,T^2/M_{\rm Pl}$ gives the decoupling temperature

$$\Gamma_{\Delta L=2} < H \;\Leftrightarrow\; T \lesssim T_{\rm dec} := \frac{1.66\sqrt{g_*}}{\pi^3}\,\frac{v^4}{M_{\rm Pl}\,\|m_\nu\|^2} \sim 2{\times}10^{12}\,\text{GeV}\,\left(\frac{0.05\ \text{eV}}{\|m_\nu\|}\right)^2, \tag{15.36}$$

so for $\|m_\nu\| \sim 0.05\,\text{eV}$, $\Delta L = 2$ processes are out of equilibrium near the electroweak epoch and do not erase a preexisting $B-L$ asymmetry. This defines the *LNV washout window*: for successful leptogenesis, sources active at $T \gtrsim T_{\rm dec}$ must subsequently freeze out before sphaleron decoupling ($T_{\rm sph} \sim 130\,\text{GeV}$).

4. Proton decay: parametrics and spectral selection rules

Widths and lifetime. Dim-6 operators yield partial widths (schematic)

$$\Gamma(p \to e^+\pi^0) \;\simeq\; \frac{m_p}{32\pi}\left(1-\frac{m_\pi^2}{m_p^2}\right)^2 \frac{|\alpha_H|^2}{\Lambda_B^4}\,|\hat{C}_B|^2\,A_{\rm RG}^2, \qquad \tau_p \;=\; \Gamma^{-1} \;\propto\; \frac{\Lambda_B^4}{|\alpha_H|^2}, \tag{15.37}$$

with hadronic matrix element $\alpha_H = \langle \pi^0 | (ud)u | p \rangle$, and $A_{\rm RG}$ the short–distance running factor.

Spectral textures and channels. Fiber–holonomy selection rules can *forbid* specific flavor structures in $\hat{C}_B$:

$$\hat{C}_B^{ijkl} \;\propto\; \sum_{\text{harmonics } h} \left[\mathcal{I}_Q^{(i)}\mathcal{I}_Q^{(j)}\mathcal{I}_Q^{(k)}\mathcal{I}_L^{(l)}\right]_h \,\delta_{\text{fiber charges}}. \tag{15.38}$$

In particular, residual $\mathbb{Z}_3$ symmetries may suppress first–generation couplings, shifting strength to $p \to \mu^+ K^0$–like channels. The *window* for Λ_B is then

$$\Lambda_B \;\gtrsim\; \left[\tau_p^{\rm min}\, m_p\, |\alpha_H|^2\, A_{\rm RG}^2\right]^{1/4} / |\hat{C}_B|^{1/2}, \tag{15.39}$$

where $|\hat{C}_B|$ inherits Gaussian suppressions from Sec. 14.1. Spectral unification (Part 3) often points to Λ_B near the gauge–unification window, with branching–ratio patterns fixed by fiber geometry.

5. $\Delta B = 2$: neutron–antineutron oscillations

Mass splitting and time scale. Dim-9 operators produce an off–diagonal Majorana mass

$$\delta m_{n\bar{n}} \;\simeq\; \frac{\beta_{n\bar{n}}}{\Lambda^5_{\Delta B=2}}\,|\hat{C}_{\Delta B=2}|\;A^{(9)}_{\rm RG}, \qquad \tau_{n\bar{n}} \;=\; \frac{1}{|\delta m_{n\bar{n}}|}, \tag{15.40}$$

with $\beta_{n\bar{n}}$ a six–quark hadronic matrix element (dimension GeV^6). Spectral selection rules can suppress $|\hat{C}_{\Delta B=2}|$ relative to $|\hat{C}_B|$, naturally decoupling n–$\bar{n}$ from proton decay (distinct overlap channels).

6. Windows for baryogenesis consistency

Sphaleron window. Let $T_{\rm sph}$ denote sphaleron freeze–out. For baryogenesis from leptogenesis, require

$$\Gamma_{\Delta L=2}(T_{\rm sph}) \;<\; H(T_{\rm sph}) \quad\Rightarrow\quad \|m_\nu\| \;\lesssim\; \left[\frac{1.66\sqrt{g_*}}{\pi^3}\,\frac{v^4}{M_{\rm Pl}\,T_{\rm sph}}\right]^{1/2}, \tag{15.41}$$

which is comfortably satisfied for $\|m_\nu\| \sim \mathcal{O}(0.05\ \mathrm{eV})$.

BNV safety window. To avoid erasure by $\Delta B = 1$ scatterings mediated by heavy resonances of mass Λ_B,

$$\Gamma_{\Delta B=1}(T) \;\sim\; n(T)\,\frac{T^2}{\Lambda_B^4}\,|\hat{C}_B|^2 \;<\; H(T) \text{ for } T \lesssim T_{\rm sph} \;\Rightarrow\; \Lambda_B \;\gtrsim\; \left[\frac{n(T_{\rm sph})}{H(T_{\rm sph})}\right]^{1/4} T_{\rm sph}^{1/2}\,/|\hat{C}_B|^{1/2}. \tag{15.42}$$

Gaussian overlap suppressions (§14.1) make this condition mild unless selection rules explicitly enhance first–generation couplings.

7. Consistency gates (S/G/TQ/D) and falsifiability

(S) Stability. All BNV/LNV operators are irrelevant (dim≥ 5); vacuum stability unaffected. Coefficients must remain perturbative ($|\hat{c}_5|, |\hat{C}_B|, |\hat{C}_{\Delta B=2}| \lesssim \mathcal{O}(1)$).

(G) Hyperbolicity/causality. No change to kinetic principal symbols; causal cones of $g_{\rm eff}$ preserved. Instanton amplitudes $e^{-S_{\rm IM}}$ are local in Euclidean time and respect reflection positivity.

(TQ) Positivity/unitarity. Forward–limit bounds exclude superluminal dispersion from WZW/CS–induced contact terms; matching to Part 5 ensures anomaly–consistent signs. Unitarity implies partial–wave bounds on contact amplitudes $\sim s/\Lambda^2$.

(D) Dissipation. Dissipative renormalizations shift overlap magnitudes but *not* topological indices; washout conditions (15.36)–(15.42) remain robust.

Kill criteria. The minimal spectral BNV/LNV sector is falsified if any hold:

1. **K1 (proton decay)**: the required Λ_B from (15.39) exceeds the spectral unification ceiling allowed by Part 3 *and* overlap suppressions are insufficient to evade bounds.
2. **K2 (washout)**: for any spectral $(\kappa_H, \alpha, \beta)$ fitting neutrino data, $\Gamma_{\Delta L=2}(T_{\rm sph}) > H(T_{\rm sph})$ unavoidably, erasing $B-L$.
3. **K3 (n–$\bar{n}$ tension)**: the predicted $\tau_{n\bar{n}}$ from (15.40) conflicts with bounds while proton–decay channels remain suppressed in ways incompatible with (15.38).

8. Summary

BNV/LNV in the spectral model originates from topological (IM), condensate (DC), and anomaly (WZW) mechanisms filtered by fiber–geometry overlaps. The Weinberg operator governs the LNV washout window, proton decay constrains Λ_B and flavor selection rules, and n–$\bar{n}$ probes orthogonal $\Delta B = 2$ structures. Because the same Gaussian/holonomy data control Yukawas and rare processes, the theory is tightly cross–linked: fits to masses/mixing fix the overlap tensors that enter (15.37) and (15.40), delimiting narrow, testable windows for discovery or exclusion.

15.4 Global likelihoods and priors

Aim. We construct a *global* statistical framework that combines rare processes, EDMs, baryon-/lepton-number violation, neutrino data, and Higgs/precision bounds into a single inference problem for the spectral parameters. We define the parameter spaces, likelihoods (including non-Gaussian one-sided limits), theory covariances (hadronic/atomic/nuclear), renormalization-group (RG) propagation, and *spectral* priors implementing S/G/TQ/D gates and geometric naturalness. The output is a posterior (or profile likelihood) over the spectral parameters and derived Wilson coefficients, together with posterior-predictive distributions for future searches.

1. Parameterization and forward map

Spectral parameters. Collect all *model* parameters in

$$\theta := \Big\{ \underbrace{\kappa_H^{(f)},\ \alpha_{L,R}^{(f)},\ \varepsilon_L^{(f)},\ \beta^{(f)},\ H_{\rm fourier}^{(f)}}_{\text{three-mode overlaps for } f=u,d,\ell,\nu},\ \underbrace{M_{R,k},\ \Lambda_{\rm LNV}}_{\text{see-saw sector}},\ \underbrace{\sin\alpha_{h\varphi},\ \kappa_f^{S,P},\ c_{\gamma,g},\ \tilde{c}_{\gamma,g}}_{\text{Higgs/phason \& WZW/CS}},\ \underbrace{\Lambda_B,\ \Lambda_{\Delta B=2}}_{\text{BNV scales}},\ \dots \Big\}. \tag{15.43}$$

Geometric holonomies (Berry phases) are included implicitly in the complex Fourier weights and Wilson lines entering the overlaps; see Parts 14, 5.

Wilson basis at $\mu = m_W$. Define the SMEFT Wilson vector $C(\mu_W)$ containing (at minimum)

$$C \;=\; \Big(C_{e\gamma},\, C_{u\gamma},\, C_{d\gamma},\, C_{uG},\, C_{dG},\, C_{H\tilde{B}},\, C_{H\tilde{W}},\, C_{H\tilde{G}},\, C_{Hud},\, C^{(3)}_{lequ},\, \dots;\; C_B^{(6)},\, C^{(9)}_{\Delta B=2},\, c_5 \Big).$$

The *matching map*

$$C(\mu_W) \;=\; \mathcal{J}\Big(\theta\Big) \tag{15.44}$$

is obtained from the overlap formulae (Secs. 14.1–14.3), Higgs/phason couplings (Sec. 8.3), and WZW/CS functionals (Part 5).

RG evolution to experiment. Let μ_i denote the characteristic scale of observable i. Linearized RG running gives

$$C(\mu_i) \;=\; \mathcal{R}(\mu_i,\mu_W)\,C(\mu_W), \qquad \mathcal{R} \;=\exp\!\left[\int_{\ln\mu_W}^{\ln\mu_i} d\ln\mu\;\Gamma(\mu)\right], \tag{15.45}$$

with anomalous dimension matrix Γ incorporating QED, QCD, and electroweak mixing (e.g. Eqs. (15.23)–(15.24)). Predictions follow as

$$y_i^{\rm th}\Big(\theta,\eta\Big) \;=\; \mathcal{H}_i[C(\mu_i),\,\eta]\,, \tag{15.46}$$

where η are *nuisance* (hadronic/atomic/nuclear) parameters and $\mathcal{H}_i$ embeds the process-specific matrix elements (e.g. Eqs. (15.5), (15.25)–(15.27), (15.37)–(15.40), (14.38)).

2. Data vector and likelihood factors

Dataset. The global analysis includes (non-exhaustive):

$$\mathcal{D} = \Big\{\text{EDMs: } (d_e,\,d_n,\,d_{\rm Hg},\,d_{\rm Ra},\ldots);\ \text{LFV: Br}(\mu\to e\gamma),\ \text{Br}(\mu\to 3e),\ \text{CR}(\mu N\to eN),\ \tau\text{-LFV};\ 0\nu\beta\beta;\ \text{prot}$$

Likelihood composition. Assuming conditional independence given θ,η,

$$\mathcal{L}(\mathcal{D}|\theta,\eta) \;=\; \prod_{i\in\mathcal{D}} \mathcal{L}_i\Big(d_i\,\Big|\,y_i^{\rm th}(\theta,\eta),\,\Sigma_i\Big)\,. \tag{15.47}$$

For measurements with central values $d_i\pm\sigma_i$, adopt correlated Gaussians (with covariance Σ_i); for one-sided limits ($d_i^{\rm lim}$ at CL α), use survival-function likelihoods

$$\mathcal{L}_i^{(\rm UL)} \;=\; \frac{1}{2}\,\operatorname{erfc}\!\left[\frac{y_i^{\rm th}(\theta,\eta)-d_i^{\rm lim}}{\sqrt{2}\,\sigma_i^{\rm eff}}\right], \tag{15.48}$$

with $\sigma_i^{\rm eff}$ chosen such that the UL reproduces the quoted CL under a Gaussian assumption. For Poisson channels (proton decay, $n-\bar n$), use

$$\mathcal{L}_i^{(\rm Poi)} \;=\; \frac{\Big(\lambda_i(\theta,\eta)\Big)^{N_i} e^{-\lambda_i(\theta,\eta)}}{N_i!}, \qquad \lambda_i = T_i\,\epsilon_i\,\Gamma_i(\theta,\eta), \tag{15.49}$$

with exposure T_i, efficiency ϵ_i, and width Γ_i from Eqs. (15.37)–(15.40).

3. Theory systematics and covariance

Nuisance hierarchy. Collect nuisance parameters in

$$\eta := \Big\{\alpha_H,\ A_{\rm RG}^{(6,9)},\ \beta_{n\bar n},\ a_q,\bar a_q,a_W,\ b_q,\bar b_q,b_W,\ \mathcal{K}_e,\mathcal{K}_S,\mathcal{K}_{\mathcal{S}},\ \mathcal{M}_{0\nu},\ \ldots\Big\}, \tag{15.50}$$

with priors from lattice/QCD sum rules and many-body atomic/nuclear calculations (quoted as Gaussians or log-normals per collaboration recommendations).

EFT truncation errors. At fixed dimension D, assign a truncation covariance to each prediction,

$$\Sigma_i^{(\mathrm{th})} \;=\; \kappa_i^2 \left(\frac{Q_i}{\Lambda_{\mathrm{EFT}}}\right)^{2(D+1-4)} \; y_i^{\mathrm{ref}\ 2}, \tag{15.51}$$

with Q_i the process scale, y_i^{ref} a reference value, and $\kappa_i = \mathcal{O}(1)$ profiled or marginalised.

Total covariance. For Gaussian channels use $\Sigma_i^{\mathrm{tot}} = \Sigma_i^{(\mathrm{exp})} + \Sigma_i^{(\mathrm{th})}$; for non-Gaussian channels propagate uncertainties via nuisance sampling.

4. Spectral priors and gates

Hard gates (constraints). Implement S/G/TQ/D gates as *support* of the prior:

$$\Pi_{\mathrm{gates}}(\theta) \;=\; \mathbb{I}\Big[(\mathrm{S}):\, \kappa_H^{(f)} > 0,\ (\mathrm{G}):\, \text{cone compat.},\ (\mathrm{TQ}):\, \text{positivity},\ (\mathrm{D}):\, |\delta\kappa| \ll 1\Big], \tag{15.52}$$

where $\mathbb{I}$ is an indicator; violations set the prior to zero.

Topological & geometric priors.

$$\text{Holonomies:}\quad \phi \in [0, 2\pi) \;\Rightarrow\; \Pi(\phi) = \frac{1}{2\pi}\ \ (\text{uniform on } S^1), \tag{15.53}$$

$$\text{Scales:}\quad \Pi(\Lambda) = \frac{1}{\Lambda}\ \ (\text{Jeffreys for } \Lambda \in [\Lambda_{\min}, \Lambda_{\max}], \tag{15.54}$$

$$\text{Small CP mixing:}\quad \sin\alpha_{h\varphi} \sim \mathcal{N}(0, \sigma_\alpha^2),\quad \sigma_\alpha \ll 1, \tag{15.55}$$

$$\text{Universality hyperprior:}\quad \kappa_H^{(u)} \approx \kappa_H^{(d)} \approx \kappa_H^{(\ell)} \approx \kappa_H^{(\nu)} \text{ with } \Pi \propto \exp\left[-\sum_{f,f'} \frac{(\kappa_H^{(f)} - \kappa_H^{(f')})^2}{2\sigma_\kappa^2}\right]. \tag{15.56}$$

Bayesian posterior and profile likelihood.

$$p(\theta, \eta|\mathcal{D}) \;\propto\; \mathcal{L}(\mathcal{D}|\theta, \eta)\ \Pi_{\mathrm{gates}}(\theta)\ \Pi(\theta)\ \Pi(\eta), \qquad \lambda(\theta) = \frac{\mathcal{L}(\mathcal{D}|\theta, \hat{\hat{\eta}}(\theta))}{\mathcal{L}(\mathcal{D}|\hat{\theta}, \hat{\eta})}. \tag{15.57}$$

Here $\hat{\hat{\eta}}(\theta)$ profiles nuisances and $(\hat{\theta}, \hat{\eta})$ denote global maximizers.

5. Linearized fast inference and correlations

Affine surrogate. For speed, linearize the prediction around a reference point (θ_0, η_0):

$$y_i^{\mathrm{th}} \;\approx\; y_{i,0} \;+\; \sum_a J_{ia}^{(\theta)}(\theta_a - \theta_{0,a}) \;+\; \sum_b J_{ib}^{(\eta)}(\eta_b - \eta_{0,b}), \tag{15.58}$$

with Jacobians extracted from the RG map (15.45) and analytic rates (15.5)–(15.40). This furnishes semi-analytic covariance propagation and exposes correlations across observables that share Wilsons (e.g. d_e and paramagnetic systems).

6. Global metrics, discovery, and exclusion

Model comparison. Use Bayes factors B_{10} between spectral sub-hypotheses (e.g. diagonal vs. circulant M_R, presence/absence of WZW CP) or likelihood-ratio tests $\Delta\chi^2 = -2\ln\lambda$ with Wilks/Chernoff prescriptions where applicable.

Prospective sensitivity. Posterior predictive distributions (PPDs) for a future experiment j follow from

$$p(d_j^{\rm fut}|\mathcal{D}) \;=\; \iint d\theta\, d\eta\; p(\theta,\eta|\mathcal{D})\; \mathcal{L}_j\Big(d_j^{\rm fut}\,\Big|\, y_j^{\rm th}(\theta,\eta)\Big)\,. \tag{15.59}$$

Discovery/exclusion contours use the profile-likelihood test statistic q_0 and CL_s for low-count Poisson channels (proton decay, n–$\bar{n}$).

7. Kill criteria in the global fit

Incompatibility tests. The spectral framework is *globally* falsified if the feasible set obeying S/G/TQ/D and priors yields

$$\min_{\theta,\eta}\; \chi^2(\theta,\eta) \;>\; \chi^2_{\rm crit}(\alpha;\,\nu), \tag{15.60}$$

for degrees of freedom ν and confidence α, *or* if Bayes factors decisively favour a nested sub-model lacking essential spectral ingredients (Jeffreys' scale). Concrete triggers include: (i) tension between EDM limits and Higgs-CP couplings required by LFV; (ii) simultaneous violation of dipole-dominance ratios and EDM correlations (Secs. 15.1–15.2); (iii) proton-decay demands on Λ_B incompatible with RG/unification windows (Sec. 3.4).

8. Implementation notes and reproducibility

Sampling. Employ nested sampling or Hamiltonian MCMC with constrained priors (15.52)–(15.56); emulate expensive nuclear/atomic kernels via Gaussian processes trained on ab-initio grids. RG and linearized surrogates (15.58) accelerate the evaluation while retaining exact calculations at the posterior modes.

Deliverables. Report (i) marginalized posteriors for key spectral parameters, (ii) credible/CL regions for derived observables, (iii) correlation matrices exposing cross-links (e.g. d_e–$h\gamma\gamma$–$\mu\to e\gamma$), and (iv) PPDs for next-generation experiments (ACME, nEDM, LEGEND, Hyper-K, NNbar).

9. Summary

The global framework maps geometric spectral data θ to a unified Wilson basis, evolves it to experimental scales, and confronts a broad array of precision and rare observables with consistent nuisances and truncation covariances. Spectral priors encode topology, scale hierarchies, and S/G/TQ/D gates, yielding a predictive, falsifiable landscape in which improvements in EDMs, LFV, and baryon-violation searches jointly sculpt the viable parameter volume and sharpen discovery prospects.

Part VI

QFT Consistency: Scattering, Positivity, and EFT

Chapter 16

LSZ, Källén–Lehmann, and the Optical Theorem

16.1 Asymptotic states on g_{eff} backgrounds

Aim. We construct the *in/out* asymptotic Fock spaces and single–particle states for quantum fields propagating on the emergent effective geometry (M, g_{eff}) of Vol. I, develop the covariant inner products and mode normalizations, and formulate precise conditions under which scattering theory (Haag–Ruelle/LSZ) is well posed despite weak curvature and slow medium variation. The section culminates in the definition of wave operators $\Omega_{\pm}$ and the background–dependent S–matrix domain, to be used in Secs. 16.2–16.4.

1. Geometric prerequisites and adiabatic regime

Global hyperbolicity and Cauchy foliation. Assume (M, g_{eff}) is globally hyperbolic with a smooth time function $t : M \to \mathbb{R}$ and Cauchy slices Σ_t. Write

$$g_{\mathrm{eff}} \;=\; -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \qquad d\Sigma^\mu \;=\; n^\mu \sqrt{h}\, d^3x, \tag{16.1}$$

with lapse N, shift N^i, induced metric h_{ij}, and future–unit normal n^μ.

Asymptotic stationarity ("Møller" regions). We posit the existence of two temporal regions $\mathcal{I}_{\pm}$ s.t.

$$\partial_t g_{\mathrm{eff}} \;\xrightarrow[t\to\pm\infty]{}\; 0, \qquad R^{\alpha}{}_{\beta\mu\nu}(g_{\mathrm{eff}}) \;\xrightarrow[t\to\pm\infty]{}\; 0, \tag{16.2}$$

uniformly on compact sets, and the spectral medium fields $\mathcal{S} = \{\rho, \eta, \chi^\mu, T, \Phi\}$ tend to constants. Define the *geometric adiabaticity* parameter

$$\epsilon_{\mathrm{geom}} \;:=\; \max\!\left\{ \frac{\ell_{\mathrm{int}}}{L_{\mathrm{curv}}},\; \frac{\ell_{\mathrm{int}}}{L_{\mathrm{var}}} \right\} \ll 1, \qquad L_{\mathrm{curv}} \sim \left\| R^{\alpha}{}_{\beta\mu\nu} \right\|^{-1/2}, \quad L_{\mathrm{var}} \sim \left\| \partial g_{\mathrm{eff}} \right\|^{-1}, \tag{16.3}$$

with ℓ_{int} the microscopic interaction (Compton) length. Scattering is meaningful for quanta with de Broglie wavelengths $\lambda \ll L_{\mathrm{curv}}, L_{\mathrm{var}}$.

2. Free fields and covariant inner products

Scalar (Klein–Gordon). For a real scalar ϕ of mass m and curvature coupling ξ,

$$P_g\,\phi \;:=\; \big(\Box_g + m^2 + \xi R[g]\big)\phi \;=\; 0, \qquad \Box_g := \frac{1}{\sqrt{-g}}\partial_\mu\!\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\right). \tag{16.4}$$

The conserved Klein–Gordon (KG) inner product on the solution space is

$$(\phi_1,\phi_2)_{\rm KG} \;:=\; i\int_{\Sigma_t} d\Sigma^\mu\; \phi_1^*\overset{\leftrightarrow}{\nabla}_\mu \phi_2, \qquad \nabla_\mu := \text{Levi–Civita}(g_{\rm eff}), \tag{16.5}$$

independent of Σ_t. Quantization yields $[\hat\phi(x),\hat\pi(y)]_{\Sigma_t} = i\delta_{\Sigma_t}(x,y)$ with $\hat\pi = \sqrt{h}\, n^\mu\partial_\mu\hat\phi$.

Spin-$\frac{1}{2}$ (Dirac). Choose a tetrad $e_a{}^\mu$ ($g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$), spin connection ω_μ^{ab}, and curved $\gamma^\mu = e_a{}^\mu\gamma^a$. The Dirac equation is

$$\big(i\gamma^\mu\nabla^{\rm S}_\mu - m\big)\psi = 0, \qquad \nabla^{\rm S}_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}. \tag{16.6}$$

The conserved inner product is

$$(\psi_1,\psi_2)_{\rm D} \;:=\; \int_{\Sigma_t} d\Sigma_\mu\; \bar\psi_1\gamma^\mu\psi_2, \qquad \bar\psi = \psi^\dagger\gamma^0, \tag{16.7}$$

again independent of Σ_t.

Vector (gauge) fields (free sector). In covariant gauge with gauge parameter α and background metric $g_{\rm eff}$,

$$\mathcal{L}_{\rm free}[A] \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\nabla_\mu A^\mu)^2 \quad \text{(Abelian)}, \tag{16.8}$$

and analogously for non–Abelian with background–covariant derivatives. Physical one–particle states will be defined in BRST cohomology (Ch. 2); here we focus on the kinematics on $g_{\rm eff}$.

3. Positive–frequency splitting and Hadamard condition

Asymptotic Killing fields. In $\mathcal{I}_\pm$ let $K^\mu_\pm$ denote the timelike Killing fields obtained as the $t\to\pm\infty$ limits of ∂_t. Positive frequency for a mode $u(x)$ is defined by

$$\mathcal{L}_{K_\pm} u \;=\; -i\omega\, u, \qquad \omega > 0, \tag{16.9}$$

ensuring $(u,u)_{\rm KG} > 0$ (or $(u,u)_{\rm D} > 0$). This yields preferred *in/out* complex structures $J_\pm$ and vacua $|0_\pm\rangle$.

Hadamard two–point functions. We adopt Hadamard states ω whose two–point function $W_\omega(x,x') = \langle\hat\phi(x)\hat\phi(x')\rangle_\omega$ has the standard short–distance form and microlocal spectrum condition:

$$\mathrm{WF}(W_\omega) \;=\; \{(x,k;x',-k') \mid (x,k)\sim(x',k'),\; k\in\mathcal{N}_x^+\}, \tag{16.10}$$

with $\mathcal{N}_x^+$ the future null cone and $(x,k)\sim(x',k')$ linked by a null geodesic. The in/out vacua $|0_\pm\rangle$ are Hadamard under (16.2) and coincide with Minkowski in the flat limit.

4. Mode solutions, wave packets, and normalization

KG modes and WKB control. Let $u_{\mathbf{p}}^{(\pm)}(x)$ solve (16.4) with asymptotics

$$u_{\mathbf{p}}^{(\pm)}(x) \;\sim\; \frac{e^{-ip\cdot x_{\pm}}}{\sqrt{2\omega_{\mathbf{p}}(2\pi)^3}} \quad \text{in } \mathcal{I}_{\pm}, \quad g_{\text{eff}}^{\mu\nu}p_\mu p_\nu = -m^2,\; p^0=\omega_{\mathbf{p}}>0. \tag{16.11}$$

Construct wave packets $u_f^{(\pm)} = \int d^3\mathbf{p}\, f(\mathbf{p}) u_{\mathbf{p}}^{(\pm)}$ with f smooth, compactly supported. Normalize by

$$\big(u_{\mathbf{p}}^{(\pm)}, u_{\mathbf{q}}^{(\pm)}\big)_{\text{KG}} = \delta^{(3)}(\mathbf{p}-\mathbf{q}), \qquad \big(u_{\mathbf{p}}^{(\pm)*}, u_{\mathbf{q}}^{(\pm)}\big)_{\text{KG}} = 0. \tag{16.12}$$

Creation/annihilation operators. Define

$$a_{\pm}(f) \;:=\; (\, u_f^{(\pm)}, \hat{\phi}\,)_{\text{KG}}, \qquad [a_{\pm}(f), a_{\pm}^{\dagger}(g)] \;=\; (f,g)_{L^2(\mathbb{R}^3)}, \tag{16.13}$$

and similarly for Dirac/spinor and vector modes using (16.7) and the BRST–compatible inner product. The *in/out* Fock spaces $\mathcal{F}_{\pm}$ are built by acting with $a_{\pm}^{\dagger}$ on $|0_{\pm}\rangle$.

5. Interactions, switching, and wave operators

Locality and time–compact switching. Let the interacting Hamiltonian density be $\mathcal{H}_{\text{int}}(x)$, local and polynomial in fields with couplings multiplied by a smooth switching function $\lambda(t)$ such that $\lambda(t)\to 0$ as $t\to\pm\infty$ and $\lambda(t)\to 1$ for $|t|<T$. Denote by $U(t_2,t_1)$ the interacting evolution and $U_0(t_2,t_1)$ the free evolution on (M, g_{eff}).

Møller (wave) operators. Define

$$\Omega_{\pm} \;:=\; \mathrm{s}\!-\!\lim_{t\to\pm\infty} U(0,t)\,U_0(t,0), \tag{16.14}$$

on the dense set of finite–particle wave–packet states. Under the adiabatic/short–range conditions (16.3) and standard bounds on $\mathcal{H}_{\text{int}}$, $\Omega_{\pm}$ exist, are isometries $\mathcal{F}_{\pm}\to\mathcal{H}$ (interacting Hilbert space), and the scattering operator is

$$S \;:=\; \Omega_-^{\dagger}\,\Omega_+ :\; \mathcal{F}_+ \to \mathcal{F}_-. \tag{16.15}$$

Asymptotic completeness ($\operatorname{Ran}\Omega_{\pm}=\mathcal{H}$) is an additional hypothesis to be checked model by model.

6. Haag–Ruelle construction on weakly curved backgrounds

Mass gap and clustering. For fields with single–particle mass $m>0$ and no zero–mass bound states, choose test functions $f^{(k)}$ with mutually separated velocity supports. Define (for scalars)

$$A_t(f) \;:=\; \int_{\Sigma_t} d\Sigma^{\mu}\, \Big(\partial_\mu u_f^{(+)}(x)\,\hat{\phi}(x) - u_f^{(+)}(x)\,\partial_\mu\hat{\phi}(x)\Big), \tag{16.16}$$

and multi–particle vectors $|\Psi_t\rangle := A_t(f^{(1)})\cdots A_t(f^{(n)})\,|0_+\rangle$. For $\epsilon_{\text{geom}}\ll 1$ one has

$$\lim_{t\to+\infty}|\Psi_t\rangle \;=\; \big|f^{(1)},\dots,f^{(n)}\big\rangle_{\text{in}}, \qquad \lim_{t\to-\infty}|\Psi_t\rangle \;=\; \big|f^{(1)},\dots,f^{(n)}\big\rangle_{\text{out}}, \tag{16.17}$$

generalizing Haag–Ruelle to g_{eff} with weak curvature; analogues hold for spinor and vector fields.

7. Kinematics: effective mass shells and flux

Local dispersion and mass shell. Define the (background) on–shell covectors by

$$\mathcal{M}_m \;:=\; \{\, p_\mu \;\big|\; g_{\rm eff}^{\mu\nu} p_\mu p_\nu = -m^2,\; p \text{ future timelike} \,\}. \tag{16.18}$$

Group velocities are $v^i = p^i/p^0$ with $p^0 > 0$ measured by $K_\pm^\mu$ in $\mathcal{I}_\pm$. WKB rays follow timelike geodesics of $g_{\rm eff}$.

Flux and single–particle normalization. For a wave packet $\Phi(x)$ of a scalar particle, the conserved current $j^\mu = \frac{i}{2}\Phi^* \overset{\leftrightarrow}{\nabla^\mu} \Phi$ yields the covariant flux through a large Σ:

$$\mathcal{F}[\Phi] \;:=\; \int_\Sigma d\Sigma_\mu\, j^\mu \;=\; 1 \quad\Longleftrightarrow\quad \text{unit normalization of the corresponding creation operator.} \tag{16.19}$$

This is the geometric replacement of the flat–space $2E_{\mathbf{p}}$ normalization.

8. Summary and gates

Summary. Under global hyperbolicity and asymptotic stationarity, one can define Hadamard *in/out* vacua, normalize covariant mode solutions using the KG/Dirac inner products, and construct Møller operators $\Omega_\pm$ that map asymptotic Fock spaces into the interacting Hilbert space on $(M, g_{\rm eff})$. Wave packets propagate along $g_{\rm eff}$ geodesics in the adiabatic regime $\epsilon_{\rm geom} \ll 1$, providing the kinematic backbone for LSZ reduction (Sec. 16.2) and unitarity/optical theorem (Sec. 16.3).

Consistency gates (S/G/TQ/D).

(S) Stability: free Hamiltonians are positive on $\mathcal{F}_\pm$; ξR is bounded below in $\mathcal{I}_\pm$; (16.20)

(G) Hyperbolicity: P_g is normally hyperbolic; principal symbol $g_{\rm eff}^{\mu\nu} k_\mu k_\nu$ is Lorentzian; (16.21)

(TQ) Positivity: two–point functions are Hadamard and obey the microlocal spectrum condition (1 (16.22)

(D) Dissipation: medium anisotropies $\delta\kappa(\chi)$ deform $g_{\rm eff}$ only adiabatically, $\epsilon_{\rm geom} \ll 1$, ensuring exis (16.23)

Outlook. With asymptotic states in hand, we next derive the covariant Källén–Lehmann representation and sum rules adapted to $g_{\rm eff}$ (Sec. 16.2), followed by unitarity cuts/optical theorem (Sec. 16.3) and the mapping between Green's functions and measurable cross sections in curved but adiabatic backgrounds (Sec. 16.4).

16.2 Spectral representation and sum rules

Aim. We generalize the Källén–Lehmann representation to quantum fields propagating on the emergent, globally hyperbolic background $(M, g_{\rm eff})$ constructed in Vol. I, Sec. 16.1. For (i) scalar, (ii) spinor, and (iii) gauge fields (in BRST cohomology) we express two–point

functions as positive superpositions of massive Green functions on $(M, g_{\rm eff})$, derive positivity properties of spectral measures, and obtain *sum rules* from equal–time commutators, Ward identities, and short–distance (Hadamard/OPE) constraints. These results provide the dispersion relations used in Sec. 16.3 and the map to cross sections in Sec. 16.4.

1. Preliminaries: two–point functions on $(M, g_{\rm eff})$

Hadamard states and Wightman distributions. Let $\hat{\Phi}$ denote a free or asymptotic interacting field (scalar for definiteness). For a Hadamard state ω, define

$$W_\omega(x,y) \;:=\; \langle \hat{\Phi}(x)\,\hat{\Phi}(y)\rangle_\omega, \qquad \Delta^{\pm}_\omega(x,y) \;:=\; \pm\Big(W_\omega(x,y) - W_\omega(y,x)\Big), \tag{16.24}$$

and the Feynman propagator $G_{F,\omega}(x,y)$ by time–ordering w.r.t. a global time function on M. All objects solve the Klein–Gordon equation (or its spin/gauge generalizations) in each entry, with distributional sources.

Massive Green kernels on $g_{\rm eff}$. For $\mu^2 \geq 0$, let $G_F^{(g)}(x,y;\mu^2)$ solve

$$\Big(\Box_g + \mu^2 + \xi R[g]\Big)_x\, G_F^{(g)}(x,y;\mu^2) \;=\; \frac{\delta^{(4)}(x,y)}{\sqrt{-g}}\,, \qquad \text{Feynman boundary conditions,} \tag{16.25}$$

with $\Box_g$ and $R[g]$ referring to $g_{\rm eff}$ (Eq. (16.4)). The kernels $G^{(g)}_{F,\pm}(x,y;\mu^2)$ (Wightman $\Delta^\pm$ analogues) are defined similarly.

2. Scalar Källén–Lehmann on curved, asymptotically stationary backgrounds

Representation. Under the assumptions of Sec. 16.1 (global hyperbolicity, Hadamard state, adiabatic in/out regions), the two–point functions admit the *curved–space* Källén–Lehmann representation

$$G_{F,\omega}(x,y) \;=\; \int_0^\infty d\mu^2\; \rho_\omega(\mu^2)\; G_F^{(g)}(x,y;\mu^2), \tag{16.26}$$

$$W_\omega(x,y) \;=\; \int_0^\infty d\mu^2\; \rho_\omega(\mu^2)\; G_+^{(g)}(x,y;\mu^2), \qquad \Delta^{\pm}_\omega(x,y) \;=\; \int_0^\infty d\mu^2\; \rho_\omega(\mu^2)\; \Delta^{\pm}_\mu(x,y), \tag{16.27}$$

where ρ_ω is a non–negative tempered measure (the *spectral density*). Eqs. (16.26)–(16.27) reduce to the flat–space formula when $g_{\rm eff} \to \eta$.

Sketch of construction. Insert the *in* completeness relation $\mathbf{1} = \sum_n |n_{\rm in}\rangle\langle n_{\rm in}|$ between fields and use covariance of the KG inner product (16.5) to organize states by invariant mass μ (defined by the geodesic WKB frequency in $\mathcal{I}_\pm$). The adiabatic Riemann–normal expansion around x shows that each contribution equals $G^{(g)}_\pm(x,y;\mu^2)$ up to curvature–suppressed contact terms; these resummate into the integral (16.26). Positivity of ρ_ω follows from positivity of W_ω (Bochner–Schwartz in the proper–time/heat–kernel representation).

Pole + continuum. For a stable particle with physical mass $m_\star^2$,

$$\rho_\omega(\mu^2) \;=\; Z_\omega\,\delta(\mu^2-m_\star^2) \;+\; \rho_\omega^{\rm cont}(\mu^2)\,\theta(\mu^2-s_0), \qquad Z_\omega\in(0,1], \tag{16.28}$$

with threshold s_0 set by the lightest multi–particle channel.

3. Sum rules from equal–time commutators and equations of motion

Normalization sum rule. Act with $\Box_g+\xi R$ on the x argument of (16.26) and use (16.25). Imposing the canonical commutator $[\hat\Phi,\hat\Pi]=i\delta_{\Sigma_t}$ yields

$$\boxed{\int_0^\infty d\mu^2\;\rho_\omega(\mu^2) \;=\; 1} \qquad \text{(field normalization)}. \tag{16.29}$$

Mass/curvature moment. Using the equation of motion $(\Box_g+m_0^2+\xi R)\hat\Phi=\cdots$ and the Hadamard short–distance form one finds, for minimally subtracted renormalization,

$$\boxed{\int_0^\infty d\mu^2\;\rho_\omega(\mu^2)\,(\mu^2-m_0^2-\xi R) \;=\; 0 \;+\; \mathcal{O}(R^2,\nabla R)}\,, \tag{16.30}$$

i.e. the first moment tracks curvature R at the insertion point (coincidence limit), with higher–order curvature corrections suppressed by $L_{\rm curv}^{-2}$.

Wave–function sum rule. Let Π_ω denote the amputated 1PI two–point function in the local momentum expansion (Riemann normal coordinates). The residue at the pole satisfies

$$Z_\omega \;=\; \Big[1-\partial_{p^2}\Pi_\omega(p^2)\Big|_{p^2=m_\star^2}\Big]^{-1} \;=\; \left[\int_0^\infty d\mu^2\;\frac{\rho_\omega(\mu^2)}{(\mu^2-m_\star^2)^2}\right]^{-1}, \tag{16.31}$$

implying $0<Z_\omega\le 1$ (for a stable particle) by positivity of ρ_ω.

4. Spinors and vector fields: projectors and BRST cohomology

Dirac fields. For ψ solving (16.6), the two–point function admits

$$S_{F,\omega}(x,y) \;=\; \int_0^\infty d\mu^2\;\rho_\omega^{(1/2)}(\mu^2)\;S_F^{(g)}(x,y;\mu^2), \qquad (i\not{\nabla}-\mu)\,S_F^{(g)}=\frac{\delta^{(4)}(x,y)}{\sqrt{-g}}, \tag{16.32}$$

with a *matrix–valued* positive spectral density $\rho_\omega^{(1/2)}\ge 0$ in the sense of spinor bilinears (positivity tested on $(\bar u,u)$).

Gauge fields. In covariant gauges the two–point function decomposes into transverse/longitudinal sectors. BRST cohomology ensures that the *physical* (transverse) part has a positive spectral density:

$$\langle A_\mu A_\nu\rangle_\omega(x,y)=\int_0^\infty d\mu^2\;\Big[\rho_\omega^{\rm T}(\mu^2)\,\Pi^{\rm T}{}_{\mu\nu}(x,y)\;G_F^{(g)}(x,y;\mu^2) \;+\; (\text{longitudinal/gauge-fixing})\Big], \tag{16.33}$$

$$k^\mu\widetilde{G}_{\mu\nu}(k)\Big|_{\rm T}=0, \qquad \rho_\omega^{\rm T}(\mu^2)\;\ge\;0 \;\text{ on BRST cohomology},$$

where $\Pi^{\rm T}$ is the parallel–transported transverse bitensor projector and tildes denote local Fourier transforms in Riemann normal coordinates. Gauge–variant pieces may violate positivity, but cancel from gauge–invariant observables (Sec. 16.3).

5. Dispersion relations in local momentum space and sum rules

Local Fourier/heat–kernel machinery. Choose Riemann normal coordinates at x and write the Schwinger proper–time form

$$G_F^{(g)}(x,y;\mu^2) \;=\; i\int_0^\infty ds\; e^{-is(\mu^2-i0)}\,K(s;x,y), \qquad K(s;x,y) \;\sim\; \frac{i\,\Delta^{1/2}}{(4\pi is)^2}\,e^{\frac{i\sigma}{2s}}\sum_{n\ge 0} a_n (is)^n, \tag{16.34}$$

with Synge world function $\sigma(x,y)$, van Vleck determinant Δ, and Hadamard–DeWitt coefficients $a_n(x,y)$ (curvature series). Inserting (16.34) into (16.26) gives a *double* spectral/proper–time representation. Matching the short–distance Hadamard form fixes moments of ρ_ω.

Short–distance (Hadamard) sum rules. Comparing the universal singular terms in the coincidence limit yields

$$\int_0^\infty d\mu^2\,\rho_\omega(\mu^2) \;=\; 1 \qquad \text{(already (16.29))}, \tag{16.35}$$

$$\int_0^\infty d\mu^2\,\rho_\omega(\mu^2)\,\ln\frac{\mu^2}{\mu_{\rm R}^2} \;=\; \mathcal{C}_\omega(\mu_{\rm R})\;, \quad \text{with } \mathcal{C}_\omega \text{ fixed by renormalization scheme and } a_1(x,x), \tag{16.36}$$

and higher curvature–dependent constraints governed by $a_{n\ge 1}$, providing *sum rules with curvature corrections*. Eq. (16.36) controls the scale dependence of wave–function renormalization.

Current–current correlators and charge sum rule. For a conserved current J^μ (global symmetry; spectral Ward identities of Ch. 2) define the polarization $\Pi(x,y) = \langle J_\mu(x) J^\mu(y)\rangle$. In local momentum space,

$$\Pi(q^2) \;=\; \frac{1}{\pi}\int_0^\infty ds\;\frac{\operatorname{Im}\Pi(s)}{s-q^2-i0}, \qquad \operatorname{Im}\Pi(s) \;=\; \pi\,\rho_{JJ}(s) \;\ge 0, \tag{16.37}$$

and the equal–time commutator (charge algebra) implies the *charge* sum rule

$$\boxed{\;\Pi(0) \;=\; \frac{1}{\pi}\int_0^\infty ds\;\frac{\rho_{JJ}(s)}{s}\;, \qquad \rho_{JJ}(s)\ge 0\;}, \tag{16.38}$$

which governs running couplings and feeds the positivity gates of Sec. 17.2.

6. Anisotropies and medium response: χ^μ dependence

Directional spectral densities. Small anisotropies from the director field χ^μ (Vol. I) deform the principal symbol: $g_{\rm eff}^{\mu\nu} \to g_{\rm eff}^{\mu\nu} + \delta\kappa^{\mu\nu}(\chi)$ (Sec. 1.1). At leading order one may write a directional density

$$\rho_\omega(\mu^2;\hat n) \;=\; \rho_\omega(\mu^2) \;+\; \delta\rho_\omega(\mu^2)\,(\hat n\cdot\hat\chi)^2 \;+\; \cdots, \qquad |\delta\rho_\omega| \;\ll\; \rho_\omega, \tag{16.39}$$

with $\hat n$ the local propagation direction (WKB). Positivity holds pointwise in $\hat n$; sum rules (16.29)–(16.30) remain valid after angular averaging, and receive controlled $\mathcal{O}(\delta\kappa)$ corrections in the directional moments.

7. Examples

(i) Scalar with a stable pole. With $\rho = Z\delta(\mu^2 - m_\star^2) + \rho^{\rm cont}$, Eq. (16.26) reads

$$G_F(x,y) \;=\; Z\, G_F^{(g)}(x,y;m_\star^2) \;+\; \int_{s_0}^{\infty} d\mu^2\, \rho^{\rm cont}(\mu^2)\, G_F^{(g)}(x,y;\mu^2)\,. \tag{16.40}$$

The residue Z is measurable via LSZ (wave–function factor); $Z \le 1$ by (16.31).

(ii) Transverse gauge propagator. For an Abelian field in a BRST gauge,

$$\widetilde{G}^{\rm T}_{\mu\nu}(q) \;=\; \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \int_0^\infty ds\; \frac{\rho^{\rm T}(s)}{s - q^2 - i0} \;+\; (\text{curvature/anisotropy}), \tag{16.41}$$

in local inertial coordinates. The sign of $\rho^{\rm T}$ controls positivity bounds on higher–derivative operators in Sec. 17.2.

(iii) Chiral current sum rules. For $V-A$ currents in a phase with (approximate) chiral symmetry, the difference $\Pi_{VV} - \Pi_{AA}$ admits curvature–corrected Weinberg–type sum rules whose leading forms read

$$\int_0^\infty ds\, \Big[\rho_{VV}(s) - \rho_{AA}(s)\Big] \;=\; f_\pi^2 \;+\; \mathcal{O}(R), \tag{16.42}$$

$$\int_0^\infty ds\; s\, \Big[\rho_{VV}(s) - \rho_{AA}(s)\Big] \;=\; \mathcal{O}(R\Lambda_{\rm had}^2), \tag{16.43}$$

providing non–perturbative constraints relevant to Part 11.

8. Summary and gates

Summary. On globally hyperbolic, asymptotically stationary $(M, g_{\rm eff})$, two–point functions admit a Källén–Lehmann representation as superpositions of massive Green functions (16.26) with non–negative spectral measures. Equal–time commutators and Hadamard short–distance structure yield normalization and moment sum rules (16.29)–(16.36), while Ward identities imply dispersive constraints for conserved currents (16.37)–(16.38). These results apply to scalars, spinors, and (transverse) gauge fields, and admit controlled χ^μ–induced anisotropy corrections.

Consistency gates (S/G/TQ/D).

(S) Stability: $\rho_\omega(\mu^2) \ge 0 \;\Rightarrow\; Z \le 1$; curvature moments respect (16.30); (16.44)

(G) Hyperbolicity: $G_F^{(g)}$ exists/unique on $(M,\, g_{\rm eff})$ (normally hyperbolic operator); (16.45)

(TQ) Positivity: W_ω positive type $\Rightarrow$ $\rho \ge 0$; BRST ensures positivity on cohomology for gauge fie (16.46)

(D) Dissipation: small $\delta\kappa(\chi)$ only deforms ρ by (16.39); adiabaticity guarantees convergence of the (16.47)

Outlook. With the spectral representation established, unitarity cuts and the optical theorem are derived in Sec. 16.3 by relating ρ (or $\mathrm{Im}\,\Pi$) to on–shell intermediate states built in Sec. 16.1, and then used to connect Green functions to measurable cross sections (Sec. 16.4).

16.3 Unitarity cuts and optical theorem

Aim. We derive the optical theorem and Cutkosky cutting rules for quantum fields propagating on the emergent, globally hyperbolic background (M, g_{eff}) introduced in Sec. 16.1. The proof proceeds in three layers: (i) algebraic unitarity of the S–matrix built from the *in/out* Fock spaces on (M, g_{eff}); (ii) Källén–Lehmann and dispersive representation of two–point functions on g_{eff} (Sec. 16.2); (iii) diagrammatic cutting rules obtained from the largest–time equation in local Riemann normal coordinates (RNC) and lifted covariantly. Gauge theories are treated in BRST cohomology, ensuring positivity of the transverse sector and ghost cancellation across cuts.

1. Algebraic unitarity and the optical theorem

Unitarity of S. Let $S : \mathcal{F}_+ \to \mathcal{F}_-$ be the scattering operator (Eq. (16.15)). By construction S is unitary on the Fock space built from Hadamard *in/out* states on (M, g_{eff}): $S^\dagger S = \mathbf{1}$. Write $S = \mathbf{1} + i\,T$; then

$$i(T^\dagger - T) \;=\; T^\dagger T \,. \tag{16.48}$$

Covariant optical theorem. Insert complete sets of multi–particle asymptotic states $\{|X_{\mathrm{out}}\rangle\}$ normalized with the KG/Dirac inner products on g_{eff} (Sec. 16.1). For a transition $i \to j$,

$$2\ \mathrm{Im}\ \mathcal{M}_{i\to j} \;=\; \sum_X \int d\Pi_X^{(g)}\ \mathcal{M}_{i\to X}\ \mathcal{M}^*_{j\to X}\,, \qquad \mathcal{M}_{i\to j} := \langle j_{\mathrm{out}}|\,T\,|i_{\mathrm{in}}\rangle\,, \tag{16.49}$$

where $d\Pi_X^{(g)}$ is the *covariant* phase–space measure on (M, g_{eff}) (defined below). For forward scattering $j = i$,

$$\boxed{\;2\ \mathrm{Im}\ \mathcal{M}_{i\to i} \;=\; \sum_X \int d\Pi_X^{(g)}\ \Big|\mathcal{M}_{i\to X}\Big|^2 \,.\;} \tag{16.50}$$

Covariant phase space. In the asymptotically stationary regions $\mathcal{I}_\pm$ with Killing vector $K^\mu_\pm$, the on–shell measure for a single particle of mass m is

$$d\Pi^{(g)}(x,p) \;:=\; \frac{d^4p}{(2\pi)^3}\ \delta\Big(g^{\mu\nu}_{\mathrm{eff}}\,p_\mu p_\nu + m^2\Big)\ \theta\Big(p_\mu K^\mu_\pm\Big)\,. \tag{16.51}$$

Multi–particle measures are products of (16.51) times the covariant delta for total 4–momentum conservation defined with respect to parallel transport along $K^\mu_\pm$ (which reduces to $\delta^{(4)}$ in local inertial frames).

2. Position–space proof via the largest–time equation

Causal decomposition. In curved spacetime, time ordering is defined w.r.t. a global time function t on M. Let $G_F = G_> + G_<$ be the Feynman propagator decomposed into $t_x > t_y$ and $t_x < t_y$ parts. Iterating this decomposition in an n–point time–ordered correlator and summing over all assignments yields the *largest–time equation* (LTE),

$$\sum_{\{\sigma_v\}} \prod_{v\in V} \sigma_v \; \mathcal{G}\Big[\{\sigma_v\}\Big] \;=\; 0, \qquad \sigma_v \in \{0,1\}, \tag{16.52}$$

where $\sigma_v = 1$ (0) marks the vertex v as circled (uncircled) and $\mathcal{G}$ is the graph value with propagators replaced by $G_>$ or $G_<$ according to the circling rules. Taking the difference of the amplitude and its complex conjugate collapses the LTE into a sum over *cut* graphs, with each internal line replaced by the *on–shell* Wightman kernel:

$$G_F(x,y) - G_F^*(x,y) \;=\; i\,\Delta(x,y) \;=\; i\int_0^\infty d\mu^2\,\rho(\mu^2)\,\Delta_\mu(x,y)\;, \tag{16.53}$$

where Δ_μ is the Pauli–Jordan function on (M, g_{eff}) and ρ is the positive spectral density (Sec. 16.2). Thus the *imaginary part* of any amplitude equals a sum over graphs with cut propagators replaced by $i\Delta_+$ (positive–frequency part), establishing (16.49).

3. Cutkosky rules in local Riemann normal coordinates

Local momentum representation. Choose RNC X^μ around a point on the scattering world tube; to leading order in curvature the propagator admits

$$\widetilde{G}_F(q) \;\simeq\; \int_0^\infty ds\;\frac{\rho(s)}{s - q^2 - i0} \;+\; \mathcal{O}\!\left(\frac{R}{q^2}, \frac{\nabla R}{q^3}\right), \tag{16.54}$$

with small curvature corrections. The *discontinuity* across the physical cut is

$$2i\;\mathrm{Im}\;\widetilde{G}_F(q) \;=\; 2\pi i\int_0^\infty ds\;\rho(s)\;\delta_+(q^2 - s)\;. \tag{16.55}$$

Cutting rules (diagrammatic form, scalar theory). For any loop amplitude $\mathcal{M}$ built from $\widetilde{G}_F$ and local vertices:

1. Draw all inequivalent *cuts* that separate external legs into two nonempty sets.
2. For each cut, replace every cut internal line by $2\pi i\,\rho(s)\,\delta_+(q^2 - s)$ (sum/integrate over s if needed).
3. Multiply the left subgraph by $\mathcal{M}_L$, the right by $\mathcal{M}_R^*$, and integrate over the covariant phase space defined by the on–shell deltas and the g_{eff} volume element.

Summing over all cuts yields $2i\;\mathrm{Im}\;\mathcal{M}$, consistent with (16.49). Curvature enters only through the local measure and $\rho(s)$; at scales $|q^2| \gg R$ the flat–space rules are recovered.

Example: one–loop bubble (covariant $2\to2$). For a scalar bubble with equal masses m,

$$2\;\mathrm{Im}\;\mathcal{M}_{2\to2} \;\simeq\; \int \frac{d^4q}{(2\pi)^4}\;(2\pi)\delta_+(q^2 - m^2)\;(2\pi)\delta_+((P-q)^2 - m^2)\;|\mathcal{M}_{2\to q,(P-q)}|^2\;, \tag{16.56}$$

where P is the total in–momentum defined with respect to $K^\mu_\pm$; the two delta functions define the *covariant* 2–body phase space on (M, g_{eff}).

4. Gauge theories: BRST cohomology and ghost cancellation

Transverse positivity. In a covariant gauge, the gauge propagator decomposes as in (16.33). Cutkosky rules are applied to the full graph, including ghosts and longitudinal modes; BRST Ward identities ensure that unphysical polarizations cancel between the two sides of the cut, leaving the physical (transverse) phase space with *positive* measure $\rho^{\mathrm{T}}(s) \geq 0$.

Optical theorem for conserved currents. For $i \to i$ forward amplitudes mediated by gauge bosons,

$$2\ \mathrm{Im}\ \mathcal{M}_{i\to i} \;=\; \int d\Pi_{\mathrm{T}}^{(g)}\ \left|\mathcal{M}_{i\to X_{\mathrm{phys}}}\right|^2, \tag{16.57}$$

where $d\Pi_{\mathrm{T}}^{(g)}$ is the transverse covariant phase space. Gauge–parameter independence follows from the LTE combined with Ward/Slavnov–Taylor identities (Sec. 2).

5. Landau equations and singularity loci on g_{eff}

Local Landau conditions. In RNC, the singularities of Feynman integrals satisfy (schematically)

$$\alpha_\ell \geq 0, \qquad \alpha_\ell\,(q_\ell^2 - m_\ell^2) = 0 \quad \text{for each line } \ell, \tag{16.58}$$

$$\sum_{\ell \in \text{loop } r} \alpha_\ell\, q_\ell \;=\; 0 \quad \text{(momentum conservation along each loop)}, \tag{16.59}$$

with q_ℓ momenta defined by parallel transport along geodesics. Solutions with all $\alpha_\ell \geq 0$ give the physical branch cuts; cutting a set of lines that satisfies (16.58)–(16.59) puts them on shell and yields the corresponding discontinuity, generalizing the Coleman–Norton picture to geodesic motion on (M, g_{eff}).

6. Schwinger–Keldysh (in–in) perspective and dissipation

In–in optical identity. For inclusive observables or slowly varying backgrounds, the Schwinger–Keldysh contour yields

$$G^R - G^A \;=\; G^> - G^< \,, \tag{16.60}$$

which is the correlator form of the optical theorem. Medium anisotropies $\delta\kappa(\chi)$ enter through modified dispersion in $G^{R/A}$ and $G^{\gtrless}$ but preserve the identity at leading order (causality). Dissipative rates extracted from $\mathrm{Im}\ \Pi^R$ are positive by the cutting rules and $\rho \geq 0$.

7. Consistency gates (S/G/TQ/D) and implications

(S) Stability. Positivity of spectral measures ρ, ρ^{T} implies $\mathrm{Im}\ G_F \geq 0$ in the physical channels and guarantees that total cross sections are non–negative via (16.50).

(G) Hyperbolicity/causality. Normal hyperbolicity of the kinetic operators on (M, g_{eff}) ensures well–posed evolution and retarded/advanced Green functions; the LTE requires global time ordering, guaranteed by global hyperbolicity.

(TQ) Positivity/unitarity. BRST cohomology removes negative–norm states from physical cuts; Ward/Slavnov–Taylor identities enforce gauge–parameter independence of (16.57). Forward–limit positivity used in Sec. 17.2 follows from (16.50).

(D) Dissipation. Anisotropy tensors $\delta\kappa(\chi)$ deform dispersion but, by causality, cannot flip the sign of Im Π^R; cut rules thus enforce non–negative spectral densities for dissipative coefficients (Kubo relations).

8. Summary

Starting from algebraic unitarity on $(M, g_{\rm eff})$, the largest–time equation and the curved–space Källén–Lehmann representation imply the *covariant optical theorem* (16.49) with phase space defined by the emergent geometry. Diagrammatic Cutkosky rules follow locally in RNC and lift covariantly; in gauge theories, BRST identities ensure that only transverse, positive–norm states propagate across cuts. These results underpin the dispersion/positivity analysis of Secs. 16.4–17.2 and the consistency gates of Part VI.

16.4 Mapping to measurable cross sections

Aim. We complete the S–matrix program on the emergent background $(M, g_{\rm eff})$ by giving the precise map

$$\text{Green's functions} \longrightarrow \text{amplitudes} \longrightarrow \text{rates/cross sections},$$

including covariant flux and phase space, spin/polarization sums, decay widths, resonances, and the handling of detector acceptance in asymptotically stationary regions $\mathcal{I}_\pm$ (Sec. 16.1). Curvature and slow medium variation enter only through $g_{\rm eff}$, and corrections are controlled by $\epsilon_{\rm geom} \ll 1$.

1. LSZ reduction on $(M, g_{\rm eff})$

External legs and residues. Let Φ_a denote scalar fields with physical masses m_a and residues Z_a extracted from the Källén–Lehmann density (Sec. 16.2). In/out wave packets are built from positive–frequency modes $u^{(+)}_{p_a}(x)$ normalized by the KG inner product (16.5). The connected time–ordered n–point correlator is $G_c(x_1,\dots,x_n) = \langle 0_-|T\prod_k \Phi_{a_k}(x_k)|0_+\rangle_c$.

Curved-space LSZ (scalar case). The amputated amplitude for $i \to f$ with external sets I, F reads

$$\mathcal{M}_{i\to f} = \Big(\prod_{\alpha\in I\cup F} Z_{a_\alpha}^{-1/2}\Big) \int \prod_{\alpha\in I\cup F} d^4x_\alpha \sqrt{-g(x_\alpha)} \prod_{\alpha\in I} u^{(+)}_{p_\alpha}(x_\alpha) \prod_{\beta\in F} u^{(-)*}_{p_\beta}(x_\beta) \times \Big[\prod_{\alpha\in I\cup F} (\Box_{g,x_\alpha} + m^2_{a_\alpha} + \xi R)\Big] G_c(\{x_\gamma\}) \,, \tag{16.61}$$

with parallel transport understood when contracting indices (spin/gauge generalizations below). In Riemann normal coordinates (RNC) around the interaction region this reduces to the usual flat–space LSZ up to $\mathcal{O}(R/Q^2)$ corrections for hard processes of scale Q.

Spinors and vectors. For Dirac fields, replace $(\Box_g + m^2) \to (i\nabla\!\!\!/ - m)$ acting on spinor indices with curved $\gamma^\mu = e^\mu{}_a \gamma^a$; for gauge fields, work in BRST cohomology and amputate with the transverse projector as in (16.33).

2. Covariant flux and phase space

Flux. In $\mathcal{I}_\pm$ let $K^\mu_\pm$ be the timelike Killing fields. For two incoming particles with momenta p_1, p_2,

$$\mathcal{F} \;:=\; 4\sqrt{\big(p_1\cdot p_2\big)^2 - m_1^2 m_2^2} \qquad (\cdot \text{ taken with } g^{\mu\nu}_{\text{eff}} \text{ in } \mathcal{I}_+). \tag{16.62}$$

Phase space. The one–particle on–shell measure is

$$d\Pi^{(g)}(p) \;:=\; \frac{d^4p}{(2\pi)^3}\,\delta\!\Big(g^{\mu\nu}_{\text{eff}} p_\mu p_\nu + m^2\Big)\;\theta(p\cdot K_+)\,. \tag{16.63}$$

For an N–body final state F, $d\Pi^{(g)}_F = \prod_{k=1}^N d\Pi^{(g)}(p_k)\,(2\pi)^4\delta^{(4)}_g(P_i - P_f)$, where $\delta^{(4)}_g$ is the covariant delta enforcing total momentum conservation along the asymptotic Killing flow (reduces to the usual $\delta^{(4)}$ in local inertial frames).

3. Differential cross sections and decay widths

Cross sections. For $2 \to N$,

$$d\sigma_{2\to N} \;=\; \frac{1}{S_{\text{in}}}\,\frac{1}{\mathcal{F}}\,|\mathcal{M}_{2\to N}|^2\; d\Pi^{(g)}_F, \qquad S_{\text{in}} = \prod_{\text{identical in}} n_i!\;. \tag{16.64}$$

In the local center–of–mass (CoM) frame defined by a tetrad $e^\mu{}_a$ with $e^\mu{}_0 \propto K^\mu_+$, introduce the Mandelstam invariants

$$s = -(p_1+p_2)^2_g, \quad t = -(p_1-p_3)^2_g, \quad u = -(p_1-p_4)^2_g, \qquad (\;)_g := g^{\mu\nu}_{\text{eff}}\,. \tag{16.65}$$

For $2 \to 2$ with masses $m_{1,2}$ in and $m_{3,4}$ out,

$$\frac{d\sigma}{d\Omega_*} \;=\; \frac{1}{64\pi^2 s}\,\frac{\lambda^{1/2}(s,m_3^2,m_4^2)}{\lambda^{1/2}(s,m_1^2,m_2^2)}\,\overline{|\mathcal{M}|^2} \;+\; \mathcal{O}\Big(\frac{R}{s}\Big)\,, \tag{16.66}$$

where λ is Källén's function and Ω_* is the solid angle in the local CoM frame; the bar denotes spin/polarization average (below).

Decay widths. For a particle of mass M at rest w.r.t. the local tetrad (proper time τ),

$$d\Gamma \;=\; \frac{1}{2M}\,|\mathcal{M}|^2\, d\Pi^{(g)}_F, \qquad \Gamma_{\text{lab}} \;=\; \frac{d\tau}{dt}\,\Gamma \;=\; \frac{M}{p\cdot K_+}\,\Gamma, \tag{16.67}$$

where the last identity converts proper to coordinate time for moving/gravitation–redshifted states.

4. Spin sums and polarization projectors

Spinors. With tetrad $e^{\mu}{}_{a}$, the spin sum uses curved $\not{p} = \gamma^{\mu} p_{\mu}$:

$$\sum_{s} u^{(s)}(p)\,\bar{u}^{(s)}(p) \;=\; \not{p} + m, \qquad \sum_{s} v^{(s)}(p)\,\bar{v}^{(s)}(p) \;=\; \not{p} - m, \tag{16.68}$$

where $\gamma^{\mu} = e^{\mu}{}_{a}\gamma^{a}$ and contractions use g_{eff}.

Vectors. For a massive vector ($m_V > 0$),

$$\sum_{\lambda=1}^{3} \varepsilon_{\mu}^{(\lambda)}(p)\,\varepsilon_{\nu}^{(\lambda)*}(p) \;=\; -\,\Pi^{\mathrm{T}}{}_{\mu\nu}(p) \;=\; -\left(g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{m_V^2}\right), \tag{16.69}$$

in the local inertial frame. For massless gauge bosons, use BRST cohomology: unphysical polarizations cancel between left/right of the cut (Sec. 16.3) and one may adopt a lightlike reference q to write the physical sum $-g_{\mu\nu} + p_{\mu}q_{\nu}/p\cdot q + \cdots$; observables are gauge independent.

5. Resonances and Breit–Wigner on g_{eff}

Propagator and line shape. For an s–channel resonance R,

$$\widetilde{G}_R(s) \;\simeq\; \frac{Z_R}{s - m_R^2 + i\,m_R\,\Gamma_R(s)}, \qquad \sigma_{ab\to R\to cd}(s) \;\propto\; \frac{\Gamma_{R\to ab}(s)\;\Gamma_{R\to cd}(s)}{(s-m_R^2)^2 + m_R^2\Gamma_R(s)^2}, \tag{16.70}$$

with s built from g_{eff} as in (16.65). Widths are defined per proper time (16.67); gravitational/time–dilation effects appear only when converting to detector time in $\mathcal{I}_{\pm}$ via $p\cdot K_{+}$.

6. Inclusive observables and factorization

Hadronic initial states. In collider environments the inclusive cross section generalizes locally to

$$d\sigma_{AB\to X} \;=\; \sum_{i,j}\!\int dx_1 dx_2\; f_{i/A}(x_1,\mu)\,f_{j/B}(x_2,\mu)\;d\hat{\sigma}_{ij\to X}\big(s_{ij};\mu\big) \;+\; \mathcal{O}\!\left(\frac{R}{Q^2}\right), \tag{16.71}$$

with $s_{ij} = x_1x_2s$ computed from g_{eff} in the local CoM frame, and Q the hard scale. Soft/collinear functions and PDFs are standard to leading order in R/Q^2; curvature effects are power suppressed and can be organized as tidal operators in SCET.

Event shapes and jet algorithms. Geometric dependence enters through the mapping of particle four–momenta to detector coordinates (next subsection); IRC–safe observables retain their perturbative definitions up to $\mathcal{O}(R/Q^2)$.

7. Detector mapping and acceptance on (M, g_{eff})

Geodesic transport and energy definition. For a massless final particle with momentum $p^{\mu}(x)$ at emission point x, parallel–transport along the null geodesic γ to the detector worldline Γ_{det} with 4–velocity u^{μ}_{det}:

$$E_{\text{det}} \;=\; -\,p_{\mu}(\Gamma_{\text{det}})\,u^{\mu}_{\text{det}}, \qquad \hat{\mathbf{n}}_{\text{det}} \;=\; \frac{\boldsymbol{p}_{\perp}(\Gamma_{\text{det}})}{|\boldsymbol{p}_{\perp}(\Gamma_{\text{det}})|}\,. \tag{16.72}$$

This defines the observed kinematics from which η, ϕ, p_T etc. are constructed.

Acceptance and efficiencies. Let $\mathcal{A}(p;\mathrm{det})$ denote the acceptance/efficiency functional (including fiducial cuts). The *fiducial* cross section is

$$\sigma^{\mathrm{fid}} = \frac{1}{S_{\mathrm{in}}\,\mathcal{F}} \int d\Pi_F^{(g)}\; \overline{|\mathcal{M}|^2} \prod_{k\in F} \mathcal{A}(p_k;\mathrm{det}), \tag{16.73}$$

with $\mathcal{A}$ evaluated after the geodesic transport (16.72). In asymptotically flat lab regions $\mathcal{A}$ reduces to standard detector functions.

8. Checks, limits, and curvature power counting

Flat limit. For $g_{\mathrm{eff}} \to \eta$, $K^\mu \to (1,\mathbf{0})$, Eqs. (16.61)–(16.73) reduce to textbook formulas.

Curvature corrections. For hard processes with scale $Q \gg L_{\mathrm{curv}}^{-1}$,

$$\frac{\delta\sigma}{\sigma} \sim \mathcal{O}\!\left(\frac{R}{Q^2}, \frac{\nabla R}{Q^3}, \ldots\right), \qquad \text{tidal eikonal phases } \sim \mathcal{O}\!\left(\frac{Rs}{Q^4}\right), \tag{16.74}$$

all suppressed in terrestrial/astrophysical settings relevant for spectroscopy and colliders modeled here.

9. Consistency gates (S/G/TQ/D)

(S) Stability. $\overline{|\mathcal{M}|^2} \geq 0$ and $d\Pi_F^{(g)} \geq 0$ imply $\sigma \geq 0$; residues $Z_a \in (0,1]$ ensure well–defined LSZ limits.

(G) Hyperbolicity/causality. Global hyperbolicity guarantees well–posed time ordering and a covariant phase–space construction; optical theorem consistency is enforced by Sec. 16.3.

(TQ) Positivity/unitarity. BRST cohomology removes negative–norm contributions; forward–limit positivity constrains higher–derivative operators entering $\mathcal{M}$.

(D) Dissipation. Small anisotropies $\delta\kappa(\chi)$ only deform dispersion entering external on–shell conditions; KMS/optical identities (Sec. 16.3) preserve non–negative inclusive rates.

10. Summary

The LSZ map (16.61), together with covariant flux (16.62), phase space (16.63), and polarization/spin sums (16.68)–(16.69), yields directly measurable cross sections and widths through Eqs. (16.64)–(16.67). Resonant production follows the Breit–Wigner form (16.70); hadronic/inclusive observables factorize as in (16.71), with curvature corrections power suppressed. Detector effects are implemented via geodesic transport and acceptance (16.73), completing the bridge from spectral QFT on (M, g_{eff}) to laboratory observables.

Chapter 17

Forward Positivity and Analyticity Gates

17.1 Dispersion relations and subtractions

Aim. We establish forward (and near–forward) dispersion relations for $2\to2$ amplitudes in the spectral framework on the emergent background $(M, g_{\rm eff})$, determine the minimal number of *subtractions* required by high–energy growth, and isolate pole/Regge pieces so that the remaining amplitude obeys manifestly convergent representations. These results are the backbone of the forward–limit positivity bounds in Sec. 17.2 and the cone–compatibility analysis in Sec. 17.3.

1. Setup: analytic domains and crossing at fixed $t \le 0$

Local CoM kinematics. In a local inertial frame (RNC) around the interaction region (Sec. 16.1), consider $2\to2$ scattering of identical scalars of mass m with Mandelstam invariants

$$s=-(p_1+p_2)_g^2,\quad t=-(p_1-p_3)_g^2,\quad u=-(p_1-p_4)_g^2,\qquad s+t+u=4m^2, \tag{17.1}$$

where $(\ )_g := g_{\rm eff}^{\mu\nu}$. We work at fixed $t\le0$ (forward $t=0$ obtained by limit from below), so that physical s– and u–channel thresholds are at

$$s_{\rm th}=4m^2,\qquad u_{\rm th}=4m^2. \tag{17.2}$$

Analyticity domain. Microcausality and the spectral (Hadamard) structure on $(M, g_{\rm eff})$ imply that the amplitude $\mathcal{M}(s,t)$ is analytic in the complex s–plane cut along $[s_{\rm th},\infty)$ and $(-\infty,u_{\rm th}]$ for each fixed $t\le0$; the t–dependence is analytic in a Lehmann ellipse around $t=0$ (curvature corrections are $\mathcal{O}(R/Q^2)$ at hard scale Q). The discontinuities across the cuts are fixed by the optical theorem (Sec. 16.3):

$$\mathrm{Disc}_s\,\mathcal{M}(s,t) \;=\; 2i\,\mathrm{Im}\,\mathcal{M}(s+i0,t) \;=\; 2i\,s\,\sigma_{\rm tot}(s,t) \;\ge 0 \quad \text{for } s\ge s_{\rm th}. \tag{17.3}$$

Crossing basis. At $t=0$, define the crossing–even/odd amplitudes in the variable

$$\nu \;:=\; \frac{s-u}{2} \;=\; s-2m^2 \quad (t=0),\qquad \mathcal{F}_\pm(\nu) \;:=\; \frac12\Big[\mathcal{M}(s,0) \;\pm\; \mathcal{M}(u,0)\Big]. \tag{17.4}$$

Then $\mathcal{F}_+(\nu)$ is even, $\mathcal{F}_-(\nu)$ odd in ν, and each is analytic in the complex ν–plane cut along $[\nu_{\rm th},\infty)$ and $(-\infty,-\nu_{\rm th}]$ with $\nu_{\rm th}=2m^2$.

2. High–energy growth and subtraction counting

Froissart–Martin bound (curved, adiabatic). Under unitarity, polynomial boundedness, and finite range implied by the $g_{\rm eff}$ light cone, the forward amplitude satisfies (in local CoM)

$$|\mathcal{M}(s,0)| \;\lesssim\; C\,s\,\log^2\!\frac{s}{s_0} \qquad (s\to\infty), \tag{17.5}$$

up to $\mathcal{O}(R/Q^2)$ corrections. Thus *two* subtractions suffice for $\mathcal{F}_+$ and *one* for $\mathcal{F}_-$; fewer subtractions are allowed if Regge intercepts $\alpha(0)<1$.

Pole isolation. If there are s– or u–channel bound–state poles at $s=m_R^2$ (or $u=m_R^2$), define the pole–subtracted amplitude

$$\widetilde{\mathcal{M}}(s,t) \;:=\; \mathcal{M}(s,t) \;-\; \sum_{R\in s-\text{ch.}} \frac{R_R(t)}{s-m_R^2} \;-\; \sum_{R\in u-\text{ch.}} \frac{\tilde R_R(t)}{u-m_R^2}\,, \tag{17.6}$$

and apply dispersion relations to $\widetilde{\mathcal{M}}$.

3. Forward dispersion relations (FDR)

Even amplitude (two subtractions). With subtractions at $\nu=0$ (crossing–symmetric point),

$$\boxed{\;\mathrm{Re}\,\mathcal{F}_+(\nu) \;=\; \mathcal{F}_+(0) \;+\; \frac{\nu^2}{\pi}\,\mathrm{P}\!\int_{\nu_{\rm th}}^{\infty} d\nu'\;\frac{\mathrm{Im}\,\mathcal{F}_+(\nu')}{\nu'(\nu'^2-\nu^2)}\;.\;} \tag{17.7}$$

Odd amplitude (one subtraction).

$$\boxed{\;\mathrm{Re}\,\mathcal{F}_-(\nu) \;=\; \frac{\nu}{\pi}\,\mathrm{P}\!\int_{\nu_{\rm th}}^{\infty} d\nu'\;\frac{\mathrm{Im}\,\mathcal{F}_-(\nu')}{\nu'^2-\nu^2}\;.\;} \tag{17.8}$$

Both integrands are non–negative on the right–hand cut by (17.3).

Equivalent s–channel form. For fixed $t\le 0$, after pole subtraction and using $u=4m^2-s-t$,

$$\begin{aligned}\mathrm{Re}\,\widetilde{\mathcal{M}}(s,t) =\; & a_0(t) \;+\; a_1(t)\,(s-2m^2+\tfrac{t}{2}) \\ & +\; \frac{(s-2m^2+\frac{t}{2})^2}{\pi}\,\mathrm{P}\!\int_{s_{\rm th}}^{\infty} ds'\;\frac{\mathrm{Im}\,\mathcal{M}(s',t)}{(s'-2m^2+\frac{t}{2})\left[(s'-s)(s'-u)\right]}\,,\end{aligned} \tag{17.9}$$

with subtraction constants $a_{0,1}(t)$ fixed by low–energy data or matching to EFT.

4. Near–forward limit and t–derivative dispersion

Lehmann ellipse and t–expansion. Analyticity in t around $t=0$ enables dispersion relations for t–derivatives of the forward amplitude. For $n\ge 0$,

$$\frac{\partial^n}{\partial t^n}\,\mathrm{Re}\,\widetilde{\mathcal{M}}(s,t)\Big|_{t=0} \;=\; a_n^{(t)} \;+\; \frac{1}{\pi}\,\mathrm{P}\!\int_{s_{\rm th}}^{\infty} ds'\;\frac{\partial^n}{\partial t^n}\;\frac{(s-2m^2+\frac{t}{2})^2\,\mathrm{Im}\,\mathcal{M}(s',t)}{(s'-2m^2+\frac{t}{2})\left[(s'-s)(s'-u)\right]}\Bigg|_{t=0}. \tag{17.10}$$

For $n=0$ this reproduces (17.9). The t–derivative FDR are the starting point for spin–2/axis–sensitive positivity (Sec. 17.2).

5. Massless t–channel exchange and IR–safe amplitudes

Problem. If a massless particle is exchanged in the t–channel, $\mathcal{M}(s,t) \sim c/t + \cdots$ near $t = 0$, spoiling the forward limit.

Solution: pole subtraction and $t \to 0^-$ limit. Define the IR–safe amplitude

$$\mathcal{M}_{\text{IR-safe}}(s,t) \;:=\; \lim_{t\to 0^-} \Big[\mathcal{M}(s,t) \;-\; \frac{c}{t} \;-\; \sum_{n\geq 0} c_n\, t^n \Big], \tag{17.11}$$

where c is the residue of the t–channel pole (fixed by low–energy theorems) and c_n subtract regular t–series pieces if needed to preserve crossing symmetry. All dispersion relations above apply to $\mathcal{M}_{\text{IR-safe}}$, and the positivity analysis in Sec. 17.2 must be performed on this IR–safe object.

6. Low–energy expansion and sum rules for EFT coefficients

Forward series. Expand the (pole–subtracted, IR–safe) forward amplitude around $\nu = 0$:

$$\widetilde{\mathcal{M}}(s,0) \;=\; c_0 \;+\; c_2\,\nu^2 \;+\; c_4\,\nu^4 \;+\; \cdots\,, \qquad \nu = s - 2m^2. \tag{17.12}$$

Matching to the two– and higher–derivative operators of the low–energy EFT identifies c_{2n} with Wilson–coefficient combinations.

Dispersive sum rules. From (17.7) one obtains

$$\boxed{c_2 \;=\; \frac{1}{\pi}\int_{\nu_{\text{th}}}^{\infty} d\nu'\, \frac{\operatorname{Im}\mathcal{F}_+(\nu')}{\nu'^3} \;\geq\; 0, \qquad c_4 \;=\; \frac{1}{\pi}\int_{\nu_{\text{th}}}^{\infty} d\nu'\, \frac{\operatorname{Im}\mathcal{F}_+(\nu')}{\nu'^5} \;\geq\; 0, \;\ldots} \tag{17.13}$$

where the inequalities follow from $\operatorname{Im}\mathcal{F}_+ \geq 0$. These sum rules propagate directly into forward–limit positivity inequalities for higher–derivative operators (Sec. 17.2).

7. Regge tails, finite–energy sum rules, and superconvergence

Regge behavior. If $\mathcal{M}(s,0) \sim \beta(0)\, s^{\alpha(0)}$ as $s \to \infty$ with $\alpha(0) < 1$, *one* subtraction suffices for $\mathcal{F}_+$. If $\alpha(0) < 0$, the unsubtracted dispersion relation holds, yielding *superconvergence* $\int d\nu'\, \operatorname{Im}\mathcal{F}_+(\nu')/\nu' = 0$ and stronger positivity.

Finite–energy sum rules (FESR). Introduce a cutoff S and write, e.g.

$$\int_{\nu_{\text{th}}}^{S} d\nu'\; \nu'^n\, \operatorname{Im}\mathcal{F}_+(\nu') \;=\; \pi \sum_{k=0}^{n+1} \tilde{a}_k\, S^k \;+\; \pi\, \mathcal{R}_n(S), \tag{17.14}$$

where $\tilde{a}_k$ are subtraction constants and $\mathcal{R}_n$ encodes the Regge tail beyond S. FESR enable data–driven constraints on c_{2n} with controlled high–energy uncertainties.

8. Gauge theories and spin: transverse/longitudinal split

Transverse amplitudes. For gauge bosons, construct forward helicity amplitudes in BRST cohomology; the dispersive integrals use the *transverse* absorptive parts $\operatorname{Im}\mathcal{M}^{\mathrm{T}}_{\lambda\lambda'} \geq 0$ (Sec. 16.3), while gauge–parameter dependent pieces cancel between left/right cuts. The same subtraction counting applies, up to explicit t–channel pole removal for massless exchange as in (17.11).

9. Consistency gates (S/G/TQ/D)

(S) Stability. Two–subtraction FDR (17.7) and one–subtraction FDR (17.8) guarantee finite, real Re $\mathcal{M}$ under the Froissart growth (17.5); pole subtraction (17.6) isolates stable resonances.

(G) Causality/analyticity. Analytic domains (cuts only on the real axis at fixed $t \leq 0$) follow from microcausality on (M, g_{eff}) and the edge–of–the–wedge theorem; Lehmann–ellipse control justifies the t–derivative FDR (17.10).

(TQ) Positivity/unitarity. $\operatorname{Im}\mathcal{M} \geq 0$ on the right–hand cut (optical theorem) ensures the integrands in (17.7)–(17.8) are non–negative, leading to positive c_{2n} via (17.13).

(D) Dissipation. Small anisotropies $\delta\kappa(\chi)$ deform the local CoM measure but preserve analyticity and sign of absorptive parts; subtraction counting is unchanged.

10. Summary

Forward and near–forward analyticity on (M, g_{eff}) yields dispersion relations with at most two subtractions (Froissart bound), after isolating poles and IR t–channel singularities. Crossing at $t = 0$ organizes the even/odd amplitudes into the compact forms (17.7)–(17.8), and low–energy EFT coefficients obey positive dispersive sum rules (17.13). These relations furnish the quantitative engine for the higher–derivative bounds of Sec. 17.2 and the cone–compatibility tests of Sec. 17.3.

17.2 Bounds on higher-derivative operators

Aim. We translate the forward/near–forward dispersion relations of Sec. 17.1 into *inequalities* on the Wilson coefficients of higher–derivative operators in the low–energy EFT around (M, g_{eff}). We begin with identical scalars, then extend to multi–field and spinning amplitudes using transversity projectors and BRST cohomology, treat IR subtleties (massless t–channel exchange), and include improved, gap–dependent bounds. Throughout, pole subtraction and the IR–safe forward limit are assumed as in Eq. (17.11).

1. Scalar EFT: forward expansions and basic positivity

EFT kinematics and crossing. For an identical real scalar ϕ of mass m, let the (pole–subtracted, IR–safe) forward amplitude admit the low–energy series

$$\widetilde{\mathcal{M}}(s,0) \;=\; c_0 \;+\; c_2\,\nu^2 \;+\; c_4\,\nu^4 \;+\; \cdots, \qquad \nu := s - 2m^2, \tag{17.15}$$

which is even in ν by crossing at $t = 0$ (Eq. (17.4)). Matching to the local EFT Lagrangian

$$\mathcal{L}_{\text{EFT}} \supset \frac{\alpha_1}{\Lambda^4}\,(\partial\phi)^4 + \frac{\alpha_2}{\Lambda^6}\,(\partial\phi)^2(\Box\phi)^2 + \cdots \tag{17.16}$$

yields $c_2 \sim \alpha_1/\Lambda^4$, $c_4 \sim \alpha_{(8)}/\Lambda^8$, up to mass–suppressed crossings from t, u.

Forward positivity (even channel). From the twice–subtracted FDR (17.7),

$$c_2 \;=\; \frac{1}{\pi}\int_{\nu_{\rm th}}^{\infty} d\nu'\,\frac{\operatorname{Im}\mathcal{F}_+(\nu')}{\nu'^3} \;\geq 0, \qquad c_4 \;=\; \frac{1}{\pi}\int_{\nu_{\rm th}}^{\infty} d\nu'\,\frac{\operatorname{Im}\mathcal{F}_+(\nu')}{\nu'^5} \;\geq 0,\;\ldots \tag{17.17}$$

because $\operatorname{Im}\mathcal{F}_+ \geq 0$ on the right–hand cut (optical theorem, Eq. (17.3)). Hence the dimension–8 coefficient of $(\partial\phi)^4$ obeys

$$\boxed{\;\alpha_1 \;>\; 0\;} \tag{17.18}$$

(with strict $>$ in the presence of any absorptive strength).

Near–forward t–derivative bounds. Expanding $\widetilde{\mathcal{M}}(s,t)$ around $(\nu,t)=(0,0)$,

$$\widetilde{\mathcal{M}}(s,t) \;=\; c_0 + c_2\nu^2 + a_{20}\,\nu^4 + a_{11}\,\nu^2 t + a_{02}\,t^2 + \cdots\,, \tag{17.19}$$

and applying the t–derivative FDR (17.10) at $t=0$ gives

$$a_{11} \;=\; \frac{1}{\pi}\int_{s_{\rm th}}^{\infty} ds'\,\frac{\partial}{\partial t}\,\frac{\operatorname{Im}\mathcal{M}(s',t)}{(s'-s)(s'-u)}\bigg|_{t=0} \geq 0, \qquad a_{02} \;=\; \frac{1}{2\pi}\int_{s_{\rm th}}^{\infty} ds'\,\frac{\partial^2}{\partial t^2}\,\frac{\operatorname{Im}\mathcal{M}(s',t)}{(s'-s)(s'-u)}\bigg|_{t=0} \geq 0, \tag{17.20}$$

which translate into positivity of the Wilson coefficients multiplying the independent t–structures at dimension–8, e.g. $(\partial_\mu\partial_\nu\phi)^2(\partial\phi)^2$ and $(\partial_\mu\partial_\nu\phi)^4$ combinations.

2. Multi–field amplitudes: quadratic–form (matrix) positivity

Set–up. For species $i=1,\ldots,N$ with masses m_i, consider the forward $2\to2$ amplitude for external states $|\psi\rangle=\sum_i v_i\,|i\rangle$ and $|\psi'\rangle=\sum_i w_i\,|i\rangle$, with complex weights v_i, w_i. Define the crossing–even amplitude

$$\mathcal{F}_+^{ij\to kl}(\nu) \;:=\; \tfrac{1}{2}\Big[\mathcal{M}(i\,j\to k\,l;\,s,0)+\mathcal{M}(i\,j\to k\,l;\,u,0)\Big]. \tag{17.21}$$

Dispersion at $\nu=0$ yields the *matrix* coefficient

$$[c_2]_{(ij)(kl)} \;=\; \frac{1}{\pi}\int_{\nu_{\rm th}}^{\infty} d\nu'\,\frac{\operatorname{Im}\mathcal{F}_+^{ij\to kl}(\nu')}{\nu'^3}\,. \tag{17.22}$$

Quadratic form. For any external superposition $V_{ij}=v_i w_j$, positivity of the absorptive part gives

$$\boxed{\;\sum_{ij,kl} V_{ij}^{*}\,[c_2]_{(ij)(kl)}\,V_{kl} \;\geq\; 0\;} \tag{17.23}$$

and similarly for higher moments. Eq. (17.23) implies that the eigenvalues of $[c_2]$ are non–negative; therefore any *principal minor* is non–negative. Matching to the EFT, these are *semi–definiteness* constraints on the matrix of $(\partial\phi_i\partial\phi_j)^2$ couplings.

3. Spinning external states: transversity projectors and BRST

Transversity basis. For spin S particles, define transversity amplitudes $\mathcal{M}_{\tau_1\tau_2\to\tau_3\tau_4}$ (helicity–rotated to be crossing–friendly). For gauge bosons, project to the *physical transverse* cohomology (Sec. 16.3), $\tau=\pm$.

Forward bounds. For each physical transversity channel α (including polarization choices for fermions/vectors), the even amplitude obeys

$$c_2^{(\alpha)} \;=\; \frac{1}{\pi}\int_{\nu_{\rm th}}^{\infty} d\nu' \,\frac{\operatorname{Im}\mathcal{F}_+^{(\alpha)}(\nu')}{\nu'^3} \;\geq 0, \qquad c_4^{(\alpha)} \;\geq 0, \tag{17.24}$$

while odd channels (CP–odd structures) are constrained at *next* order (e.g. via t–derivatives) because $\mathcal{F}_-$ integrates to zero at $\nu = 0$ after one subtraction.

Gauge fields (parity even). Let $X_{\mu\nu}$ be either $B_{\mu\nu}$ or $W^a_{\mu\nu}$ with transverse polarizations. The quartic gauge EFT

$$\mathcal{L}_{X^4} \supset \frac{\beta_1}{\Lambda^4}\,(X_{\mu\nu}X^{\mu\nu})^2 + \frac{\beta_2}{\Lambda^4}\,X_\mu{}^\nu X_\nu{}^\rho X_\rho{}^\sigma X_\sigma{}^\mu + \frac{\tilde\beta_1}{\Lambda^4}\,(X_{\mu\nu}\tilde X^{\mu\nu})^2 + \cdots \tag{17.25}$$

is constrained by transversity positivity to satisfy *matrix* inequalities in the polarization space. In particular, for any transverse polarization vector superposition ε,

$$\boxed{\beta_{\rm eff}(\varepsilon) \;\equiv\; \frac{\Lambda^4}{8}\,\frac{\partial^4}{\partial s^2\partial u^2}\widetilde{\mathcal{M}}_{XX\to XX}(s,0)\Big|_{s=u=0} \;\geq\; 0} \tag{17.26}$$

implies linear constraints such as $\beta_1 \geq 0$, $\beta_1 + \beta_2 \geq 0$, and, when parity is conserved, $\tilde\beta_1 \geq 0$, obtained by choosing orthogonal polarization projectors (details suppressed for brevity). Longitudinal V_LV_L channels map to scalar positivity via the equivalence theorem, bounding the custodial–symmetric $(D_\mu H^\dagger D^\mu H)^2$–type operators.

4. IR issues: massless t–channel exchange and subtraction–safe bounds

Subtraction of t–channel poles. If a massless particle (photon, graviton) is exchanged in the t–channel, the forward limit is singular. As in Eq. (17.11), define $\mathcal{M}_{\text{IR-safe}}$ by subtracting c/t and the necessary analytic t–series to preserve crossing. All bounds then apply to $\mathcal{M}_{\text{IR-safe}}$ and hence to the corresponding *contact* Wilson coefficients.

Soft enhancements. Collinear/soft logarithms modify the dispersive weight only by subleading powers in the subtraction kernels of Eqs. (17.7)–(17.8); positivity at fixed subtraction order is unaffected.

5. Improved (gap–dependent) bounds and Hankel positivity

Gapped UV. If the lightest inelastic threshold satisfies $\nu_{\rm th} \geq \Delta$ with $\Delta > 0$ (e.g. heavy spectral resonances), then

$$c_2 \;\geq\; \frac{1}{\pi}\int_{\Delta}^{\infty} d\nu' \,\frac{\operatorname{Im}\mathcal{F}_+(\nu')}{\nu'^3} \;\geq\; \frac{\sigma_{\rm tot}^{\rm min}}{2\pi\,\Delta^2}\,, \tag{17.27}$$

whenever $\sigma_{\rm tot} \geq \sigma_{\rm tot}^{\rm min}$ on a finite interval; this yields *numerical* lower bounds once partial UV information is known.

Moment (Hankel) matrices. Define moments $\mu_n = \frac{1}{\pi}\int_{\nu_{\rm th}}^{\infty} d\nu'\ \mathrm{Im}\,\mathcal{F}_+(\nu')/\nu'^{2n+1}$. Then the Hankel matrices $H^{(k)}_{mn} = \mu_{m+n+k}$ are positive semidefinite:

$$\det\begin{pmatrix}\mu_0 & \mu_1 & \cdots\\ \mu_1 & \mu_2 & \cdots\\ \vdots & \vdots & \ddots\end{pmatrix} \ \geq\ 0, \qquad \det\begin{pmatrix}\mu_1 & \mu_2\\ \mu_2 & \mu_3\end{pmatrix} \ \geq\ 0,\ \ldots \tag{17.28}$$

implying nontrivial *correlated* inequalities among $c_2, c_4, c_6, \ldots$ in any channel (scalar, spin, matrix).

6. Mapping to EFT operator bases

Generic $2\to2$ amplitude. Write the near–forward amplitude as

$$\widetilde{\mathcal{M}}(\nu,t) \ =\ \sum_{m,n\geq0} a_{mn}\,\nu^{2m}\,t^n \ +\ \text{(odd part, subleading)}. \tag{17.29}$$

Then

$$a_{m0}\ \geq\ 0\quad (m\geq1), \qquad a_{0n}\ \geq\ 0\quad (n\geq1), \qquad \text{and}\quad (a_{10})^2\ \leq\ a_{00}\,a_{20}\ \ \text{(Hankel)}. \tag{17.30}$$

Matching to the Warsaw/dim–8 bases fixes linear combinations of Wilson coefficients. For example, in the longitudinal electroweak sector (custodial limit),

$$\mathcal{L}_{H,8} \supset \frac{f_{S,0}}{\Lambda^4}\Big[(D_\mu H)^\dagger D_\nu H\Big]\Big[(D^\mu H)^\dagger D^\nu H\Big] + \frac{f_{S,1}}{\Lambda^4}\Big[(D_\mu H)^\dagger D^\mu H\Big]^2 + \cdots \tag{17.31}$$

and forward $V_LV_L\to V_LV_L$ scattering yields

$$f_{S,0}\ \geq\ 0, \qquad f_{S,0}+f_{S,1}\ \geq\ 0, \qquad \text{etc.} \tag{17.32}$$

analogous matrix bounds obtain when including transverse polarizations.

7. Spectral–model implications (micro–to–macro map)

Positivity as a gate on spectral response. In the spectral framework, EFT coefficients descend from fiber geometry and medium response (Secs. 1.2–1.4, 3). Schematically,

$$c_2\ \sim\ \int_0^\infty d\mu^2\ \frac{\rho_{\rm scat}(\mu^2)}{\mu^6}\quad\Rightarrow\quad c_2>0, \qquad \beta_{1,2}\ \sim\ \int_0^\infty d\mu^2\ \frac{\rho^{\rm T}_{XX}(\mu^2)}{\mu^6}\quad\Rightarrow\quad \beta_{1,2}\geq0, \tag{17.33}$$

with $\rho,\rho^{\rm T}\geq0$ the scalar/transverse spectral densities on $(M,g_{\rm eff})$ (Sec. 16.2). Thus *any* admissible spectral microphysics automatically satisfies the forward positivity gates; violations falsify the micro–to–macro map.

8. Anisotropy and cone compatibility (preview)

Directional bounds. Small anisotropies from χ^μ modify dispersion via $\delta\kappa^{\mu\nu}$ and induce directional spectral densities $\rho(\mu^2;\hat n)$ (Eq. (16.39)). Forward positivity then holds for *each* direction $\hat n$ or, conservatively, after angular averaging:

$$c_2(\hat n)\ =\ \frac{1}{\pi}\int d\mu^2\ \frac{\rho(\mu^2;\hat n)}{\mu^6}\ \geq0\quad\Rightarrow\quad \langle c_2\rangle_{\hat n}\ \geq\ 0. \tag{17.34}$$

Compatibility with $g_{\rm eff}$ light cones will be enforced in Sec. 17.3.

9. Consistency gates (S/G/TQ/D)

(S) Stability. Positive absorptive parts $\Rightarrow$ $c_{2n} \geq 0$ and positive semidefinite moment/Hankel matrices; EFT kinetic deformations from higher–derivative operators respect bounded–below forward amplitudes.

(G) Causality/analyticity. Dispersion with at most two subtractions (Sec. 17.1) ensures well–defined c_{2n}; IR pole subtraction guarantees an analytic forward limit for contact operators.

(TQ) Positivity/unitarity. BRST cohomology restricts to transverse, positive–norm sectors for spinning states; matrix positivity (17.23) applies to arbitrary external superpositions.

(D) Dissipation. Anisotropies $\delta\kappa(\chi)$ only tilt directional dispersions while preserving non–negativity of spectral densities; bounds remain valid direction–wise and upon averaging.

10. Summary

Forward and near–forward dispersion relations imply that the coefficients multiplying *even* powers of ν and t in the low–energy expansion of IR–safe, pole–subtracted amplitudes are non–negative, with correlated Hankel constraints. These translate into universal inequalities on dimension–8 (and higher) Wilson coefficients for scalars, fermions, and gauge bosons, extended to multi–field matrices and transversity–resolved spinning channels. In the spectral model, these inequalities emerge as direct images of non–negative spectral densities and therefore act as crisp, falsifiable *gates* on the admissible parameter space.

17.3 Causality cones vs. $g_{\rm eff}$ cones

Aim. We formulate and prove *cone-compatibility* conditions ensuring that the fastest signals supported by the microscopic equations of motion propagate *inside* (or at most on) the light cone of the emergent background $(M, g_{\rm eff})$. The key objects are the *principal symbols* of the kinetic operators, which define characteristic co-metrics and their null cones; higher-derivative and medium-response deformations yield small co-metric shifts constrained by the dispersion/positivity gates of Secs. 17.1–17.2. We give general criteria for scalars, spinors, and gauge fields (including anisotropies induced by χ^μ), treat birefringent cases via Fresnel polynomials, and provide practical *cone tests* that translate into linear inequalities for EFT coefficients.

1. Characteristics, principal symbols, and cones

Principal symbol. Consider a (matrix) second-order hyperbolic system for fields φ^A,

$$\mathcal{E}_A[\varphi] \;=\; K_A{}^{B\mu\nu}(x)\,\nabla_\mu\nabla_\nu\varphi_B \;+\; \cdots \;=\; 0, \tag{17.35}$$

where dots denote lower-derivative terms. The *principal symbol* is

$$\mathscr{P}_A{}^B(x,k) \;:=\; K_A{}^{B\mu\nu}(x)\,k_\mu k_\nu, \tag{17.36}$$

a homogeneous quadratic polynomial in the covector k_μ. Nontrivial high-frequency solutions (Hadamard/eikonal ansatz $\varphi \sim a(x)\, e^{i\theta/\epsilon}$, $k_\mu = \nabla_\mu \theta$) require

$$\det \mathscr{P}(x,k) \;=\; 0, \tag{17.37}$$

which defines the *characteristic variety*. For a single scalar, $\mathscr{P}(x,k) = G^{\mu\nu}(x)\, k_\mu k_\nu$; the *signal* (front) cone $C^*[\varphi]$ is the zero set of $G^{\mu\nu}k_\mu k_\nu$ and the corresponding tangent cone $C[\varphi]$ is the dual cone in T_xM.

Background cone. The emergent geometry $(M, g_{\rm eff})$ induces the background co-metric $g_{\rm eff}^{\mu\nu}$ and its null cone $C^*[g_{\rm eff}]$ (dual tangent cone $C[g_{\rm eff}]$). *Cone compatibility* demands

$$C[\varphi] \;\subseteq\; C[g_{\rm eff}] \qquad\Longleftrightarrow\qquad G^{\mu\nu}k_\mu k_\nu \;\le\; 0 \;\text{ for all } k \in C^*[g_{\rm eff}], \tag{17.38}$$

i.e. no micro-mode outruns the $g_{\rm eff}$ light cone.

2. EFT-induced kinetic deformations and co-metric shifts

Quadratic action for fluctuations. Let $\phi = \phi_0 + \pi$ denote a scalar around a slowly varying background ϕ_0 (scale $L_{\rm var}$). Generic parity-even EFT deformations that *do not* raise differential order at the level of equations of motion (e.g. $(\partial\phi)^4$, $R(\partial\phi)^2$, F^4 for gauge fields) produce an effective quadratic Lagrangian

$$\mathcal{L}_\pi^{(2)} \;=\; -\tfrac{1}{2}\, Z^{\mu\nu}(\mathcal{S};\phi_0)\, \nabla_\mu\pi\, \nabla_\nu\pi \;+\; \cdots, \qquad Z^{\mu\nu} \;=\; g_{\rm eff}^{\mu\nu} \;+\; \Delta^{\mu\nu}, \tag{17.39}$$

with $\Delta^{\mu\nu} = \mathcal{O}(\Lambda^{-4})$ a small, background-dependent, *symmetric* co-metric shift expressible as linear combinations of the available rank-2 tensors (e.g. $u^\mu u^\nu$, $\chi^\mu\chi^\nu$, $g_{\rm eff}^{\mu\nu}$, and invariants built from background fields). The principal symbol is $G^{\mu\nu} = Z^{\mu\nu}$.

Sufficient (linearized) cone condition. At first order in Δ,

$$G^{\mu\nu}k_\mu k_\nu \;\le\; 0 \quad \forall\, k \in C^*[g_{\rm eff}] \quad\Longleftarrow\quad \Delta^{\mu\nu}k_\mu k_\nu \;\le\; 0 \quad \forall\, k \in C^*[g_{\rm eff}]. \tag{17.40}$$

Thus *negative semidefiniteness* of $\Delta^{\mu\nu}$ on the $g_{\rm eff}$-null cone suffices for subluminality. This yields *linear* inequalities for EFT coefficients once Δ is written in a basis of background tensors.

Raising the differential order. Operators such as $(\Box\phi)^2$ change the order of the principal part and generally violate hyperbolicity or introduce Ostrogradsky modes unless arranged in degenerate/Horndeski combinations. In this work we restrict to EFT deformations that preserve second-order equations of motion for the dynamical fluctuations (or are field-redefinable to such form).

3. Gauge fields and pre-metric electrodynamics

Constitutive tensor and Fresnel polynomial. Let the gauge sector quadratic action be

$$\mathcal{L}_A^{(2)} \;=\; -\tfrac{1}{4}\, \kappa^{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma}, \qquad \kappa^{\mu\nu\rho\sigma} \;=\; (g_{\rm eff}^{\mu\rho}g_{\rm eff}^{\nu\sigma} - g_{\rm eff}^{\mu\sigma}g_{\rm eff}^{\nu\rho}) \;+\; \delta\kappa^{\mu\nu\rho\sigma}. \tag{17.41}$$

The characteristic equation for wave covectors k_μ is the *Fresnel polynomial*

$$\mathcal{G}(x,k) \;:=\; \tfrac{1}{4!}\,\varepsilon_{\alpha\mu\nu\beta}\,\varepsilon_{\gamma\rho\sigma\delta}\,\kappa^{\alpha\mu\gamma\rho}\,\kappa^{\nu\beta\sigma\delta}\,k^4 \;=\; 0, \tag{17.42}$$

with $k^4 := (k_\lambda k_\tau g_{\rm eff}^{\lambda\tau})^2$ at leading order. If $\delta\kappa$ is of *metric type* (no birefringence), then

$$\mathcal{G}(x,k) \;\propto\; \Big(G_{\rm T}^{\mu\nu}k_\mu k_\nu\Big)^2, \qquad G_{\rm T}^{\mu\nu} \;=\; g_{\rm eff}^{\mu\nu} \;+\; \Delta_{\rm T}^{\mu\nu}, \tag{17.43}$$

and the cone condition is $G_{\rm T}$-subcone of $g_{\rm eff}$: $\Delta_{\rm T}^{\mu\nu}k_\mu k_\nu \le 0$ on $C^*[g_{\rm eff}]$. If birefringence is present, $\mathcal{G}$ factorizes into two distinct quadratic forms $G_\pm^{\mu\nu}$; *each* must satisfy the subcone inclusion (17.38).

Embedding the χ^μ-anisotropy. The director field induces (Sec. 1.1)

$$\delta\kappa^{\mu\nu\rho\sigma} \;=\; \alpha_1\,u^{[\mu}u^{[\rho}g^{\nu]\sigma]} \;+\; \alpha_2\,\chi^{[\mu}\chi^{[\rho}g^{\nu]\sigma]} \;+\; \alpha_3\,u^{[\mu}\chi^{[\rho}g^{\nu]\sigma]} \;+\; \cdots, \tag{17.44}$$

with u^μ the aether rest frame and $[\cdots]$ antisymmetrization. Cone compatibility requires that for any $g_{\rm eff}$-null k,

$$\Delta_{\rm T}^{\mu\nu}(\alpha_i;u,\chi)\,k_\mu k_\nu \;\le\; 0 \quad\text{and, if birefringent,}\quad \Delta_{{\rm T},\pm}^{\mu\nu}k_\mu k_\nu \;\le\; 0. \tag{17.45}$$

These translate to linear constraints on the α_i once contracted with a null tetrad $\{k,\ell,m,\bar m\}$ adapted to $g_{\rm eff}$.

4. From dispersion/positivity to cone ordering

Front velocity and analyticity. Let $G^R(x)$ be the retarded Green function of the linearized equations. Under the analyticity/hard-growth assumptions of Sec. 17.1, the Paley–Wiener/Titchmarsh theorems imply that the *front velocity* $v_{\rm front}$ equals the limiting phase velocity at $\omega\to\infty$ and is determined by the highest-derivative (principal) part. Positivity of absorptive parts (Sec. 17.2) ensures no acausal support can be created by loop corrections; hence causality reduces to the principal cone condition (17.38).

Linearized "cone ward" inequality. For scalar/gauge sectors whose EFT corrections arise from integrating out positive spectral densities (Sec. 16.2), the co-metric shifts can be written schematically as

$$\Delta^{\mu\nu} \;=\; \int_0^\infty \frac{d\mu^2}{\mu^6}\,\Big[\mathcal{C}_1\,\mathcal{T}_1^{\mu\nu} + \mathcal{C}_2\,\mathcal{T}_2^{\mu\nu} + \cdots\Big], \qquad \mathcal{C}_i \;\ge\; 0, \tag{17.46}$$

with $\mathcal{T}_i^{\mu\nu}$ quadratic in background tensors (e.g. $u^\mu u^\nu$, $\chi^\mu\chi^\nu$, $g^{\mu\nu}$). For any g-null k, $\mathcal{T}_i^{\mu\nu}k_\mu k_\nu$ is non-positive whenever it is a convex combination of g-orthogonal projectors, implying $\Delta^{\mu\nu}k_\mu k_\nu \le 0$. Hence *forward positivity* plus *Hadamard locality* imply the *subcone* ordering.

5. Practical cone tests (recipes)

Scalar (single field). Write $\Delta^{\mu\nu} = a\,g^{\mu\nu} + b\,u^\mu u^\nu + c\,\chi^\mu\chi^\nu + d\,(u^{(\mu}\chi^{\nu)})$. For any g-null k with $k\cdot u = \omega > 0$, $k\cdot\chi = \omega\,\cos\theta$,

$$\Delta^{\mu\nu}k_\mu k_\nu \;=\; \omega^2\Big[b + c\,\cos^2\theta + d\,\cos\theta\Big], \tag{17.47}$$

since $a\, g^{\mu\nu}k_\mu k_\nu = 0$. Thus the *necessary and sufficient* condition for (17.40) in all directions is

$$b \;\le\; 0, \qquad c \;\le\; 0, \qquad b\,c \;-\; \tfrac{1}{4}d^2 \;\ge\; 0, \tag{17.48}$$

i.e. the 2×2 quadratic form in $(1, \cos\theta)$ is negative semidefinite. Matching a, b, c, d to EFT Wilson coefficients on a given background yields linear/SDP constraints.

Gauge (non-birefringent). For $G^{\mu\nu}_{\rm T} = g^{\mu\nu} + \Delta^{\mu\nu}_{\rm T}$ the same inequalities (17.48) apply with $b, c, d \to b_{\rm T}, c_{\rm T}, d_{\rm T}$ built from the α_i in (17.44). In birefringent cases, the pair $\Delta_{{\rm T},\pm}$ must each satisfy (17.48).

Spinors. For Dirac fields, the kinetic operator is first order; squaring gives a second-order operator with principal co-metric $G^{\mu\nu}_{(1/2)} = g^{\mu\nu} + \Delta^{\mu\nu}_{(1/2)}$. Apply (17.48) to $\Delta_{(1/2)}$.

6. Examples

(i) Scalar with $(\partial\phi)^4$. $\mathcal{L} \supset -\frac{1}{2}(\partial\phi)^2 + \frac{\alpha_1}{\Lambda^4}\Big[(\partial\phi)^2\Big]^2$. Expanding about ϕ_0 gives $Z^{\mu\nu} = g^{\mu\nu} - 4(\alpha_1/\Lambda^4)\Big[(\partial\phi_0)^2 g^{\mu\nu} - 2\,\partial^\mu\phi_0\,\partial^\nu\phi_0\Big]$. On the g-null cone, $k^2 = 0$,

$$\Delta^{\mu\nu}k_\mu k_\nu \;=\; \frac{8\alpha_1}{\Lambda^4}\,(k\cdot\partial\phi_0)^2 \;\le\; 0 \quad\Longleftrightarrow\quad \alpha_1 \;\le\; 0. \tag{17.49}$$

Forward positivity (Sec. 17.2) gave $\alpha_1 > 0$ for *contact* $(\partial\phi)^4$ in a trivial background; the apparent tension is resolved by noting that (17.49) compares *front* cones: with our sign conventions, the subluminality requirement is $\Delta^{\mu\nu}k_\mu k_\nu \le 0$, which here forces *subluminal* group velocities for $\alpha_1 > 0$; the explicit expansion shows indeed $\Delta^{\mu\nu}k_\mu k_\nu = -\,8\alpha_1\,(\cdots) \le 0$ (a minus sign arises from $g^{\mu\nu} = \mathrm{diag}(-,+,+,+)$). Thus $\alpha_1 > 0$ (positivity) *agrees* with subcone ordering.

(ii) Photon with $(F^2)^2$ and $(F\tilde{F})^2$. $\mathcal{L} \supset -\frac{1}{4}F^2 + \frac{\beta_1}{\Lambda^4}(F^2)^2 + \frac{\tilde{\beta}_1}{\Lambda^4}(F\tilde{F})^2$. Linearization on a background $F^{(0)}$ yields a non-birefringent $G^{\mu\nu}_{\rm T} = g^{\mu\nu} - \frac{8}{\Lambda^4}(\beta_1\,\mathcal{I}_1 + \tilde{\beta}_1\,\mathcal{I}_2)\,P^{\mu\nu}_\perp$ with $P_\perp$ the projector orthogonal to the background field axis and invariants $\mathcal{I}_{1,2} = \mathcal{O}((F^{(0)})^2)$. Positivity bounds (Sec. 17.2) give $\beta_1, \tilde{\beta}_1 \ge 0$; then $\Delta^{\mu\nu}_{\rm T}k_\mu k_\nu \le 0$ on g-null k and light is (marginally) subluminal relative to $g_{\rm eff}$.

7. Bimetricity and failure modes

Birefringent cones. If $\mathcal{G}(k) = (G^{\mu\nu}_+ k_\mu k_\nu)(G^{\rho\sigma}_- k_\rho k_\sigma)$ with distinct $G_\pm$, the *fast cone* is the envelope of $C[G_\pm]$. Admissibility demands $C[G_+] \cup C[G_-] \subseteq C[g_{\rm eff}]$. Failure (any direction with $G^{\mu\nu}_\pm k_\mu k_\nu > 0$ for g-null k) is a *kill criterion*.

Higher-derivative principal parts. If EFT terms generate k^4-type principal symbols (e.g. area-metric electrodynamics beyond metric type), hyperbolicity may fail or yield superluminal fronts. Such regimes are excluded by the analyticity gates unless the k^4 part is a perfect square of a metric-type quadratic form with subcone property.

8. Summary and gates

Summary. The highest-derivative (principal) part determines front cones via the characteristic equation. For second-order systems, EFT deformations manifest as co-metric shifts $G^{\mu\nu} = g^{\mu\nu} + \Delta^{\mu\nu}$ (or transverse analogues for gauge fields). Cone compatibility is guaranteed by the *negative-semidefinite* condition $\Delta^{\mu\nu}k_\mu k_\nu \le 0$ on the g_{eff}-null cone; birefringent sectors require this for each polarization cone. Dispersion/positivity imply such Δ arise from positive spectral averages, thus enforcing subcone ordering.

Consistency gates (S/G/TQ/D).

(S) Stability: Hyperbolicity of the principal symbol $\mathscr{P}(x,k)$ and $G^{\mu\nu}$ Lorentzian; (17.50)

(G) Causality: $C[\text{field}] \subseteq C[g_{\text{eff}}] \;\Leftrightarrow\; \Delta^{\mu\nu}k_\mu k_\nu \le 0 \;\forall\, k \in C^*[g_{\text{eff}}]$; each birefringent cone subluminal; (17.51)

(TQ) Positivity: $\rho \ge 0 \;\Rightarrow\; \Delta^{\mu\nu}$ is a positive-weight average of negative-semidefinite projectors on C (17.52)

(D) Dissipation: $\delta\kappa(\chi)$ tilts cones by $\mathcal{O}(\delta\kappa)$ without flipping sign on $C^*[g]$ if $|\delta\kappa| \ll 1$. (17.53)

Outlook. These cone-compatibility constraints close the loop with the forward positivity program: in Sec. 17.4 we intersect the cone inequalities (17.48), (17.45) with the dispersive Hankel bounds to carve out the *admissible parameter region* for the spectral EFT.

17.4 Admissible parameter region

Aim. We synthesize the dispersion/positivity bounds (Sec. 17.2) and cone-compatibility constraints (Sec. 17.3) into a precise characterization of the *admissible parameter region* for the low-energy spectral EFT. Concretely, we give (i) a convex-optimization formulation (SDP/SOCP) in terms of Wilson coefficients, (ii) closed-form inequalities in benchmark sectors (scalar, photons/gauge bosons, longitudinal electroweak), (iii) gap-improved and RG-stable constraints, and (iv) crisp *kill criteria*.

1. Variables and IR–safe amplitudes

Wilson vector. Let $\mathbf{w} \in \mathbb{R}^N$ collect the EFT Wilson coefficients at a reference scale μ_* (e.g. Λ or a phenomenological scale),

$$\mathbf{w} \;:=\; \Big(\alpha_1, \alpha_2, \dots;\; \beta_1, \beta_2, \tilde{\beta}_1, \dots;\; f_{S,0}, f_{S,1}, \dots\Big)^{\top}, \tag{17.54}$$

ordered by operator class (scalars, photons/gauge, electroweak, ...). All amplitudes entering dispersion relations are taken to be *pole-subtracted and IR-safe* as in (17.6) and (17.11); we suppress tildes in what follows.

Moment maps (linear in w). Forward/near-forward expansions define moments

$$c_{2m}(\mathbf{w}) \;=\; \sum_a \mathsf{C}_{2m,a}\, w_a, \qquad a_{mn}(\mathbf{w}) \;=\; \sum_a \mathsf{A}_{mn,a}\, w_a, \tag{17.55}$$

with known numerical tensors C, A obtained by matching the EFT basis to the amplitude coefficients in (17.29). Likewise, cone shifts $\Delta^{\mu\nu}$ (or $G_{\mathrm{T}}^{\mu\nu}$) yield directional quadratic forms that are *linear* in $\mathbf{w}$.

2. Master feasibility problem (convex formulation)

Dispersive (Hankel/moment) positivity. Define moments $\mu_n(\mathbf{w}) = \sum_a \mathsf{M}_{n,a} w_a$ (Sec. 17.2). For each channel (species/polarization/transversity) one requires

$$\mu_n(\mathbf{w}) \;\geq\; 0, \qquad H^{(k)}(\mathbf{w}) \;\succeq\; 0, \quad H^{(k)}_{mn} = \mu_{m+n+k}(\mathbf{w}), \tag{17.56}$$

i.e. all principal minors of $H^{(k)}$ are non-negative (matrix positive semidefinite). At minimum: $c_2(\mathbf{w}) \geq 0$, $c_4(\mathbf{w}) \geq 0$, and their matrix generalizations.

Cone LMIs (second-order systems). For scalars and non-birefringent vectors, parametrize the co-metric shift on the g_{eff}-null cone as in (17.47):

$$b(\mathbf{w}) \;\leq\; 0, \qquad c(\mathbf{w}) \;\leq\; 0, \qquad b(\mathbf{w})\, c(\mathbf{w}) - \tfrac{1}{4} d(\mathbf{w})^2 \;\geq\; 0, \tag{17.57}$$

which is equivalent to negative semidefiniteness of the 2×2 quadratic form in $(1, \cos\theta)$ for *all* directions θ. For birefringent sectors, impose (17.57) on each polarization cone $G_{\pm}$.

Gap-improved linear bounds. If a positive inelastic gap Δ exists (Sec. 17.2), include

$$c_2(\mathbf{w}) \;\geq\; \frac{\sigma_{\min}}{2\pi\,\Delta^2}, \qquad c_4(\mathbf{w}) \;\geq\; \frac{\sigma_{\min}}{4\pi\,\Delta^4}, \tag{17.58}$$

for any interval where $\sigma_{\mathrm{tot}} \geq \sigma_{\min} > 0$ is established (the second inequality follows from the same logic on the next moment).

Admissible region (definition). The *admissible parameter region* $\mathcal{A}$ is the closed convex set

$$\mathcal{A} \;:=\; \Big\{ \mathbf{w} \in \mathbb{R}^N \;\Big|\; (17.56),\ (17.57) \text{ (all channels), and optional } (17.58) \text{ hold} \Big\}. \tag{17.59}$$

Any $\mathbf{w} \notin \mathcal{A}$ is *excluded* by first principles (analyticity/causality/unitarity).

3. Closed-form fences in benchmark sectors

(i) Scalar quartic-derivative sector. For $\mathcal{L} \supset \frac{\alpha_1}{\Lambda^4}(\partial\phi)^4 + \cdots$,

$$\boxed{\alpha_1 \;\geq\; 0} \quad \text{(forward positivity)}, \qquad \boxed{\text{cone: } \Delta^{\mu\nu} k_\mu k_\nu \leq 0 \;\Rightarrow\; \alpha_1 \geq 0}. \tag{17.60}$$

Including $(\partial\phi)^2(\nabla_\mu\nabla_\nu\phi)^2$-type structures after field redefinitions yields additional non-trivial $a_{11}(\mathbf{w}) \geq 0$, $a_{02}(\mathbf{w}) \geq 0$ from (17.20).

(ii) Photons / Abelian gauge (parity even). For

$$\mathcal{L} \supset -\tfrac{1}{4}F^2 + \frac{\beta_1}{\Lambda^4}(F^2)^2 + \frac{\beta_2}{\Lambda^4} F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma F_\sigma{}^\mu + \frac{\tilde{\beta}_1}{\Lambda^4}(F\tilde{F})^2, \tag{17.61}$$

transverse positivity and non-birefringent cone ordering imply

$$\boxed{\beta_1 \;\geq\; 0, \qquad \beta_1 + \beta_2 \;\geq\; 0, \qquad \tilde{\beta}_1 \;\geq\; 0}. \tag{17.62}$$

(iii) Longitudinal electroweak (custodial limit). With the dim-8 basis of (17.31),

$$\boxed{f_{S,0} \;\geq\; 0, \qquad f_{S,0} + f_{S,1} \;\geq\; 0} \tag{17.63}$$

from forward $V_L V_L$ positivity; cone ordering (no superluminal V_L) does not weaken these and constrains additional mixed-derivative coefficients when present.

(iv) Matrix/multispecies bounds. For multi-field $(\partial\phi_i\partial\phi_j)^2$ couplings with matrix $[c_2]_{(ij)(kl)}$ (Eq. (17.22)),

$$[c_2](\mathbf{w}) \;\succeq\; 0 \quad\Longrightarrow\quad \text{all eigenvalues } \geq 0,\ \text{all principal minors } \geq 0, \tag{17.64}$$

and the same for higher moments/Hankel matrices. These imply, e.g., positivity of any *linear* combination probed by external-state superpositions.

4. RG stability and flow windows

Dispersive stability along the flow. At leading order, $c_{2n}(\mu)$ are proportional to IR-convergent dispersive moments of $\operatorname{Im}\mathcal{M}$ and therefore remain ≥ 0 under RG evolution (their μ-dependence reshuffles subtraction constants but not the sign of positive moments). Hence, if $\mathbf{w}(\mu_0) \in \mathcal{A}$ at some scale μ_0, then

$$\mathbf{w}(\mu) \in \mathcal{A} \quad \text{for all } \mu \text{ in the window where EFT and analytic assumptions hold,} \tag{17.65}$$

up to small violations from thresholds that are explicitly accounted for via matching (Sec. 3.2).

Gap-improved tightening. If the spectral gap $\Delta(\mu)$ increases along the flow (e.g. decoupling), the numerical lower bounds in (17.58) *tighten*, shrinking $\mathcal{A}$.

5. Practical pipeline (algorithm)

1. **Choose basis & subtraction scheme.** Fix an EFT operator basis and construct IR-safe, pole-subtracted amplitudes.
2. **Map to moments.** Compute $\mathsf{C}_{2m,a}$, $\mathsf{A}_{mn,a}$ in (17.55) (symbolically or with on-shell tools).
3. **Assemble LMIs.** Build Hankel matrices $H^{(k)}(\mathbf{w})$ and cone quadratic forms (coefficients b, c, d) for each channel.
4. **Add gap/phenomenology.** If available, include (17.58); optionally add experimental ellipsoids (EW precision, EDMs) as additional convex cuts.
5. **Solve feasibility/optimize.** Use SDP/SOCP to check $\mathbf{w} \in \mathcal{A}$ or to maximize/minimize linear functionals (e.g. a target coefficient) subject to $\mathbf{w} \in \mathcal{A}$.

6. Kill criteria and boundary diagnostics

Kill criteria. Any of the following *immediately* excludes a parameter point $\mathbf{w}$:

(K1) $c_{2n}(\mathbf{w}) < 0$ in any physical channel;

(K2) $H^{(k)}(\mathbf{w}) \;\not\succeq\; 0$ (negative principa

(K3) $\exists\, g$-null $k:\ \Delta^{\mu\nu}(\mathbf{w})\, k_\mu k_\nu > 0$ (superluminal front);

(K4) birefringent fast cone exits $C[g_{\text{eff}}$

(17.66)

Boundary structure. The boundary $\partial\mathcal{A}$ consists of unions of (i) linear faces from $c_{2n} = 0$, $a_{mn} = 0$, (ii) curved/algebraic pieces from $\det H^{(k)} = 0$, and (iii) quadratic cones from $b = 0$, $c = 0$, $bc - \frac{1}{4}d^2 = 0$. Approaching $\partial\mathcal{A}$ often correlates with *soft* emergent resonances saturating dispersive integrals (narrow-width dominance) or marginal cone tilts (front velocity $\to$ luminal).

7. Worked mini-examples

Scalar + photon (two-parameter slice). Take $\mathbf{w} = (\alpha_1, \beta_1)$ with all others zero. Then

$$\mathcal{A}_{\{\alpha_1,\beta_1\}} \;=\; \Big\{(\alpha_1,\beta_1)\;\Big|\;\alpha_1 \geq 0,\;\beta_1 \geq 0\Big\}, \tag{17.67}$$

and cone LMIs add no further restriction (they coincide with $\alpha_1, \beta_1 \geq 0$ for metric-type deformations).

Electroweak longitudinal (two-parameter slice). For $\mathbf{w} = (f_{S,0}, f_{S,1})$,

$$\mathcal{A}_{\{f_{S,0},f_{S,1}\}} \;=\; \Big\{(f_{S,0},f_{S,1})\;\Big|\;f_{S,0} \geq 0,\;f_{S,0} + f_{S,1} \geq 0\Big\}, \tag{17.68}$$

with possible *numerical* lower bounds from gap information if VBS inelastic thresholds are known.

8. Summary

We have cast the analyticity/positivity and cone-compatibility requirements into a *convex* feasibility problem in the space of Wilson coefficients. The admissible region $\mathcal{A}$ defined in (17.59) is bounded by (i) dispersive Hankel positivity, (ii) linear near-forward inequalities, and (iii) cone LMIs ensuring subluminal fronts relative to g_{eff}. Gap information tightens bounds numerically; RG evolution preserves admissibility within the EFT window. Violations of any moment, Hankel, or cone condition are *kill criteria* that falsify the parameter point (and, if structural, the underlying microphysical hypothesis).

Chapter 18

Precision QED/QCD and Atomic Spectra

18.1 Lamb shift, $g-2$, and running α

Aim. This section develops the precision-QED interface of the spectral framework. We (i) derive the running of the electromagnetic coupling $\alpha(q^2)$ from the Källén–Lehmann representation of the photon self–energy including *spectral* contributions from the internal fiber, (ii) map these corrections into the anomalous magnetic moments $a_\ell = (g_\ell - 2)/2$ of charged leptons through vacuum polarization (VP) and light–by–light (LbL) kernels, and (iii) propagate them to bound-state shifts of hydrogenic levels (Lamb shift). All results are expressed as dispersive integrals against positive spectral densities, which immediately yield convex bounds (Sec. 18.2) and consistency gates (S/G/TQ/D).

1. Running coupling from spectral photon self–energy

Renormalized photon two–point function. Let the renormalized vacuum polarization be

$$\Pi_R(q^2) \;:=\; \Pi(q^2) - \Pi(0)\,, \qquad \alpha(q^2) \;=\; \frac{\alpha(0)}{1 - \Pi_R(q^2)}\,. \tag{18.1}$$

Causality and unitarity imply a subtracted dispersion relation

$$\Pi_R(q^2) \;=\; \frac{q^2}{\pi}\,\mathrm{P}\!\int_{s_{\rm th}}^{\infty} ds\; \frac{\mathrm{Im}\,\Pi(s)}{s\,(s-q^2)}\,, \tag{18.2}$$

with $\mathrm{Im}\,\Pi(s) = \frac{\alpha(0)}{3}\,R(s)$ proportional to the observable ratio $R(s) = \sigma(e^+e^- \to \text{hadrons} + \text{leptons} + \cdots)/\sigma_{\rm pt}$.

Decomposition of absorptive strength. In the spectral model we write

$$\mathrm{Im}\,\Pi(s) \;=\; \mathrm{Im}\,\Pi_{\rm SM}(s) \;+\; \mathrm{Im}\,\Pi_{\rm spec}(s)\,, \qquad \mathrm{Im}\,\Pi_{\rm spec}(s) \;=\; \frac{\alpha(0)}{3}\,R_{\rm spec}(s) \;\geq 0, \tag{18.3}$$

where $R_{\rm spec}$ encodes *positive* spectral weight from fiber–induced charged composites or continuum thresholds.[1]

[1] Positivity is a direct corollary of the optical theorem and the Källén–Lehmann representation on $(M, g_{\rm eff})$ (Part VI).

Beta function and decoupling. At scales μ above a spectral threshold M_a, the contribution to the QED beta function reads

$$\mu\frac{d}{d\mu}\,\alpha(\mu) \;=\; \beta_{\rm QED}^{\rm SM}(\mu) \;+\; \frac{\alpha(\mu)^2}{3\pi}\sum_a Q_a^2\,\vartheta\Big(\mu; M_a\Big)\,, \tag{18.4}$$

where Q_a are effective charges of the spectral excitations and $\vartheta(\mu; M_a)$ is a smooth decoupling function ($\vartheta\to 0$ for $\mu\ll M_a$, $\vartheta\to 1$ for $\mu\gg M_a$), computable from the same $R_{\rm spec}(s)$ used in (18.3). Thus $\alpha(q^2)$ is tightened (screened) monotonically by *nonnegative* spectral weight.

2. Leptonic $g-2$ from spectral kernels

Master VP formula. To leading order in the insertion of Π_R, the contribution of vacuum polarization to the lepton anomaly a_ℓ is

$$a_\ell^{\rm VP} \;=\; \frac{\alpha(0)}{\pi}\int_0^1 dx\;(1-x)\,\Big[-\,\Pi_R\Big(q_x^2\Big)\Big], \qquad q_x^2 \;:=\; -\,\frac{x^2}{1-x}\,m_\ell^2\,. \tag{18.5}$$

Inserting (18.2) and interchanging integrals gives a positive kernel representation

$$a_\ell^{\rm VP} \;=\; \frac{\alpha(0)^2}{3\pi^2}\int_{s_{\rm th}}^\infty\frac{ds}{s}\;R(s)\;K_{\rm VP}\!\left(\frac{s}{m_\ell^2}\right), \qquad K_{\rm VP}(r) \;=\; \int_0^1 dx\;\frac{x^2(1-x)}{x^2+(1-x)r}\,, \tag{18.6}$$

with $0 < K_{\rm VP}(r) \le 1/3$ decreasing in r. The *spectral* increment is

$$\delta a_\ell^{\rm VP,\,spec} \;=\; \frac{\alpha(0)^2}{3\pi^2}\int_{s_{\rm th}}^\infty\frac{ds}{s}\;R_{\rm spec}(s)\;K_{\rm VP}\!\left(\frac{s}{m_\ell^2}\right) \;\ge\; 0\,. \tag{18.7}$$

Hence VP contributions from spectral weight are *nonnegative* for any lepton ℓ.

Schwinger, EW, and LbL pieces. The full anomaly is

$$a_\ell \;=\; \frac{\alpha(0)}{2\pi} \;+\; a_\ell^{\rm VP} \;+\; a_\ell^{\rm LbL} \;+\; a_\ell^{\rm EW} \;+\; \cdots\,, \tag{18.8}$$

where $a_\ell^{\rm LbL}$ receives contributions from four–photon contact terms at low energy. In the spectral EFT,

$$\mathcal{L}_{\gamma^4}^{(8)} \;=\; \frac{1}{\Lambda^4}\Big[\,\beta_1\,(F_{\mu\nu}F^{\mu\nu})^2 \;+\; \beta_2\,F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\sigma F_\sigma{}^\mu \;+\; \tilde\beta_1\,(F_{\mu\nu}\tilde F^{\mu\nu})^2\Big], \tag{18.9}$$

induces, at leading order,[2]

$$\delta a_\ell^{\rm LbL,\,spec} \;\sim\; \frac{\alpha(0)}{\pi}\,\frac{m_\ell^2}{\Lambda^4}\,\Big(c_1\,\beta_1 + c_2\,\beta_2 + \tilde c_1\,\tilde\beta_1\Big), \qquad c_i > 0, \tag{18.10}$$

which is *nonnegative* for $\beta_{1,2}, \tilde\beta_1 \ge 0$ (forward positivity, Sec. 17.2). Therefore the net spectral shift $\delta a_\ell^{\rm spec} = \delta a_\ell^{\rm VP,\,spec} + \delta a_\ell^{\rm LbL,\,spec}$ is typically positive.

[2]Model–independent scaling from the Euler–Heisenberg effective action; the numerical coefficients depend on the LbL kernel used (muon/electron).

3. Lamb shift as a dispersive potential

Bound-state Hamiltonian and Uehling potential. In hydrogenic systems (nucleus charge Z, reduced mass m_r) the leading vacuum–polarization correction to the Coulomb potential is the Uehling potential,

$$V_U(r) \;=\; -\,\frac{Z\alpha(0)}{r}\;\frac{2\alpha(0)}{3\pi}\int_1^\infty d\xi\;\frac{\sqrt{\xi^2-1}}{\xi^2}\left(1+\frac{1}{2\xi^2}\right)e^{-2m_e r\xi}\,. \tag{18.11}$$

More generally, using (18.2),

$$V_{\rm VP}(r) \;=\; -\,\frac{Z\alpha(0)}{r}\;\frac{1}{\pi}\int_{s_{\rm th}}^\infty\frac{ds}{s}\;\mathrm{Im}\,\Pi(s)\;e^{-r\sqrt{s}}\left(1+\frac{1}{2\sqrt{s}\,r}\right), \tag{18.12}$$

so that any *additional* spectral weight produces

$$\delta V_{\rm spec}(r) \;=\; -\,\frac{Z\alpha(0)}{r}\;\frac{\alpha(0)}{3\pi^2}\int_{s_{\rm th}}^\infty\frac{ds}{s}\;R_{\rm spec}(s)\;e^{-r\sqrt{s}}\left(1+\frac{1}{2\sqrt{s}\,r}\right)\;\le 0. \tag{18.13}$$

The sign is *attractive* (more screening).

Energy shifts and S–wave dominance. The first–order shift of an $n\ell j$ level is

$$\Delta E^{\rm VP}_{n\ell j} \;=\; \langle n\ell j|V_{\rm VP}|n\ell j\rangle \;=\; \int_0^\infty 4\pi r^2 dr\;|R_{n\ell}(r)|^2\,V_{\rm VP}(r), \tag{18.14}$$

with hydrogenic radial wavefunction $R_{n\ell}(r)$. For $\ell=0$, $|R_{n0}(0)|^2=\frac{(Z\alpha m_r)^3}{\pi n^3}$ is nonzero, yielding the characteristic S-wave enhancement of the Lamb shift. Using (18.13) we obtain a positive kernel integral

$$\delta E^{\rm spec}_{nS} \;=\; -\,\frac{4(Z\alpha m_r)^3}{3\pi n^3}\;\frac{\alpha(0)^2}{\pi}\;\int_{s_{\rm th}}^\infty\frac{ds}{s^{3/2}}\;R_{\rm spec}(s)\;F_{nS}\!\left(\frac{\sqrt{s}}{Z\alpha m_r}\right), \tag{18.15}$$

with F_{nS} a known, positive, dimensionless function decreasing with its argument.[3] Hence *additional spectral weight always increases the absolute Lamb shift splitting* $E_{2S}-E_{2P}$ in hydrogen and muonic hydrogen.

Bethe logarithm and vertex terms. Beyond VP, the standard self–energy (Bethe–log) and recoil/finite–size pieces enter,

$$\Delta E^{\rm SE}_{n\ell} \;=\; \frac{\alpha(0)}{\pi}\,(Z\alpha)^4\,m_r\,\frac{1}{n^3}\left[\ln\frac{1}{(Z\alpha)^2}+\beta_{n\ell}+\cdots\right], \tag{18.16}$$

with $\beta_{n\ell}$ the Bethe logarithm. Spectral corrections modify α and, if heavy charged states couple to the electron, generate higher–dimension form factors $\propto(q^2/\Lambda^2)$ entering $\Delta E^{\rm SE}_{n\ell}$; such effects can be re–expressed as positive dispersive moments analogous to (18.6) and are subleading for $\Lambda\gg m_r$.

[3] For the $1S$ level, $F_{1S}(y)=\int_0^\infty dx\,x^2e^{-2x}\,(1+\frac{1}{2yx})e^{-xy}$; similar closed forms exist for $n=2$.

4. QCD links: hadronic VP for a_μ and $\Delta\alpha$

Hadronic vacuum polarization (HVP). The hadronic piece of a_μ is purely dispersive

$$a_\mu^{\rm HVP} \;=\; \frac{\alpha(0)^2}{3\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}\,R_{\rm had}(s)\,K_{\rm VP}\!\left(\frac{s}{m_\mu^2}\right), \tag{18.17}$$

and the same $R_{\rm had}$ controls the running $\Delta\alpha_{\rm had}(M_Z^2)$ via (18.2). In the spectral model, hadronic modifications appear *only* through standard QCD dynamics (Part IV); any extra spectral sector charged under $U(1)$ would add a distinct, positive $R_{\rm spec}(s)$ piece.

5. Bounds from signs and kernels (no numerics needed)

Sign structure. Equations (18.7) and (18.15) show that *any* additional positive spectral weight increases a_ℓ and increases the magnitude of the Lamb shift (more screening). Therefore:

$$\delta a_\ell^{\rm spec} \;\geq 0, \qquad \delta(E_{2S}-E_{2P})^{\rm spec} \;\geq 0. \tag{18.18}$$

Model–independent envelope bounds. Let $R_{\rm spec}(s)\leq \mathcal{N}\,\theta(s-M^2)$ (unit–normalized step at M). Then, from (18.7),

$$\delta a_\ell^{\rm VP,\,spec} \;\leq\; \frac{\alpha(0)^2}{9\pi^2}\,\mathcal{N}\,\log\!\left(1+\frac{m_\ell^2}{M^2}\right), \tag{18.19}$$

which falls as M^{-2} for $M\gg m_\ell$. From (18.15), using $F_{nS}(y)\leq F_{nS}(0)=1$,

$$0 \;\leq\; \delta E_{nS}^{\rm spec} \;\leq\; \frac{4(Z\alpha m_r)^3}{3\pi n^3}\,\frac{\alpha(0)^2}{\pi}\,\mathcal{N}\int_{M^2}^{\infty}\frac{ds}{s^{3/2}} \;=\; \frac{8(Z\alpha m_r)^3}{3\pi n^3}\,\frac{\alpha(0)^2}{\pi}\,\frac{\mathcal{N}}{M}\,. \tag{18.20}$$

Equations (18.19)–(18.20) are convenient for turning experimental residuals into scale bounds on M *without* committing to detailed spectral shapes.

6. Consistency gates (S/G/TQ/D) and scheme independence

(S) Stability. Positive spectral densities guarantee that the dressed photon propagator preserves the correct sign of the Källén–Lehmann measure; no ghost poles appear.

(G) Causality/analyticity. The once–subtracted dispersion (18.2) follows from micro-causality; running α and the derived VP/LbL kernels inherit the analyticity domain used in Part VI.

(TQ) Positivity/unitarity. $\mathrm{Im}\,\Pi\geq 0$ implies $\delta a_\ell^{\rm VP,\,spec}\geq 0$ and $\delta V_{\rm spec}(r)\leq 0$; the Euler–Heisenberg coefficients satisfy $\beta_{1,2},\tilde{\beta}_1\geq 0$ (Sec. 17.2), which ensures $\delta a_\ell^{\rm LbL,\,spec}\geq 0$.

(D) Dissipation. Small anisotropies from χ^μ (treated systematically in Sec. 18.3) do not alter the sign of dispersive kernels and therefore preserve bounds; any directional variation enters at subleading order.

Scheme independence. Physical shifts δa_ℓ and δE are renormalization–scheme independent: scheme changes reshuffle subtractions in (18.2) but leave the dispersive integrals invariant.

7. Summary

The precision observables a_ℓ and Lamb shifts can be written as *positive* dispersive functionals of the same absorptive input that governs the running of $\alpha(q^2)$. In the spectral model, additional internal–fiber spectral weight $R_{\rm spec}(s) \geq 0$ monotonically increases a_ℓ and the magnitude of Uehling–driven level shifts, yielding immediate, convex constraints on spectral thresholds and Euler–Heisenberg coefficients. These results provide a clean, number–ready bridge from spectral microphysics to high–precision QED/QCD tests, which we quantify and globalize in Secs. 18.2–18.4.

18.2 Spectral corrections and bounds

Aim. We convert the dispersive representations of Sec. 18.1 into *model–independent bounds* on spectral contributions to precision QED observables. Our inputs are: (i) the positive spectral weight $R_{\rm spec}(s) \geq 0$ entering the photon vacuum polarization, (ii) the positive Euler–Heisenberg (EH) quartic–photon couplings $\beta_{1,2}, \tilde{\beta}_1 \geq 0$ controlling light–by–light (LbL) kernels, and (iii) experimental residuals w.r.t. a reference SM prediction. We derive (A) *integral moment* bounds, (B) *scale* bounds for heavy thresholds or narrow structures, and (C) *correlated* constraints from the combined set $\{a_e, a_\mu, \Delta\alpha(q^2), \text{Lamb shifts}\}$.

1. Residuals and positive kernels

Residual definitions. Let the measured–minus–reference residuals be

$$\Delta a_\ell \;:=\; a_\ell^{\rm exp} - a_\ell^{\rm ref}, \qquad \Delta_{\rm Lamb}(nS) \;:=\; \Big[E_{nS} - E_{nP}\Big]^{\rm exp} - \Big[E_{nS} - E_{nP}\Big]^{\rm ref}, \tag{18.21}$$

and for the running coupling at a space–like or time–like scale Q^2,

$$\Delta\alpha(Q^2) \;:=\; \alpha^{-1}(Q^2)_{\rm ref} - \alpha^{-1}(Q^2)_{\rm exp}. \tag{18.22}$$

From Sec. 18.1,

$$\Delta a_\ell \;=\; \delta a_\ell^{\rm VP,\,spec} \;+\; \delta a_\ell^{\rm LbL,\,spec} \;+\; \cdots \;\geq\; 0 \quad \text{(each term nonnegative)}, \tag{18.23}$$

$$\Delta_{\rm Lamb}(nS) \;=\; \delta E_{nS}^{\rm spec} - \delta E_{nP}^{\rm spec} \;\simeq\; \delta E_{nS}^{\rm spec} \;\geq\; 0, \tag{18.24}$$

$$\Delta\alpha(Q^2) \;=\; \frac{\alpha(0)}{\pi}\,{\rm P}\!\int_{s_{\rm th}}^{\infty} ds\,\frac{R_{\rm spec}(s)}{s - Q^2} \;\geq\; 0 \quad (Q^2 > 0), \tag{18.25}$$

where the inequalities follow from $R_{\rm spec} \geq 0$ and the positivity of the VP/LbL kernels.

Positive kernels. Using Eqs. (18.6), (18.10), (18.15), and (18.2), define three spectral moments

$$\mu_{-1}[w] \;:=\; \int_{s_{\rm th}}^{\infty} \frac{ds}{s}\,R_{\rm spec}(s)\,w(s), \quad \mu_{-2} \;:=\; \int \frac{ds}{s^2} R_{\rm spec}(s), \quad \mu_{-\frac{3}{2}} \;:=\; \int \frac{ds}{s^{3/2}} R_{\rm spec}(s), \tag{18.26}$$

with $w(s) \geq 0$ a known monotone weight (e.g. $K_{\rm VP}(s/m_\ell^2)$ or $|Q^2|/(s+|Q^2|)$ for space–like $Q^2 < 0$). All three are nonnegative and finite under the mild falloff of $R_{\rm spec}$.

2. Integral (moment) bounds

Vacuum polarization to a_ℓ. From (18.7) we have

$$\Delta a_\ell^{\rm VP} \equiv \delta a_\ell^{\rm VP,\,spec} = \frac{\alpha(0)^2}{3\pi^2}\,\mu_{-1}\Big[K_{\rm VP}\Big(\frac{s}{m_\ell^2}\Big)\Big], \qquad 0 < K_{\rm VP}(r) \le \frac{1}{3}, \quad \frac{dK_{\rm VP}}{dr} < 0. \tag{18.27}$$

Hence

$$\boxed{0 \le \Delta a_\ell^{\rm VP} \le \frac{\alpha(0)^2}{9\pi^2}\,\mu_{-1}\,, \quad \mu_{-1} := \int \frac{ds}{s}\,R_{\rm spec}(s).} \tag{18.28}$$

Lamb shifts. Equation (18.15) yields

$$\Delta_{\rm Lamb}(nS) \simeq \delta E_{nS}^{\rm spec} = \frac{4(Z\alpha m_r)^3}{3\pi n^3}\,\frac{\alpha(0)^2}{\pi}\int_{s_{\rm th}}^\infty \frac{ds}{s^{3/2}}\,R_{\rm spec}(s)\,F_{nS}\Big(\frac{\sqrt{s}}{Z\alpha m_r}\Big), \tag{18.29}$$

with $0 < F_{nS}(y) \le 1$ decreasing in y. Therefore

$$\boxed{0 \le \Delta_{\rm Lamb}(nS) \le \frac{4(Z\alpha m_r)^3}{3\pi n^3}\,\frac{\alpha(0)^2}{\pi}\,\mu_{-\frac{3}{2}}\,.} \tag{18.30}$$

Running α. At space–like $Q^2 > 0$, from (18.2) and (18.25),

$$\Delta\alpha(-Q^2) = \frac{\alpha(0)}{\pi}\int_{s_{\rm th}}^\infty ds\,\frac{R_{\rm spec}(s)}{s+Q^2} = \frac{\alpha(0)}{\pi}\,\mu_{-1}\Big[\frac{Q^2}{s+Q^2}\Big], \tag{18.31}$$

bounded by

$$0 \le \Delta\alpha(-Q^2) \le \frac{\alpha(0)}{\pi}\,\mu_{-1}. \tag{18.32}$$

Combining (18.28) and (18.32) eliminates μ_{-1}:

$$\boxed{\Delta a_\ell^{\rm VP} \le \frac{\alpha(0)}{9\pi}\,\Delta\alpha(-Q^2) \qquad (\forall\, Q^2 > 0).} \tag{18.33}$$

Any violation for a given ℓ and Q^2 falsifies a putative $R_{\rm spec}$.

3. Heavy–threshold (asymptotic) regime and mass scaling

Large–s kernels. For $s \gg m_\ell^2$, $K_{\rm VP}(s/m_\ell^2) = \frac{1}{3}\frac{m_\ell^2}{s} + \mathcal{O}(m_\ell^4/s^2)$. Thus if all spectral weight lies above a scale $M^2 \gg m_\ell^2$,

$$\Delta a_\ell^{\rm VP} = \frac{\alpha(0)^2}{9\pi^2}\,m_\ell^2\int_{M^2}^\infty ds\,\frac{R_{\rm spec}(s)}{s^2} + \mathcal{O}\Big(\frac{m_\ell^4}{M^4}\Big), \tag{18.34}$$

and, parametrically,

$$\frac{\Delta a_e^{\rm VP}}{\Delta a_\mu^{\rm VP}} \simeq \frac{m_e^2}{m_\mu^2} \qquad (M^2 \gg m_\mu^2). \tag{18.35}$$

Hence a spectral explanation of a muon–specific anomaly with tiny electron effect prefers M *well above* m_μ; conversely, comparable Δa_e and Δa_μ would point to low–lying spectral weight $M \lesssim m_\mu$.

Lamb shift scaling. For $M \gg Z\alpha m_r$ (hydrogenic scales), (18.29) gives

$$\Delta_{\rm Lamb}(nS) \;\approx\; \frac{4(Z\alpha m_r)^3}{3\pi n^3}\,\frac{\alpha(0)^2}{\pi}\int_{M^2}^{\infty}\frac{ds}{s^{3/2}}\,R_{\rm spec}(s) \;\sim\; \frac{\alpha(0)^2}{\pi}\,\frac{(Z\alpha m_r)^3}{n^3}\,\frac{\mathcal{N}_{\rm eff}}{M}, \tag{18.36}$$

with $\mathcal{N}_{\rm eff}$ a dimensionless measure of the total spectral weight. Thus atomic spectroscopy is particularly sensitive to *low* masses $\sim M^{-1}$.

4. Narrow structures and step–like spectra

Delta–like (narrow) resonance. Let $R_{\rm spec}(s) = \mathcal{N}\,\delta(s - M^2)$ (effective normalization $\mathcal{N} > 0$). Then

$$\Delta a_\ell^{\rm VP} = \frac{\alpha(0)^2}{3\pi^2}\,\frac{\mathcal{N}}{M^2}\,K_{\rm VP}\!\left(\frac{M^2}{m_\ell^2}\right), \tag{18.37}$$

$$\Delta_{\rm Lamb}(nS) = \frac{4(Z\alpha m_r)^3}{3\pi n^3}\,\frac{\alpha(0)^2}{\pi}\,\frac{\mathcal{N}}{M^3}\,F_{nS}\!\left(\frac{M}{Z\alpha m_r}\right), \tag{18.38}$$

$$\Delta\alpha(-Q^2) = \frac{\alpha(0)}{\pi}\,\frac{\mathcal{N}}{M^2+Q^2}. \tag{18.39}$$

Given any one residual and a choice of (M, ℓ, n, Q), the others are *predicted*, providing sharp cross–checks.

Step–like continuum. For $R_{\rm spec}(s) = \mathcal{N}\,\theta(s - M^2)$,

$$\Delta a_\ell^{\rm VP} \le \frac{\alpha(0)^2}{9\pi^2}\,\mathcal{N}\,\log\!\left(1+\frac{m_\ell^2}{M^2}\right), \tag{18.40}$$

$$\Delta_{\rm Lamb}(nS) \le \frac{8(Z\alpha m_r)^3}{3\pi n^3}\,\frac{\alpha(0)^2}{\pi}\,\frac{\mathcal{N}}{M}, \tag{18.41}$$

$$\Delta\alpha(-Q^2) \le \frac{\alpha(0)}{\pi}\,\mathcal{N}\,\log\!\left(1+\frac{Q^2}{M^2}\right). \tag{18.42}$$

These *envelope* inequalities allow immediate scale inferences.

5. Light–by–light (EH) sector and positivity

EH coefficients from a_ℓ. From (18.10),

$$\Delta a_\ell^{\rm LbL} \;\equiv\; \delta a_\ell^{\rm LbL,\,spec} \;\simeq\; \frac{\alpha(0)}{\pi}\,\frac{m_\ell^2}{\Lambda^4}\,\Big(c_1\,\beta_1 + c_2\,\beta_2 + \tilde{c}_1\,\tilde{\beta}_1\Big), \qquad c_i > 0, \tag{18.43}$$

with the *positivity* bounds (Part 17)

$$\beta_1 \;\ge\; 0, \qquad \beta_1 + \beta_2 \;\ge\; 0, \qquad \tilde{\beta}_1 \;\ge\; 0. \tag{18.44}$$

Hence, for any admissible $(\beta_1, \beta_2, \tilde{\beta}_1)$,

$$0 \;\le\; \Delta a_\ell^{\rm LbL} \;\le\; \frac{\alpha(0)}{\pi}\,\frac{m_\ell^2}{\Lambda^4}\,\Big(c_1\,\beta_1 + c_2\,(\beta_1+\beta_2) + \tilde{c}_1\,\tilde{\beta}_1\Big), \tag{18.45}$$

so a measured *upper* limit on Δa_e or Δa_μ yields a convex cap on the $(\beta_1, \beta_2, \tilde{\beta}_1)$ polytope.

6. Correlated constraints from $\{a_e, a_\mu, \Delta\alpha, \mathbf{Lamb}\}$

Two–lepton ratio test. Using (18.27) for both $\ell = e, \mu$,

$$\frac{\Delta a_e^{\mathrm{VP}}}{\Delta a_\mu^{\mathrm{VP}}} = \frac{\int \frac{ds}{s} R_{\mathrm{spec}}(s)\, K_{\mathrm{VP}}\left(\frac{s}{m_e^2}\right)}{\int \frac{ds}{s} R_{\mathrm{spec}}(s)\, K_{\mathrm{VP}}\left(\frac{s}{m_\mu^2}\right)}. \tag{18.46}$$

If R_{spec} is supported on $[M^2, \infty)$, then K_{VP} is monotone decreasing and one obtains

$$\frac{K_{\mathrm{VP}}\big(M^2/m_e^2\big)}{K_{\mathrm{VP}}\big(M^2/m_\mu^2\big)} \le \frac{\Delta a_e^{\mathrm{VP}}}{\Delta a_\mu^{\mathrm{VP}}} \le \frac{K_{\mathrm{VP}}(+\infty)}{K_{\mathrm{VP}}(+\infty)} = 1, \tag{18.47}$$

with the left inequality saturating for ultra–narrow spectra. In the heavy–threshold regime ($M \gg m_\mu$) this tightens to (18.35).

Running vs. $g-2$. Eliminate μ_{-1} between (18.28) and (18.32) to obtain the *universal* inequality (18.33). A combined analysis with $\Delta_{\mathrm{Lamb}}(nS)$ gives, for any $Q^2 > 0$ and n,

$$\Delta_{\mathrm{Lamb}}(nS) \le \frac{4(Z\alpha m_r)^3}{3\pi n^3} \frac{\alpha(0)}{\pi} \int_{s_{\mathrm{th}}}^{\infty} \frac{ds}{s^{1/2}(s+Q^2)} R_{\mathrm{spec}}(s), \tag{18.48}$$

which, together with the measured $\Delta\alpha(-Q^2)$, restricts the low–s part of R_{spec} by Hölder inequalities.

7. Convex feasibility in spectral moment space

Moment variables. Introduce a finite *moment vector*

$$\mathbf{m} := \Big(\mu_{-1},\ \mu_{-2},\ \mu_{-\frac{3}{2}},\ \mu_{-1}[w_e],\ \mu_{-1}[w_\mu],\ \dots\Big)^{\top}, \tag{18.49}$$

where $w_\ell(s) = K_{\mathrm{VP}}(s/m_\ell^2) \in (0, 1/3]$ and additional entries can include step–filtered or windowed moments to reflect known threshold structure. All entries are linear in R_{spec} and satisfy $\mathbf{m} \ge 0$ component–wise.

Feasibility problem. Given experimental caps $X_i^{\max}$ on a set of residuals $X_i \in \{\Delta a_\ell, \Delta\alpha, \Delta_{\mathrm{Lamb}}, \dots\}$, the *admissible spectral-moment polytope* is

$$\mathcal{M} := \Big\{ \mathbf{m} \ge 0 \,\Big|\, \mathsf{L}\,\mathbf{m} \le \mathbf{X}^{\max},\ \text{and Hankel/moment positivity of Sec. 17.2} \Big\}, \tag{18.50}$$

with a known nonnegative matrix L encoding the kernels of Eqs. (18.27), (18.29), (18.31). Any EFT parameter point $(\beta_i, \dots)$ mapping to a $\mathbf{m} \notin \mathcal{M}$ is *excluded*.

8. EH sector: bounds on $(\beta_1, \beta_2, \tilde{\beta}_1)$

From Δa_e and Δa_μ. Combining (18.43) for $\ell = e, \mu$ gives two linear inequalities

$$\begin{pmatrix} c_1 & c_2 & \tilde{c}_1 \\ c_1 \frac{m_\mu^2}{m_e^2} & c_2 \frac{m_\mu^2}{m_e^2} & \tilde{c}_1 \frac{m_\mu^2}{m_e^2} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \tilde{\beta}_1 \end{pmatrix} \le \frac{\pi}{\alpha(0)} \frac{\Lambda^4}{m_e^2} \begin{pmatrix} \Delta a_e^{\mathrm{LbL}} \\ \Delta a_\mu^{\mathrm{LbL}} \end{pmatrix}, \tag{18.51}$$

to be intersected with the positivity wedge (18.44). This produces a compact convex region for $(\beta_1, \beta_2, \tilde{\beta}_1)$ at a chosen Λ.

9. Kill criteria and consistency gates

Kill criteria. Any of the following invalidates a spectral explanation:

$$\begin{aligned}
&\text{(K1)}\quad \Delta a_\ell < 0 \text{ or } \Delta_{\mathrm{Lamb}}(nS) < 0 \text{ (contradicts positive kernels);}\\
&\text{(K2)}\quad \Delta a_\ell^{\mathrm{VP}} > \frac{\alpha(0)}{9\pi}\,\Delta\alpha(-Q^2) \text{ for some } Q^2 > 0 \text{ (violates (18.33));}\\
&\text{(K3)}\quad \text{Moment vector } \mathbf{m} \notin \mathcal{M} \text{ (fails convex constraints or Hankel positivity).}
\end{aligned} \tag{18.52}$$

Consistency gates (S/G/TQ/D).

$$\textbf{(S)}: \; R_{\mathrm{spec}} \geq 0 \;\Rightarrow\; \text{no ghost poles; kernels bounded.} \tag{18.53}$$

$$\textbf{(G)}: \; \text{Dispersive integrals obey microcausal analyticity; subtraction scheme cancels IR poles.} \tag{18.54}$$

$$\textbf{(TQ)}: \; \text{Forward positivity implies } \beta_{1,2}, \tilde{\beta}_1 \geq 0 \text{ and } \Delta a_\ell, \Delta_{\mathrm{Lamb}}, \Delta\alpha \geq 0. \tag{18.55}$$

$$\textbf{(D)}: \; \chi^\mu\text{-induced anisotropies enter } \mathcal{O}(\delta\kappa) \text{ without flipping kernel signs.} \tag{18.56}$$

10. Summary

Positive–kernel dispersive formulas convert precision data into convex, largely shape–independent bounds on the spectral input. Moment inequalities constrain the *integrated* size of R_{spec}; heavy–threshold scaling relates a_e and a_μ via m_ℓ^2; Lamb shifts are most sensitive to low masses through a $1/M$ tail; and $\Delta\alpha$ ties VP effects to running–coupling measurements. The EH sector is simultaneously bounded by a_e and a_μ within the positivity wedge. Together, these results provide a tight, falsifiable interface between internal–fiber microphysics and precision QED/QCD phenomenology.

18.3 Atomic clock constraints and δc

Aim. High–precision clocks and cavity oscillators probe tiny departures of photon and matter dispersion from the emergent background (M, g_{eff}). In the spectral framework, anisotropic gauge response—induced by the director χ^μ and encoded in the constitutive tensor $\kappa^{\mu\nu\rho\sigma}$—can tilt the electromagnetic light cone relative to g_{eff} by an amount we parameterize as $\delta c(\hat{\boldsymbol{n}}, t)$, where $\hat{\boldsymbol{n}}$ is the propagation (or cavity–axis) direction in the lab. This section derives: (i) a covariant definition of δc from the principal symbol (no birefringence case and its extension), (ii) the mapping from δc and slow drifts of α to clock–comparison signals using sensitivity coefficients, (iii) sidereal/annual modulation templates from the rotation between the Sun–centered inertial frame and the laboratory, and (iv) convex, experiment–ready bounds on the underlying spectral parameters.

1. Photon dispersion and the definition of δc

Principal cone for the transverse photon. The quadratic gauge action (cf. Eq. (17.41)) reads

$$\mathcal{L}_A^{(2)} \;=\; -\frac{1}{4}\,\kappa^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \qquad \kappa^{\mu\nu\rho\sigma} \;=\; g_{\mathrm{eff}}^{\mu\rho} g_{\mathrm{eff}}^{\nu\sigma} - g_{\mathrm{eff}}^{\mu\sigma} g_{\mathrm{eff}}^{\nu\rho} \;+\; \delta\kappa^{\mu\nu\rho\sigma}. \tag{18.57}$$

For *metric-type* deformations (no birefringence at leading order) the Fresnel polynomial factorizes as (cf. Eq. (17.43))

$$\mathcal{G}(x,k) \;\propto\; \left(G_{\rm T}^{\mu\nu}k_\mu k_\nu\right)^2, \qquad G_{\rm T}^{\mu\nu} \;=\; g_{\rm eff}^{\mu\nu} + \Delta_{\rm T}^{\mu\nu}, \tag{18.58}$$

so the phase/front cone is governed by $G_{\rm T}$. In the lab rest frame with $g_{\rm eff}^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and wave covector $k^\mu = (\omega, \boldsymbol{k})$, the null condition $G_{\rm T}^{\mu\nu}k_\mu k_\nu = 0$ gives, to first order in $\Delta_{\rm T}$,

$$\omega^2 \;=\; |\boldsymbol{k}|^2\left[1 + \delta n^{ij}\,\hat{k}_i\hat{k}_j\right], \qquad \delta n^{ij} \;:=\; \Delta_{\rm T}^{ij} + \Delta_{\rm T}^{00}\,\hat{k}^i\hat{k}^j - 2\,\Delta_{\rm T}^{0(i}\hat{k}^{j)}. \tag{18.59}$$

The (directional) phase/group velocity shift is then

$$\boxed{\frac{\delta c}{c}(\hat{\boldsymbol{n}}) \;\equiv\; \frac{\omega}{|\boldsymbol{k}|} - 1 \;=\; \frac{1}{2}\,\delta n^{ij}\,\hat{n}_i\hat{n}_j \;+\; \mathcal{O}(\Delta_{\rm T}^2), \qquad \hat{\boldsymbol{n}} := \frac{\boldsymbol{k}}{|\boldsymbol{k}|}.} \tag{18.60}$$

When birefringence is present, $\mathcal{G} \propto (G_+k^2)(G_-k^2)$ and one obtains two shifts $\delta c_\pm(\hat{\boldsymbol{n}})$; the analysis below applies to each polarization separately.

Spectral parameterization. From Eq. (17.44), small χ^μ–induced anisotropies yield

$$\Delta_{\rm T}^{\mu\nu} \;=\; a_1\,u^\mu u^\nu + a_2\,\chi^\mu\chi^\nu + a_3\,u^{(\mu}\chi^{\nu)} + a_0\,g_{\rm eff}^{\mu\nu}, \tag{18.61}$$

with u^μ the aether rest frame, and a trace convention chosen so that a_0 drops out of (18.60). Thus

$$\frac{\delta c}{c}(\hat{\boldsymbol{n}}) \;=\; \frac{1}{2}\Big[\,b + c\,(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\chi}})^2 + d\,(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\chi}})\Big], \qquad (b,c,d) \text{ linear in } (a_1,a_2,a_3), \tag{18.62}$$

the same (b,c,d) that entered the cone–ordering inequalities (Sec. 17.3). Cone compatibility requires the quadratic form in $(1,\cos\theta)$ to be negative semidefinite on the $g_{\rm eff}$ null cone, bounding (b,c,d).

2. Clock and cavity observables: sensitivity map

Atomic transitions. The frequency of a transition A depends on a small set of "constants" $X \in \{\alpha,\ m_e/\Lambda_{\rm QCD},\ m_q/\Lambda_{\rm QCD}, \ldots\}$ and, if photon propagation enters the apparatus response, on δc along relevant axes. Linearizing,

$$\frac{\delta\nu_A}{\nu_A} \;=\; K_\alpha^{(A)}\,\frac{\delta\alpha}{\alpha} + K_e^{(A)}\,\frac{\delta(m_e/\Lambda_{\rm QCD})}{(m_e/\Lambda_{\rm QCD})} + K_q^{(A)}\,\frac{\delta(m_q/\Lambda_{\rm QCD})}{(m_q/\Lambda_{\rm QCD})} + K_c^{(A)}\,\delta c_{\rm eff}^{(A)} + \cdots. \tag{18.63}$$

The K–coefficients are transition–specific (atomic structure) and can be taken as constants for a given species. The effective $\delta c_{\rm eff}^{(A)}$ captures how the experimental signal inherits the photon velocity shift: for a cavity–stabilized laser,

$$\nu_{\rm cav} \;=\; \frac{m\,c}{2n\,L} \quad\Longrightarrow\quad \frac{\delta\nu_{\rm cav}}{\nu_{\rm cav}} \;=\; \frac{\delta c}{c} - \frac{\delta n}{n} - \frac{\delta L}{L}, \tag{18.64}$$

so that, in a differential comparison where material drifts largely cancel, $\delta c/c$ imprints directly. For purely atomic references (no cavity), $K_c^{(A)} \approx 0$ and the signal arises from $\delta\alpha/\alpha$ etc.

Differential readout. Comparing two references A, B eliminates common–mode drifts:

$$y_{AB}(t) \;:=\; \frac{\nu_A-\nu_B}{\nu_{\rm ref}} \;\simeq\; \sum_X \Big(K_X^{(A)}-K_X^{(B)}\Big)\,x_X(t) \;+\; \Big(K_c^{(A)}\,\delta c_{\rm eff}^{(A)}(t)-K_c^{(B)}\,\delta c_{\rm eff}^{(B)}(t)\Big), \tag{18.65}$$

with $x_X(t)$ the fractional variation of X. In cavity–cavity comparisons with orthogonal axes $\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2$, Eq. (18.60) predicts a clean *quadrupolar* angular dependence.

3. Sidereal/annual templates (frame rotation)

Frames and rotation. Let (T, X, Y, Z) denote a Sun–centered inertial frame with Z along the Earth's spin axis. The lab frame (t, x, y, z) with colatitude ζ is related by a time–dependent rotation $R_i{}^I(t)$ at the sidereal frequency $\omega_\oplus$ and, for orbital effects, by an annual boost/rotation at $\Omega_\oplus$. A symmetric, traceless spatial tensor C_{IJ} (the spatial part of $\Delta_{\rm T}$) gives in the lab

$$\delta c/c \;=\; \tfrac{1}{2}\,\hat n_i \hat n_j\, C_{ij}(t), \qquad C_{ij}(t) \;=\; R_i{}^I(t)\, R_j{}^J(t)\, C_{IJ}. \tag{18.66}$$

Expanding $C_{ij}(t)$ into harmonics yields the standard modulation

$$\frac{\delta c}{c}(\hat{\boldsymbol{n}}, t) \;=\; A_0(\hat{\boldsymbol{n}}) + \sum_{m=1}^{2}\Big[A_m(\hat{\boldsymbol{n}})\cos(m\omega_\oplus t) + B_m(\hat{\boldsymbol{n}})\sin(m\omega_\oplus t)\Big] \;+\; \mathcal{O}(\Omega_\oplus), \tag{18.67}$$

with coefficients linear in C_{IJ} and depending on the lab's latitude and the cavity/beam orientation. For two orthogonal cavities ($\hat{\boldsymbol{n}}_{1,2}$ in the horizontal plane) one finds a dominant $m = 2$ line with amplitude

$$\mathcal{A}_2 \;=\; \frac{1}{4}\Big|\Lambda^{IJ}(\zeta, \hat{\boldsymbol{n}}_{1,2})\, C_{IJ}\Big|, \qquad \Lambda^{IJ} \text{ a known geometric tensor}, \tag{18.68}$$

providing a direct linear constraint on C_{IJ} from the measured sideband.

4. Bounds: from sideband limits to spectral coefficients

Linear constraints on C_{IJ}. A non-observation at level $|\mathcal{A}_m| \le A_m^{\rm max}$ implies

$$\Big|\Lambda_m^{IJ}\, C_{IJ}\Big| \;\le\; 4\, A_m^{\rm max} \qquad (m = 1, 2), \tag{18.69}$$

for each measured harmonic (and geometry). Gathering several orientations and latitudes produces an overconstrained linear system for the five independent components of the traceless C_{IJ}, solvable by least squares or bounded by linear programming to obtain a convex *elliptic* allowed region in C–space.

Map to (b, c, d) and to spectral (a_1, a_2, a_3). Using Eq. (18.62), the measured bounds translate to

$$|b| \;\le\; b_{\rm max}, \qquad |c| \;\le\; c_{\rm max}, \qquad |d| \;\le\; d_{\rm max}, \tag{18.70}$$

with $(b_{\rm max}, c_{\rm max}, d_{\rm max})$ determined by the most sensitive sidebands and orientations. Since (b, c, d) are linear in (a_1, a_2, a_3), this yields a convex polytope for the underlying spectral couplings of Eq. (18.61). Intersecting with the *cone-compatibility* LMIs (Sec. 17.3) further shrinks the allowed region.

Cavity–vs–atomic and mixed comparisons. Let A be a cavity and B an optical clock ($K_c^{(B)} \approx 0$). From (18.65) and (18.67), the sideband amplitude at $m = 2$ is

$$|\mathcal{A}_2^{(AB)}| \;=\; |K_c^{(A)}|\,\mathcal{A}_2 \;\le\; A_2^{\max} \quad\Rightarrow\quad \left|\Lambda^{IJ} C_{IJ}\right| \;\le\; 4\,A_2^{\max}/|K_c^{(A)}|. \tag{18.71}$$

This bound is robust against slow drifts of α or m_e, which enter only the DC term via $K_{\alpha,e}^{(A,B)}$.

5. Gravitational potential coupling and annual templates

Parametrization. In addition to anisotropy, clocks constrain possible scalar responses of the medium to the gravitational potential Φ (environmental dependence),

$$\frac{\delta\alpha}{\alpha} \;=\; k_\alpha\,\Delta\Phi, \qquad \frac{\delta c}{c} \;=\; k_c\,\Delta\Phi, \qquad \Delta\Phi(t) \;\simeq\; \Phi_\odot\,\frac{e\,\cos(\Omega_\oplus t - \varphi_0)}{1-e^2} + \cdots, \tag{18.72}$$

where e is Earth's orbital eccentricity and $\Phi_\odot$ the solar potential at 1 AU. Comparing two clocks with contrasting K_α produces a clean annual line at $\Omega_\oplus$ with amplitude $\propto (K_\alpha^{(A)} - K_\alpha^{(B)})k_\alpha$, while cavity–atomic comparisons with fixed axis isolate k_c.

Bound propagation. Annual amplitude limits $A_\Omega^{\max}$ imply

$$|k_\alpha| \;\le\; \frac{A_\Omega^{\max}}{|K_\alpha^{(A)} - K_\alpha^{(B)}|\,|\Delta\Phi|_{\rm amp}}, \qquad |k_c| \;\le\; \frac{A_\Omega^{\max}}{|K_c^{(A)}|\,|\Delta\Phi|_{\rm amp}}, \tag{18.73}$$

providing *isotropic* (DC) constraints complementary to the anisotropic C_{IJ} bounds.

6. Consistency with dispersion/positivity and cones

Signs and convexity. The dispersion/positivity gates (Part 17) require EH coefficients $\beta_{1,2}, \tilde\beta_1 \ge 0$ and imply that loop–induced index shifts from integrating out *isotropic* spectral weight are nonnegative in the sense of (17.33). The *anisotropic* pieces, governed by (a_1, a_2, a_3), must satisfy the cone LMIs $b \le 0$, $c \le 0$, $bc - \frac{1}{4}d^2 \ge 0$ (Sec. 17.3). Hence any experimental inference that would require $b > 0$ or $c > 0$, or $bc - \frac{1}{4}d^2 < 0$, is a *kill criterion* for the corresponding spectral hypothesis.

Birefringence. If two polarization cones exist, the sideband analysis applies to each $\delta c_\pm$; differential polarimetry or dual–mode cavities can bound $|\delta c_+ - \delta c_-|$, which directly limits the *skewon/axion*–type parts of $\delta\kappa$ and thus specific linear combinations of (a_i).

7. Worked templates

Orthogonal cavity pair (Michelson–Morley type). Take two cavities with axes $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$ orthogonal in the horizontal plane; measure $y(t) = (\nu_1 - \nu_2)/\nu_{\rm ref}$. Using (18.60)–(18.67),

$$y(t) \;=\; \frac{1}{2}\Big(\delta n^{ij}\hat n_{1i}\hat n_{1j} - \delta n^{ij}\hat n_{2i}\hat n_{2j}\Big) \;=\; \sum_{m=0}^{2}\Big[\mathcal{C}_m\cos(m\omega_\oplus t) + \mathcal{S}_m\sin(m\omega_\oplus t)\Big], \tag{18.74}$$

with $(\mathcal{C}_m, \mathcal{S}_m)$ linear in C_{IJ}. Upper limits on the $m = 2$ quadrupole lines yield Eq. (18.69).

Cavity vs optical atom. With a vertical cavity ($\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$) against an optical ion clock, the signal is dominantly at $m = 1, 2$ with geometry factors depending on the lab's colatitude ζ. A null at $m = 1$ but limit $A_2^{\max}$ implies $|C_{XX} - C_{YY}| \lesssim \mathcal{O}(A_2^{\max})$.

8. Summary (experiment–ready bounds)

- The directional light–speed shift is $\delta c/c(\hat{\boldsymbol{n}}) = \frac{1}{2}\,\hat{n}_i\hat{n}_j C_{ij}(t)$ with C_{ij} a traceless spatial tensor linear in the spectral anisotropy parameters (a_1, a_2, a_3).
- Clock/cavity comparisons measure harmonics at $\omega_\oplus$ and $2\omega_\oplus$ whose amplitudes are *linear* in C_{IJ} [Eqs. (18.67)–(18.68)]. Non-observation yields convex *linear* bounds [Eq. (18.69)].
- Annual modulations constrain isotropic couplings to the gravitational potential, k_α and k_c [Eq. (18.73)].
- Intersecting these bounds with cone LMIs and positivity wedges gives a small, falsifiable domain for (a_1, a_2, a_3) (anisotropy) and for EH coefficients (isotropic nonlinear optics).

Consistency gates (S/G/TQ/D).

(S) : No tachyonic modes; G_{T} Lorentzian; birefringence, if present, is small and stable. (18.75)

(G) : Cone inclusion $C[A] \subseteq C[g_{\mathrm{eff}}] \;\Rightarrow\; b \le 0,\; c \le 0,\; bc - \frac{1}{4}d^2 \ge 0.$ (18.76)

(TQ) : Positive spectral densities ensure EH coefficients $\beta_{1,2}, \tilde{\beta}_1 \ge 0$ and preserve kernel signs. (18.77)

(D) : χ^μ–induced dissipation is $\mathcal{O}(\delta\kappa)$ and does not generate sidebands by itself; it only rescales amp (18.78)

18.4 Global consistency checks

Aim. We assemble the precision-QED/QCD and clock constraints into a *unified* set of first-principles checks that every admissible spectral scenario must satisfy. The construction has two layers: (i) a *convex feasibility tier* implementing analyticity/positivity and cone ordering (basis/scale independent), and (ii) a *likelihood tier* combining experimental data with correlated systematics. We provide explicit maps among variables, formulate null tests and sum rules that do not require numerics, and state kill criteria.

1. Parameterization and forward maps

Wilson/spectral vectors. Collect the EFT and spectral inputs at a reference scale μ_* into

$$\mathbf{w} \;:=\; (\beta_1, \beta_2, \tilde{\beta}_1;\; a_1, a_2, a_3;\; \ldots)^\top, \qquad \mathbf{m} \;:=\; \left(\mu_{-1}, \mu_{-2}, \mu_{-\frac{3}{2}}, \mu_{-1}[w_e], \mu_{-1}[w_\mu], \ldots\right)^\top, \tag{18.79}$$

where $\mathbf{m}$ denotes positive spectral moments of $R_{\mathrm{spec}}(s)$ (Sec. 18.2). The anisotropy tensor C_{ij} entering clocks/cavities is linear in (a_1, a_2, a_3) (Sec. 18.3).

Observables. Pack the precision observables into

$$\mathbf{X} := \Big(\Delta a_e,\ \Delta a_\mu,\ \Delta\alpha(-Q_k^2)_{k=1..K},\ \Delta_{\rm Lamb}(nS)_{\{n,Z\}},\ \mathcal{A}_m^{\rm clock}({\rm geoms})\Big)^{\top}. \tag{18.80}$$

There exist *linear* maps $\mathsf{L}_{\rm disp}$ and $\mathsf{L}_{\rm clock}$ such that

$$\mathbf{X} \;=\; \mathsf{L}_{\rm disp}\,\mathbf{m} \;+\; \mathsf{L}_{\rm LbL}\,\mathbf{w} \;+\; \mathsf{L}_{\rm clock}\,C(\mathbf{w}) \;+\; \mathcal{O}({\rm subleading}), \tag{18.81}$$

with $\mathsf{L}_{\rm disp}$ built from the positive kernels of Sec. 18.1 and $\mathsf{L}_{\rm LbL}$ from Eq. (18.9). Subleading pieces (recoil, higher orders) can be added as nuisance terms with priors.

2. Convex feasibility tier (basis/scale independent)

Analyticity/positivity. From Secs. 17.2 and 18.2:

$$\mathbf{m} \;\geq 0, \qquad H^{(k)}(\mathbf{m}) \;\succeq\; 0, \qquad \beta_1 \geq 0,\ \ \beta_1+\beta_2 \geq 0,\ \ \tilde\beta_1 \geq 0. \tag{18.82}$$

Cone ordering (no superluminal fronts). For each sector (scalar/transverse gauge/longitudinal) the co-metric shifts satisfy the linear matrix inequalities (LMIs)

$$b(\mathbf{w}) \;\leq 0, \qquad c(\mathbf{w}) \;\leq 0, \qquad b(\mathbf{w})\,c(\mathbf{w}) - \tfrac{1}{4}d(\mathbf{w})^2 \;\geq 0 \quad \text{and similarly for each birefringent cone.} \tag{18.83}$$

Clock linear bounds. For each measured sidereal harmonic $\mathcal{A}_m$ in a given geometry,

$$\Big|\Lambda_m^{IJ}\,C_{IJ}(\mathbf{w})\Big| \;\leq\; A_m^{\rm max}, \tag{18.84}$$

with known geometry tensor Λ_m^{IJ} (Sec. 18.3).

Master feasible set. Define

$$\mathcal{F} \;:=\; \Big\{(\mathbf{w},\mathbf{m}) \;\Big|\; (18.82),\ (18.83),\ (18.84)\ \text{hold}\ \Big\}, \tag{18.85}$$

a closed convex set (intersection of cones and half-spaces). Any point outside $\mathcal{F}$ is excluded *independently* of experimental central values.

3. Likelihood tier and inference

Data vector and covariance. Let $\mathbf{X}^{\rm exp}$ and covariance Σ (including correlations across $\Delta\alpha$, a_ℓ, and Lamb shifts). Predictions $\mathbf{X}^{\rm th}(\mathbf{w},\mathbf{m})$ are given by (18.81). The Gaussian log-likelihood reads

$$-2\ln\mathcal{L} \;=\; \Big(\mathbf{X}^{\rm exp}-\mathbf{X}^{\rm th}\Big)^{\top}\Sigma^{-1}\Big(\mathbf{X}^{\rm exp}-\mathbf{X}^{\rm th}\Big) \;+\; {\rm const.} \tag{18.86}$$

We then solve the constrained optimization

$$\min_{(\mathbf{w},\mathbf{m})\in\mathcal{F}}\ \Big[-2\ln\mathcal{L}\Big] \quad \text{or} \quad \max_{(\mathbf{w},\mathbf{m})\in\mathcal{F}}\ \mathcal{L}, \tag{18.87}$$

with convex (or mixed convex–linear) structure: the feasibility $\mathcal{F}$ is convex; $\mathbf{X}^{\rm th}$ is linear in $\mathbf{m}$ and affine in $\mathbf{w}$ at leading order. Profiled posteriors or confidence regions follow from standard likelihood-ratio contours.

Priors and nuisance. If nuisance $\boldsymbol{\nu}$ (e.g. nuclear radii, hadronic kernels) enter linearly, one can include Gaussian priors and analytically profile them out, preserving convexity.

4. Null tests and algebraic sum rules (no numerics)

Sign and ordering tests. Independently of covariances:

$$\Delta a_\ell \ \geq 0, \qquad \Delta_{\mathrm{Lamb}}(nS) \ \geq 0, \qquad \Delta a_\ell^{\mathrm{VP}} \ \leq \ \frac{\alpha(0)}{9\pi}\,\Delta\alpha(-Q^2) \ \ (\forall Q^2>0), \tag{18.88}$$

and for heavy thresholds ($M \gg m_\mu$),

$$\frac{\Delta a_e^{\mathrm{VP}}}{\Delta a_\mu^{\mathrm{VP}}} \ \simeq \ \frac{m_e^2}{m_\mu^2} \quad (\text{up to } M^{-2} \text{ corrections}). \tag{18.89}$$

Violation of any relation in (18.88)–(18.89) kills the spectral hypothesis producing R_{spec}.

Clock quadrupole test. For orthogonal cavities in the horizontal plane, the leading sidereal amplitude must be at $2\omega_\oplus$ with phase fixed by the latitude geometry. Observation of a dominant $1\omega_\oplus$ line with the wrong phase (after accounting for boosts) contradicts the traceless C_{ij} form and requires birefringent/skewon pieces outside the metric-type sector.

Lamb–a_e correlation. Eliminating $\mu_{-3/2}$ with a step-function upper envelope yields

$$\Delta_{\mathrm{Lamb}}(1S) \ \lesssim \ \frac{8(Z\alpha m_r)^3}{3\pi}\,\frac{\alpha(0)}{\pi}\,\frac{\Delta\alpha(-Q^2)}{Q} \qquad (Q \text{ chosen near } Z\alpha m_r), \tag{18.90}$$

a rough yet basis-independent cross-check on low-mass spectral tails.

5. Scheme and scale checks

Basis invariance. Under linear field redefinitions or operator basis changes $(\mathbf{w},\mathbf{m}) \mapsto (\mathsf{T}\mathbf{w},\mathsf{S}\mathbf{m})$, the feasible set $\mathcal{F}$ and the inequalities (18.82)–(18.84) remain invariant because Hankel positivity and cone LMIs are tensorial statements.

RG passage and thresholds. Between scales $\mu < M_{\mathrm{th}}$ and $\mu > M_{\mathrm{th}}$, match $\mathbf{w}$ and update $\mathbf{m}$ by removing/adding spectral support as required by R_{spec}; the sign properties and LMIs are preserved, and admissibility (§17.4) is stable across the flow.

6. Worked audit pathway (recipe)

1. **Assemble inputs:** choose $\{Q_k\}$, $\{nS\}$, clock geometries; compile $\mathbf{X}^{\mathrm{exp}}$ and Σ.
2. **Build maps:** construct $\mathsf{L}_{\mathrm{disp}}$ (kernels K_{VP}, F_{nS}) and $\mathsf{L}_{\mathrm{clock}}$ (geometry tensors).
3. **Feasibility pass:** check $(\mathbf{w},\mathbf{m}) \in \mathcal{F}$; if not, *kill*.
4. **Likelihood fit:** solve (18.87); report best fit, profiled intervals, and pull vectors.
5. **Cross-validation:** verify null tests (18.88)–(18.89) and template phases/amplitudes in clocks.

7. Kill criteria and diagnostic flags

Kill criteria. Any of the following excludes a hypothesis (independent of priors):

$$\begin{aligned}
&\text{(K1)}\quad \mathbf{m} \not\geq 0 \text{ or } H^{(k)} \not\succeq 0 \quad \text{(violates analyticity/positivity)};\\
&\text{(K2)}\quad b > 0 \text{ or } c > 0 \text{ or } bc - \tfrac{1}{4}d^2 < 0 \quad \text{(superluminal fronts)};\\
&\text{(K3)}\quad \Delta a_\ell < 0,\ \Delta_{\mathrm{Lamb}}(nS) < 0,\ \text{or } \Delta a_\ell^{\mathrm{VP}} > \frac{\alpha}{9\pi}\Delta\alpha(-Q^2);\\
&\text{(K4)}\quad \text{Clock sidebands fail the linear geometry constraints (18.84).}
\end{aligned} \tag{18.91}$$

Diagnostic pulls. On the boundary of $\mathcal{F}$, typical saturations are: $c_2 \to 0$ (forward limit), $bc - \frac{1}{4}d^2 \to 0$ (marginal cone), or a single C_{IJ} direction saturating an $A_m^{\max}$ plane; each indicates which microscopic structure (spectral weight vs anisotropy vs nonlinear optics) controls the fit.

8. Summary

Global consistency is achieved by intersecting three convex domains: (i) dispersive/Hankel positivity for spectral moments, (ii) cone-ordering LMIs for propagation on (M, g_{eff}), and (iii) linear clock/cavity bounds on anisotropy—supplemented by simple algebraic null tests that provide immediate falsification. A constrained likelihood built on this feasible set yields robust, scheme-independent inferences on $(\beta_1, \beta_2, \tilde{\beta}_1)$, (a_1, a_2, a_3), and on integrated properties of R_{spec}, while preserving the first-principles gates (S/G/TQ/D). This closes the precision loop of the spectral model: any surviving parameter region is simultaneously causal, unitary, dispersive, and experimentally viable.

Part VII

Phenomenology, Pipelines, and Discovery Space

Chapter 19

Collider and Fixed-Target Probes

19.1 LHC/HL-LHC: Higgs and EW fits

Aim. We build the collider interface of the spectral framework by mapping the microscopic parameters of the internal fiber and director sector — summarized here by the vector

$$\mathbf{p}_{\text{spec}} := \big(\kappa_G, \kappa_y, \lambda_3, \lambda_4;\ a_1, a_2, a_3;\ \cdots\big)^{\top}, \tag{19.1}$$

into low-energy effective couplings that control Higgs, electroweak (EW), and diboson processes at the LHC/HL-LHC. The map proceeds through the gauge-invariant spectral EFT (Part VI) and can be expressed either in a *κ-framework* for rates or in the *Warsaw/SILH SMEFT* bases for differential observables. We derive closed-form expressions for signal strengths, oblique/EWPO shifts, and anomalous gauge couplings; then formulate a global likelihood respecting the S/G/TQ/D gates and the positivity/cone LMIs derived previously.

1. From spectral parameters to EFT couplings

EFT dictionary. At scales $E \ll \Lambda$ the spectral model induces a SMEFT Lagrangian

$$\mathcal{L}_{\text{SMEFT}} \;=\; \mathcal{L}_{\text{SM}} \;+\sum_i \frac{C_i}{\Lambda^2}\,\mathcal{O}_i \;+\; \sum_j \frac{C_j^{(8)}}{\Lambda^4}\,\mathcal{O}_j^{(8)} \;+\cdots, \tag{19.2}$$

with Wilson vectors $\mathbf{C}^{(6)}$, $\mathbf{C}^{(8)}$ that are *linear* in $\mathbf{p}_{\text{spec}}$ at leading order:

$$\mathbf{C}^{(d)} \;=\; \mathsf{M}^{(d)}\,\mathbf{p}_{\text{spec}}\,, \qquad d = 6, 8, \tag{19.3}$$

where $\mathsf{M}^{(d)}$ are calculable matching matrices (Vol. IV, Chs. 8, 17). For collider phenomenology we will repeatedly use the subset

$$\begin{aligned}
&\mathcal{O}_H = (H^\dagger H)^3, \quad \mathcal{O}_{HD} = (H^\dagger D_\mu H)^*(H^\dagger D^\mu H), \quad \mathcal{O}_{H\Box} = (H^\dagger H)\Box(H^\dagger H),\\
&\mathcal{O}_{HW} = H^\dagger H\, W^a_{\mu\nu} W^{a\mu\nu}, \quad \mathcal{O}_{HB} = H^\dagger H\, B_{\mu\nu} B^{\mu\nu}, \quad \mathcal{O}_{HWB} = H^\dagger \tau^a H\, W^a_{\mu\nu} B^{\mu\nu},\\
&\mathcal{O}_{GG} = H^\dagger H\, G^A_{\mu\nu} G^{A\mu\nu}, \quad \mathcal{O}_{y_f} = (H^\dagger H)\,\bar{f}_L H f_R + \text{h.c.}, \quad \mathcal{O}_{3W} = \epsilon^{abc} W^{a\nu}{}_{\mu} W^{b\rho}{}_{\nu} W^{c\mu}{}_{\rho}.
\end{aligned} \tag{19.4}$$

Spectral anisotropies (a_1, a_2, a_3) contribute to the neutral-current sector through cone-compatible tensors (Sec. 17.3); at collider energies they manifest as tiny direction-independent shifts in the effective kinetic terms and (if retained) birefringent structures

suppressed by the LMIs. We therefore treat collider kinematics in the *metric-type* limit and relegate anisotropy to precision/clock tests (Part 18).

κ-parameters. For inclusive Higgs rates, define

$$\kappa_V := \frac{g_{hVV}}{g_{hVV}^{\rm SM}}, \qquad \kappa_f := \frac{y_f}{y_f^{\rm SM}}, \qquad c_g,\ c_\gamma \text{ via } \ \mathcal{L} \supset \frac{\alpha_s}{12\pi}\frac{c_g}{v} h\, G^A_{\mu\nu}G^{A\mu\nu} + \frac{\alpha}{8\pi}\frac{c_\gamma}{v} h\, F_{\mu\nu}F^{\mu\nu}, \tag{19.5}$$

with $v = (\sqrt{2}G_F)^{-1/2}$. To leading order in v^2/Λ^2 one has the linear relations

$$\delta\kappa_V := \kappa_V - 1 \ = \ \frac{v^2}{\Lambda^2}\Big(\alpha_V^H C_H + \alpha_V^{HD} C_{HD} + \alpha_V^{H\Box} C_{H\Box}\Big), \tag{19.6}$$

$$\delta\kappa_f := \kappa_f - 1 \ = \ \frac{v^2}{\Lambda^2}\Big(\alpha_f\, C_{y_f} + \beta_f^{H\Box} C_{H\Box} + \beta_f^{HD} C_{HD}\Big), \tag{19.7}$$

$$c_g = \frac{v^2}{\Lambda^2}\,\gamma_g\, C_{GG}, \qquad c_\gamma = \frac{v^2}{\Lambda^2}\Big(\gamma_W C_{HW} + \gamma_B C_{HB} + \gamma_{WB} C_{HWB}\Big), \tag{19.8}$$

with calculable coefficients α, β, γ fixed by matching and field redefinitions (canonical normalization).

2. Higgs signal strengths

Inclusive rates. For a production mode i and decay f, the signal strength is

$$\mu_{i\to h\to f} \ := \ \frac{\sigma_i}{\sigma_i^{\rm SM}} \times \frac{\Gamma_f}{\Gamma_f^{\rm SM}} \times \frac{\Gamma_{\rm tot}^{\rm SM}}{\Gamma_{\rm tot}}, \tag{19.9}$$

with

$$\frac{\sigma_{\rm ggF}}{\sigma_{\rm ggF}^{\rm SM}} \simeq \Big|\kappa_t + c_g\Big|^2, \qquad \frac{\sigma_{\rm VBF}}{\sigma_{\rm VBF}^{\rm SM}} \simeq \kappa_V^2, \qquad \frac{\sigma_{VH}}{\sigma_{VH}^{\rm SM}} \simeq \kappa_V^2, \qquad \frac{\sigma_{t\bar{t}h}}{\sigma_{t\bar{t}h}^{\rm SM}} \simeq \kappa_t^2, \tag{19.10}$$

$$\frac{\Gamma_{b\bar{b}}}{\Gamma_{b\bar{b}}^{\rm SM}} \simeq \kappa_b^2, \quad \frac{\Gamma_{\tau\tau}}{\Gamma_{\tau\tau}^{\rm SM}} \simeq \kappa_\tau^2, \quad \frac{\Gamma_{WW}}{\Gamma_{WW}^{\rm SM}} \simeq \kappa_V^2, \quad \frac{\Gamma_{ZZ}}{\Gamma_{ZZ}^{\rm SM}} \simeq \kappa_V^2,$$

$$\frac{\Gamma_{\gamma\gamma}}{\Gamma_{\gamma\gamma}^{\rm SM}} \simeq \frac{|A_W\kappa_V + A_t\kappa_t + 8\pi c_\gamma/\alpha|^2}{|A_W + A_t|^2}, \tag{19.11}$$

where A_W, A_t are SM loop amplitudes. The total width scales as

$$\frac{\Gamma_{\rm tot}}{\Gamma_{\rm tot}^{\rm SM}} \ \simeq \ \sum_f {\rm BR}_f^{\rm SM}\,\frac{\Gamma_f}{\Gamma_f^{\rm SM}} \ + \ {\rm BR}_{\rm inv}^{\rm spec}, \tag{19.12}$$

allowing for invisible or undetected spectral decays (if kinematically open). Equations (19.9)–(19.12) define a closed algebraic system mapping $\mathbf{p}_{\rm spec}$ to the set $\{\mu\}$.

Off-shell and lineshape. In $gg \to ZZ \to 4\ell$ above threshold, interference of the Higgs-mediated amplitude with the continuum probes κ_t and c_g *off shell*. To linear order,

$$\frac{d\sigma}{dm_{ZZ}}\Big/\frac{d\sigma}{dm_{ZZ}}\Big|_{\rm SM} \ \simeq \ 1 + \xi_1(m_{ZZ})\,\delta\kappa_t + \xi_2(m_{ZZ})\,c_g + \cdots, \tag{19.13}$$

with computable kernels $\xi_{1,2}$; incorporating these shapes breaks degeneracies with $\Gamma_{\rm tot}$.

3. Electroweak precision and oblique parameters

Oblique set. At tree level in SMEFT,

$$S \;=\; 4 s_W c_W \,\frac{v^2}{\Lambda^2}\, C_{HWB}, \qquad T \;=\; -\frac{v^2}{2\alpha_{\mathrm{em}}\Lambda^2}\, C_{HD}, \qquad W \;=\; \frac{m_W^2}{\Lambda^2}\, C_{2W}, \qquad Y \;=\; \frac{m_W^2}{\Lambda^2}\, C_{2B}, \tag{19.14}$$

where $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$. In the spectral model (Part 8), custodial alignment enforces $T \simeq 0$ at leading order, while S is controlled by the geometric mixing encoded in C_{HWB} and ultimately by the fiber metric ratios that fix θ_W (Sec. 7.4). Consistency requires the fit to satisfy positivity and cone LMIs; in particular, any superluminal cone tilt is forbidden (Sec. 17.3).

Z-pole and W-mass. Vertex redefinitions and input-scheme choices imply linear relations

$$\frac{\delta m_W}{m_W} \;=\; \alpha_W^S\, S + \alpha_W^T\, T + \alpha_W^W\, W + \alpha_W^Y\, Y, \qquad \delta g_{Zf} \;=\; \sum_i \zeta_{fi}\, \frac{v^2}{\Lambda^2}\, C_i, \tag{19.15}$$

with known coefficients α, ζ once the input set is fixed (e.g. $\{G_F, m_Z, \alpha\}$). These formulae bind the spectral geometry to electroweak pole observables.

4. Dibosons, TGCs, and VBS (summary)

Anomalous TGCs. LEP/LHC parameterizations use $(\delta g_1^Z, \delta\kappa_\gamma, \lambda_Z)$ with the SMEFT map

$$\delta g_1^Z \;\sim\; \frac{v^2}{\Lambda^2}\,(c_{HW} + c_{HWB}), \qquad \delta\kappa_\gamma \;\sim\; \frac{v^2}{\Lambda^2}\,(c_{HW} + c_{HB} + c_{HWB}), \qquad \lambda_Z \;\sim\; \frac{v^2}{\Lambda^2}\, c_{3W}, \tag{19.16}$$

up to fixed numerical factors. Vector-boson scattering (VBS) probes complementary dim-8 operators $f_{S,0}, f_{S,1}$ (Sec. 17.2); positivity requires $f_{S,0} \geq 0$, $f_{S,0} + f_{S,1} \geq 0$.

5. Likelihood construction and profiling

Data vector. Collect the experimental inputs into

$$\mathbf{X}_{\mathrm{coll}} \;:=\; \Big(\mu_{i\to f}\ (\text{bins}),\ d\sigma/dm_{ZZ},\ d\sigma/dp_T^H,\ S, T, W, Y,\ \text{aTGC bins},\ \text{VBS bins}\Big)^{\top}. \tag{19.17}$$

Predictions $\mathbf{X}_{\mathrm{th}}(\mathbf{p}_{\mathrm{spec}})$ follow from (19.3)–(19.16) including acceptance effects (below). The Gaussian (or piecewise-Gaussian) likelihood with covariance Σ is

$$-2\ln\mathcal{L}_{\mathrm{coll}} \;=\; \Big(\mathbf{X}_{\mathrm{coll}}^{\mathrm{exp}} - \mathbf{X}_{\mathrm{th}}\Big)^{\top} \Sigma^{-1} \Big(\mathbf{X}_{\mathrm{coll}}^{\mathrm{exp}} - \mathbf{X}_{\mathrm{th}}\Big) \;+\; \sum_k \frac{(\nu_k - \nu_k^{(0)})^2}{\sigma_{\nu_k}^2}, \tag{19.18}$$

where ν_k are nuisance parameters for theory (PDF/scale), detector systematics, and unfolding. We minimize (19.18) subject to the admissible region constraints (Part 17), i.e.

$$(\mathbf{p}_{\mathrm{spec}}, \mathbf{C}) \;\in\; \mathcal{A} \quad \text{(Hankel positivity, cone LMIs, custodial selection).} \tag{19.19}$$

Acceptance and EFT power counting. For differential bins we use *linearized* EFT predictions

$$\sigma_{\rm bin} \;\simeq\; \sigma_{\rm SM,bin} \;+\; \sum_i \frac{C_i}{\Lambda^2}\,\sigma^{\rm int}_{i,\rm bin} \;+\; \Theta_{\rm quad} \sum_{i\le j} \frac{C_i C_j}{\Lambda^4}\,\sigma^{\rm quad}_{ij,\rm bin}, \tag{19.20}$$

with $\Theta_{\rm quad}$ a switch controlling quadratic terms in high-energy bins; positivity and unitarity provide a natural regulator by excluding regions where (19.20) would violate the forward bounds (Part 17).

6. Worked algebra and illustrative constraints

Inclusive Higgs fit (closed form). In the no-invisible limit, the total width factor simplifies:

$$\frac{\Gamma_{\rm tot}}{\Gamma^{\rm SM}_{\rm tot}} \simeq \sum_{f=b,\tau,c,WW,ZZ,gg,\gamma\gamma} {\rm BR}^{\rm SM}_f\, r_f(\kappa_V,\kappa_f,c_g,c_\gamma), \tag{19.21}$$

with $r_{b,\tau,c}=\kappa_f^2$, $r_{WW,ZZ}=\kappa_V^2$, $r_{gg}=|\kappa_t+c_g|^2\,/\,|1|^2$, $r_{\gamma\gamma}$ from (19.11). Plugging (19.10)–(19.21) into (19.9) yields analytic $\mu_{i\to f}(\mathbf{p}_{\rm spec})$ suitable for fast scanning.

EWPO ellipse. Combining m_W and Z-pole with (19.14)–(19.15) produces an (S,T) ellipse; in the custodial spectral limit $T\simeq 0$ the fit reduces to a single linear constraint on C_{HWB} and thus on the geometric mixing of Sec. 7.4.

aTGC–VBS interplay. From unitarity and positivity, large $\lambda_Z \propto C_{3W}$ is bounded by forward-limit constraints in $VV\to VV$ scattering; simultaneously, $f_{S,0}, f_{S,1}$ are required to be nonnegative. A combined fit thus tends to favor small C_{3W} unless compensated by positive dim-8 terms — a pattern characteristic of causal UV completions.

7. Kill criteria (collider layer)

(K1) $\exists$ bin with $\sigma_{\rm th}<0$ from linearized EFT (violates positivity) when $\Theta_{\rm quad}=0$;
(K2) Predicted (S,T,W,Y) falls outside the EWPO ellipse under custodial assumptions;
(K3) Cone LMIs broken at collider energy (superluminal fronts in fitted region);
(K4) $\mu_{i\to f}$ requires $|\kappa_V|>1+\mathcal{O}(1)$ while geometry of Sec. 7.4 fixes $\kappa_V\simeq 1$ (incompatibility). (19.22)

8. Summary

We have provided the end-to-end algebra to translate spectral microphysics into LHC/HL-LHC observables: a linear matching to SMEFT, rate-level κ expressions for Higgs channels, oblique/EWPO relations, and anomalous gauge couplings for diboson/VBS. A convex, constraint-preserving likelihood completes the pipeline, ensuring that any fitted point respects analyticity/positivity and cone ordering established in Parts 17 and 18. This sets the stage for future-collider extrapolations (Sec. 19.2) and spectral dark-portal searches (Sec. 19.3).

19.2 Future colliders: ILC/CLIC/FCChh/CEPC

Aim. We provide an analytical, collider-agnostic framework to propagate spectral-model parameters into forecasted sensitivities for future facilities: lepton Higgs factories (ILC/CEPC at $\sqrt{s} \sim 240$–$500\,\mathrm{GeV}$), multi-TeV linear e^+e^- (CLIC), and pp at tens of TeV (FCC-hh). Our treatment is fully EFT-based, respects the S/G/TQ/D gates, and incorporates cone-ordering LMIs. The key outputs are: (i) closed-form Fisher information for polarized e^+e^- processes (ZH, WW, $f\bar{f}$), (ii) energy-growth diagnostics for SMEFT directions, (iii) unitarity/positivity envelopes for VBS and di-Higgs at hadron machines, and (iv) a global profiling recipe combining machines.

1. EFT interface and scaling with energy and luminosity

Linearized SMEFT response. For any binned observable b with prediction $N_b(\boldsymbol{\theta})$ depending on SMEFT parameters $\boldsymbol{\theta} = (C_i/\Lambda^2, C_j^{(8)}/\Lambda^4, \ldots)$, the leading expansion reads

$$N_b \;=\; N_b^{\mathrm{SM}} \;+\; \sum_i \frac{C_i}{\Lambda^2}\, S_{i,b} \;+\; \Theta_{\mathrm{quad}} \sum_{i\le j} \frac{C_i C_j}{\Lambda^4}\, Q_{ij,b} \;+\; \sum_k \frac{C_k^{(8)}}{\Lambda^4}\, S_{k,b}^{(8)} \;+\; \cdots\,, \tag{19.23}$$

with interference shapes $S_{i,b} \propto s$ for operators linear in field strengths or derivatives, and $S_{k,b}^{(8)} \propto s^2$ at fixed angle (s the partonic c.m. energy). The statistical Fisher matrix for Poisson counts is

$$\mathcal{I}_{ij} \;=\; \sum_b \frac{1}{N_b^{\mathrm{SM}} + \sigma_{b,\mathrm{sys}}^2}\, \frac{S_{i,b}\, S_{j,b}}{\Lambda^4}\,, \qquad \delta\theta_i \simeq \sqrt{(\mathcal{I}^{-1})_{ii}}, \tag{19.24}$$

which scales as $\delta\theta_i \sim (L\,\langle s\rangle)^{-1/2}$ for interference-dominated directions (here L is the integrated luminosity). Positivity and cone LMIs restrict the admissible sign/size of combinations of (C_i), effectively regularizing high-energy tails.

2. Lepton colliders: polarized $e^+e^- \to ZH$, WW, $f\bar{f}$

Polarization projectors and chiral couplings. Let (P_-, P_+) be the longitudinal polarizations of e^- and e^+, respectively. The polarized cross section decomposes as

$$\sigma(P_-, P_+) \;=\; \frac{1}{4}\Big[(1{+}P_-)(1{-}P_+)\,\sigma_{LR} + (1{-}P_-)(1{+}P_+)\,\sigma_{RL}\Big], \tag{19.25}$$

where LR (RL) denotes $e_L^- e_R^+$ ($e_R^- e_L^+$) helicities. This projects SMEFT deformations carried by left/right Z couplings and four-fermion operators, enabling near-orthogonal constraints.

$e^+e^- \to ZH$ (Higgsstrahlung). At tree level,

$$\sigma_{ZH} \;=\; \sigma_{ZH}^{\mathrm{SM}} \left[1 + 2\,\delta\kappa_Z + \frac{s}{\Lambda^2}\,\Xi_{ZH}(\cos\theta; C_{HWB}, C_{HD}, C_{H\Box}, C_{HL}^{(1,3)}, C_{HE})\right], \tag{19.26}$$

where $\delta\kappa_Z$ encodes the custodial-preserving hZZ rescaling and Ξ_{ZH} is a known angular polynomial whose coefficients depend on beam polarization through (19.25). Differential binning in $\cos\theta$ plus (P_-, P_+) disentangles $\delta\kappa_Z$ from contact terms, turning the per-mille total-rate precision of a Higgs factory into per-mille constraints on linear SMEFT directions.

$e^+e^- \to W^+W^-$ and anomalous TGCs. The helicity amplitudes receive contributions

$$\mathcal{M} \;=\; \mathcal{M}_{\rm SM} \;+\; \frac{s}{\Lambda^2}\left(\alpha_1\,\delta g_1^Z + \alpha_2\,\delta\kappa_\gamma\right) \;+\; \frac{s}{m_W^2}\,\frac{v^2}{\Lambda^2}\,\alpha_3\,\lambda_Z \;+\; \cdots\,, \tag{19.27}$$

with $(\delta g_1^Z, \delta\kappa_\gamma, \lambda_Z)$ mapped to Warsaw coefficients by Eq. (19.16). The λ_Z term grows as s in the amplitude (thus s^2 in rate), so shape information at multi-TeV CLIC energies is crucial; positivity of forward amplitudes (Part 17) caps the allowed growth and enforces C_{3W} correlations with dim-8 VBS operators.

$e^+e^- \to f\bar{f}$ and four-fermion. Away from the Z pole,

$$\frac{d\sigma}{d\cos\theta} \;=\; \frac{d\sigma_{\rm SM}}{d\cos\theta} + \frac{s}{\Lambda^2}\sum_X c_X\,\mathcal{P}_X(\cos\theta;P_\pm) \;+\; \mathcal{O}\!\left(\frac{s^2}{\Lambda^4}\right), \tag{19.28}$$

where X runs over $C_{HL}^{(1,3)}, C_{HE}, C_{HQ}^{(1,3)}, C_{HU}, C_{HD}$ and semileptonic $C_{\ell q}, C_{\ell u}, \ldots$; the polynomials $\mathcal{P}_X$ are orthogonalized by using angular moments and polarization flips. CLIC's energy lever arm turns these interference terms into powerful constraints scaling as $1/\sqrt{L\,s}$.

Top threshold and Yukawa. At $\sqrt{s} \approx 2m_t$, the threshold lineshape depends on $(m_t^{1S}, \Gamma_t, \alpha_s, y_t)$; EFT deformations enter as small shifts,

$$\Delta\sigma_{t\bar{t}}^{\rm thr} \;\simeq\; \frac{\partial\sigma}{\partial m_t}\Delta m_t + \frac{\partial\sigma}{\partial\Gamma_t}\Delta\Gamma_t + \frac{\partial\sigma}{\partial y_t}\Delta y_t + \sum_i \frac{C_i}{\Lambda^2}S_{i,\rm thr}, \tag{19.29}$$

with derivatives known from N^{3}LO threshold calculations. This provides an orthogonal handle on y_t complementary to di-Higgs at hadron colliders.

3. Hadron colliders: FCC-hh and energy-growing channels

Di-Higgs production and Higgs self-coupling. The $gg \to hh$ amplitude contains the triangle ($\propto \kappa_t\kappa_\lambda$) and box ($\propto \kappa_t^2$) pieces, plus EFT contact terms (c_g, $\mathcal{O}^{(8)}$). A convenient polynomial parameterization is

$$\frac{\sigma_{hh}}{\sigma_{hh}^{\rm SM}} \;=\; A\,\kappa_t^4 + B\,\kappa_t^2\kappa_\lambda^2 + C\,\kappa_t^3\kappa_\lambda + D\,\kappa_t^2 c_g + E\,\kappa_t\kappa_\lambda c_g + F\,c_g^2 \;+\; \cdots\,, \tag{19.30}$$

with A–F known from LO matrix elements and reweighted to NLO/NNLO. Binning in m_{hh} and $\cos\theta^*$ disentangles κ_λ from (κ_t, c_g); high-m_{hh} tails amplify dim-8 contact terms, which are kept within the EFT domain by unitarity and positivity gates (below).

Vector-boson scattering (VBS). Longitudinal $VV \to VV$ is sensitive to dim-8 operators $f_{S,0}, f_{S,1}$:

$$\mathcal{A}_{LL\to LL} \;\simeq\; \mathcal{A}_{\rm SM} \;+\; \frac{s^2}{\Lambda^4}\left(\alpha_{S,0}\,f_{S,0} + \alpha_{S,1}\,f_{S,1}\right), \qquad a_0 \;\approx\; \frac{1}{16\pi s}\int_{-s}^{0} dt\,\mathcal{A}(s,t)\,. \tag{19.31}$$

Partial-wave unitarity $|\mathrm{Re}\,a_0| \le \frac{1}{2}$ implies the envelope

$$\left|\alpha_{S,0}\,f_{S,0} + \alpha_{S,1}\,f_{S,1}\right| \;\lesssim\; 8\pi\,\frac{\Lambda^4}{s^2}\,, \tag{19.32}$$

to be enforced bin-by-bin when fitting high-mass VBS distributions. Forward-limit positivity further constrains $(f_{S,0}, f_{S,1})$ to a convex wedge (Part 17).

High-p_T Higgs and dibosons. In $pp \to Hj$ and $pp \to VV$, interference with $\mathcal{O}_{HW,HB,HWB}$ yields tails scaling as p_T^2/Λ^2:

$$\frac{d\sigma}{dp_T} \simeq \frac{d\sigma_{\rm SM}}{dp_T}\left[1 + \frac{p_T^2}{\Lambda^2}\sum_{i\in\{HW,HB,HWB\}} \omega_i(p_T)\, C_i\right], \tag{19.33}$$

with analytic ω_i at parton level and modest PDF/scale dressing. Cone LMIs eliminate pathological regions that would imply superluminal fronts when extrapolated to asymptotic energies.

4. EFT validity: truncation, unitarity, and positivity

Truncation criterion. We require, for every bin with characteristic $\hat{s}$,

$$\left|\sum_i \frac{C_i}{\Lambda^2} S_{i,b}\right| \gg \left|\sum_{i\le j} \frac{C_i C_j}{\Lambda^4} Q_{ij,b}\right| \quad \text{or} \quad \Theta_{\rm quad} = 1 \text{ with } |a_0| \le \tfrac{1}{2}, \tag{19.34}$$

and reject fits that rely on uncontrolled quadratic dominance.

Cone-ordering LMIs. Any fitted point must satisfy the cone LMIs of Sec. 17.3 for the transverse gauge sector; violations imply acausal propagation and *kill* the hypothesis, independent of collider likelihood.

5. Machine-combination profiling

Block-diagonal likelihood. Let $\mathcal{L}_{\rm ILC}$, $\mathcal{L}_{\rm CLIC}$, $\mathcal{L}_{\rm CEPC}$, $\mathcal{L}_{\rm FCC}$ be Gaussian (or piecewise-Gaussian) likelihoods with their covariances and nuisance sets. The global objective is

$$-2\ln\mathcal{L}_{\rm global} = \sum_{X\in\{\rm ILC,CLIC,CEPC,FCC\}} \left[-2\ln\mathcal{L}_X\right], \qquad \text{subject to positivity/LMIs/unitarity.} \tag{19.35}$$

Lepton colliders fix contact and TGC directions at per-mille to per-cent via polarization and angular moments; hadron colliders lift flat directions tied to energy growth (VBS, hh). The intersection yields robust bounds on the spectral parameter vector $\mathbf{p}_{\rm spec}$ via the linear map (19.3).

6. Worked analytic templates

Higgs-factory Fisher for ZH. For n_θ bins in $\cos\theta$ and two polarization settings,

$$\mathcal{I}_{ij}^{ZH} = \sum_{\alpha\in\{(P_-^{(1)},P_+^{(1)}),(P_-^{(2)},P_+^{(2)})\}} \sum_{k=1}^{n_\theta} \frac{L\,\epsilon_k\,\sigma_k^{\rm SM}(\alpha)}{1+\delta_{k,\rm sys}^2}\,\frac{1}{\Lambda^4}\,\partial_i \ln\sigma_k(\alpha)\,\partial_j\ln\sigma_k(\alpha), \tag{19.36}$$

with $\partial_i \ln\sigma_k$ linear in $(C_{HWB}, C_{HD}, C_{H\Box}, C_{HL}^{(1,3)}, C_{HE}, \delta\kappa_Z)$. Choosing opposite-sign $(P_-^{(1)}, P_+^{(1)}) = -(P_-^{(2)}, P_+^{(2)})$ maximizes determinant $\det\mathcal{I}$.

CLIC shape for WW. Define normalized angular moments $M_\ell = \int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\,(d\sigma/d\cos\theta)/\sigma$. To leading order,

$$\Delta M_\ell \;=\; \frac{s}{\Lambda^2}\Big(\alpha_{\ell 1}\,\delta g_1^Z + \alpha_{\ell 2}\,\delta\kappa_\gamma\Big) + \frac{s}{m_W^2}\,\frac{v^2}{\Lambda^2}\,\alpha_{\ell 3}\,\lambda_Z\,, \tag{19.37}$$

and the three moments $\ell = 0, 1, 2$ suffice to isolate the trio $(\delta g_1^Z, \delta\kappa_\gamma, \lambda_Z)$ at each energy/polarization.

VBS unitarity envelope. For a given VBS bin with median $\hat{s}$, enforce (19.32); if the unconstrained fit prefers $|f_{S,0}|$ above the envelope, the projection onto the positivity wedge selects the closest admissible point, preventing spurious sensitivity from EFT breakdown.

7. Discovery/kill criteria for the future program

(K1) Any best-fit point violates partial-wave bound (19.32) in a used bin $\Rightarrow$ invalidate the bin or th

(K2) Cone LMIs broken (superluminal fronts) $\Rightarrow$ exclude hypothesis independently of fit quality;

(K3) Disjoint lepton/hadron fits in the same SMEFT basis (no overlap within CL intervals) $\Rightarrow$ inco

(K4) Higgs-factory per-mille ZH and WW shapes demand C_{HWB}, C_{HD} outside geometric ranges fixe

(19.38)

8. Summary

Future lepton colliders provide polarization-assisted, interference-dominated access to SMEFT directions with clean energy scaling and minimal systematics; multi-TeV lepton machines lift TGC/four-fermion directions through s-growth; hadron colliders exploit VBS and di-Higgs tails to probe self-interactions and dim-8 contact terms. Within the spectral framework, the linear matching (19.3), Fisher constructs (19.36) and (19.37), and the unitarity/positivity envelopes (19.32)–(19.34) together deliver a robust, convex, and causality-safe projection of discovery reach and exclusion power across ILC/CLIC/CEPC/FCChh.

19.3 Dark portals from spectral sectors

Aim. We construct and analyze the leading *spectral portals* that couple the Standard-Model (SM) sector to additional degrees of freedom living on (or arising from) the internal spectral fiber $\mathcal{F}$. The derivation proceeds from the geometric data of Part I and the EFT/positivity gates of Parts 17 and 18. We give canonical parameterizations, map them to collider and fixed-target observables, and state admissibility and kill criteria consistent with S/G/TQ/D gates.

1. Spectral origin and portal taxonomy

Product isometries and mixed curvatures. Let the fiber admit an isometry product $G_{\rm vis} \times G_{\rm dark}$ with Killing sets $\{\xi_A\}$ and $\{\zeta_\alpha\}$. The kinetic term of Sec. 1.1, $\mathcal{L}_{\rm int}^{(2)} \sim \kappa_y\, g_{\rm eff}^{\mu\nu} h_{IJ}(y)\, \mathcal{D}_\mu y^I \mathcal{D}_\nu y^J$, induces, after integrating out fast internal modes and expanding around a background y_0, cross-quadratic forms in the corresponding connections $\mathcal{A}_\mu^A$ and $\mathcal{X}_\mu^\alpha$:

$$\Delta\mathcal{L}_{\rm mix} \;=\; -\frac{\epsilon_{A\alpha}}{2}\,\mathrm{Tr}\Big[F^A_{\mu\nu}\,\mathcal{Q}^{A\alpha}\Big]\,X^{\mu\nu}_\alpha \;\equiv\; -\frac{\epsilon}{2}\,F^Y_{\mu\nu}X^{\mu\nu} \;+\; \cdots\,, \tag{19.39}$$

where $F^Y_{\mu\nu}$ is the hypercharge field strength and $X_{\mu\nu}$ the Abelian field strength of a dark $U(1)_D \subset G_{\rm dark}$. The dimensionless mixing

$$\epsilon \;=\; \kappa_y \left\langle\, h_{IJ}(y_0)\, \xi^I_Y(y_0)\, \zeta^J_D(y_0) \,\right\rangle_{\mathcal{F}} \tag{19.40}$$

is an *overlap* of Killing directions in the fiber metric, averaged over the fast modes; small misalignment naturally yields $\epsilon \ll 1$.

Scalar (Higgs) portal. Spectral phasons (phase modes of Φ) or radial excitations S of internal textures couple to the SM through the stiffness-induced invariant

$$\mathcal{L}_{H\text{-portal}} \;=\; -\lambda_{HS}\,(H^\dagger H)\,S^2 \;-\; \kappa_{HS}\, v\, h\, S \;+\; \cdots, \tag{19.41}$$

where $H = (0, (v+h)/\sqrt{2})^\top$, v is fixed by electroweak symmetry breaking, and the linear term κ_{HS} appears when the spectral background breaks the $S \to -S$ parity (e.g. by a phason bias). Diagonalizing the (h, S) mass matrix gives a mixing angle

$$\tan 2\theta \;=\; \frac{2\,\kappa_{HS}\, v}{m_h^2 - m_S^2}\,, \qquad h_1 = \cos\theta\, h + \sin\theta\, S, \quad h_2 = -\sin\theta\, h + \cos\theta\, S, \tag{19.42}$$

with h_1 the 125 GeV-like state for $|\theta| \ll 1$.

Vector (kinetic) portal. For a dark photon A'_μ with field strength $X_{\mu\nu}$ and mass from either a dark Higgs S_D or a Stückelberg fiber phase, the canonically normalized Lagrangian is

$$\mathcal{L}_{\rm kin\ mix} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} - \frac{\epsilon}{2}\,F_{\mu\nu}X^{\mu\nu} + \frac{1}{2}m^2_{A'}A'_\mu A'^\mu \;+\; \cdots, \tag{19.43}$$

so that A' couples to the electromagnetic current $J^\mu_{\rm em}$ with strength ϵe after EWSB.

Neutrino (Yukawa) portal. Neutral spectral fermions N (zero modes localized on fiber defects; Sec. 13.1) admit

$$\mathcal{L}_{\nu\text{-portal}} \;=\; -y_\alpha\, \bar{L}_\alpha H N \;-\frac{1}{2}\, M_N\, \bar{N}^c N \;+\; \text{h.c.}, \tag{19.44}$$

leading to active–sterile mixing $U_{\alpha N} \simeq y_\alpha v/M_N$ and a light-neutrino mass $m_\nu \simeq (yv)^2/M_N$ (spectral see-saw; Sec. 14.3).

2. Portal parameter set and spectral map

Minimal collider-facing parameters. We collect the leading parameters as

$$\mathbf{p}_{\rm portal} := \Big(\sin\theta,\; m_S,\; \lambda_{HS}\;;\; \epsilon,\; m_{A'}\;;\; |U_{\alpha N}|^2,\; m_N\Big), \tag{19.45}$$

with spectral origins

$$\sin\theta \sim \frac{\kappa_{HS}\, v}{|m_h^2 - m_S^2|}\,, \qquad \kappa_{HS} \propto \kappa_y\, \langle \partial_y \Phi \rangle\,, \tag{19.46}$$

$$\epsilon \sim \kappa_y\, \langle h_{IJ}\, \xi^I_Y\, \zeta^J_D \rangle_{\mathcal{F}}\,, \quad m^2_{A'} \sim g_D^2\, \langle (S_D)^2 \rangle \text{ or Stückelberg mass from a fiber phase,} \tag{19.47}$$

$$|U_{\alpha N}| \sim \frac{|y_\alpha|\, v}{M_N}\,, \qquad y_\alpha \propto \text{overlap integrals on } \mathcal{F} \text{ (Sec. 14.1).} \tag{19.48}$$

Positivity and cone LMIs bound ϵ-induced deformations (no superluminal cones; Sec. 17.3).

3. Production and decay: analytic rates

Higgs portal scalar S. For $m_S < m_h/2$, the invisible (or exotic) Higgs branch

$$\Gamma(h \to SS) \;=\; \frac{\lambda_{hSS}^2}{32\pi m_h}\sqrt{1-\frac{4m_S^2}{m_h^2}}, \qquad \lambda_{hSS} \;\equiv\; \lambda_{HS} v\cos\theta - (m_h^2 - m_S^2)\sin\theta/(2v), \tag{19.49}$$

contributes to $\mathrm{BR}_{\mathrm{inv/exo}}$ in Eq. (19.12). If S mixes with h by angle θ, its SM-like partial widths scale as

$$\Gamma(S \to f\bar f,\, VV,\, gg,\, \gamma\gamma) \;=\; \sin^2\theta\; \Gamma_h^{\mathrm{SM}}(m_h \to m_S), \tag{19.50}$$

so that the proper decay length is $c\tau_S \propto \sin^{-2}\theta$ and *displaced* signatures arise naturally for $\sin^2\theta \ll 10^{-8}$ at GeV masses.

Dark photon A'. To leading order,

$$\Gamma(A' \to f\bar f) \;=\; \frac{N_c Q_f^2\,\alpha\,\epsilon^2}{3}\, m_{A'}\,\beta_f\left(1+\frac{2m_f^2}{m_{A'}^2}\right), \qquad \beta_f = \sqrt{1-\frac{4m_f^2}{m_{A'}^2}}, \tag{19.51}$$

and hadronic modes can be included through $R(s)$ as in Sec. 18.1. Production mechanisms:

$$pp:\ \text{Drell–Yan}\ (q\bar q \to \gamma^*/Z^* \to A')\ \propto\ \epsilon^2; \qquad e^+e^-:\ \text{radiative return}\ (e^+e^- \to \gamma A')\ \propto\ \epsilon^2. \tag{19.52}$$

For $m_{A'} < m_h$, exotic $h \to A'A'$ or $h \to ZA'$ arise if a dark Higgs mixes with h (portal coupling $\propto \sin\theta\, g_D$).

Heavy neutral lepton N. For $m_N \ll m_W$ the total width scales as

$$\Gamma_N \;\simeq\; \frac{G_F^2 m_N^5}{96\pi^3}\sum_\alpha |U_{\alpha N}|^2 \quad\Rightarrow\quad c\tau_N \;\approx\; 1\,\mathrm{m}\left(\frac{10^{-8}}{\sum|U_{\alpha N}|^2}\right)\left(\frac{1\,\mathrm{GeV}}{m_N}\right)^5, \tag{19.53}$$

with production in meson decays ($K, D, B \to \ell N X$), $pp \to W^* \to \ell N$, and $e^+e^- \to Z^* \to \nu N$.

4. Collider and fixed-target signatures

Higgs/exotic at the LHC and lepton colliders. Key channels:

$$h \to SS \to (b\bar b)(b\bar b),\ (b\bar b)(\tau^+\tau^-),\ 4\ell,\ \text{or invisible}, \tag{19.54}$$

$$h \to A'A' \to 4\ell,\ 2\ell 2\pi,\ \text{displaced leptons if } c\tau_{A'} \gg \mathrm{mm}, \tag{19.55}$$

$$h \to ZA' \to \ell^+\ell^- + \ell^+\ell^-\ (\text{off-shell } Z),\ \text{rate}\ \propto \epsilon^2\sin^2\theta.$$

Lepton colliders (Sec. 19.2) add recoil-mass tagging of $e^+e^- \to Zh(\to \text{invisible/exotic})$, giving model-independent bounds on $\mathrm{BR}_{\mathrm{inv/exo}}$.

Mono-X and VBF+$\not{E}_T$. Invisible portals give $pp \to (Z/W/h/\mathrm{j}) + \not{E}_T$ with rates set by λ_{HS} (via h-strahlung) or ϵ (via A'-strahlung). Spectral positivity implies the missing-momentum tails respect the EFT-forward bounds (no negative bins).

Displaced and LLP programs. For $c\tau$ in mm–km, displaced vertices and far detectors are decisive:

$$N_{\rm dec} \;=\; N_{\rm prod} \times \mathcal{A} \times \Big(e^{-L_{\rm min}/\bar{\ell}} - e^{-L_{\rm max}/\bar{\ell}}\Big), \qquad \bar{\ell} = \gamma\beta\, c\tau, \tag{19.56}$$

with acceptance $\mathcal{A}$ and geometric baselines $(L_{\rm min}, L_{\rm max})$. Eq. (19.56) is used identically for S, A', and N with their respective lifetimes.

Fixed-target. Production via proton bremsstrahlung and meson decays (for A') or semileptonic channels (for N) scales as $\propto \epsilon^2$ or $\propto |U_{\alpha N}|^2$; event yields again follow (19.56). The geometric $1/L^2$ loss favors light, long-lived states.

5. Precision and positivity gates for portals

Electroweak and Higgs bounds. Mixing with h modifies couplings $\kappa_V, \kappa_f \to \cos\theta\, \kappa_{V,f}$ and adds $\mathrm{BR}_{\rm inv/exo}$ through (19.49); thus

$$\textbf{Higgs-fit gate:} \qquad \cos^2\theta \sum_f \mathrm{BR}_f^{\rm SM}\, r_f \;+\; \mathrm{BR}_{\rm inv/exo} \;\le 1, \tag{19.57}$$

with r_f as in Eq. (19.11).

Cone-ordering with kinetic mixing. Diagonalizing (19.43) modifies the photon co-metric by $\mathcal{O}(\epsilon^2)$. The *cone LMIs* (Sec. 17.3) impose

$$b_\gamma(\epsilon) \;\le 0, \qquad c_\gamma(\epsilon) \;\le 0, \qquad b_\gamma c_\gamma - \tfrac{1}{4} d_\gamma^2 \;\ge 0 \quad \Rightarrow \quad |\epsilon| \;\lesssim\; \epsilon_{\rm cone}^{\rm max}, \tag{19.58}$$

with $\epsilon_{\rm cone}^{\rm max}$ a calculable geometric bound (typically much weaker than experimental limits but *absolute*).

Forward positivity. Integrating out a scalar S at tree level contributes $(H^\dagger H)^2$ and $(H^\dagger H)F^2$ operators with coefficients

$$C_H > 0, \qquad C_{HB/HW} \propto \sin^2\theta \ge 0, \tag{19.59}$$

consistent with forward-limit positivity (Part 17). Any portal fit demanding $C_{HB/HW} < 0$ is excluded.

6. Worked analytic planes and lifetimes

Higgs-portal plane $(m_S, \sin^2\theta)$**.** For $m_S < m_h/2$,

$$\mathrm{BR}(h \to SS) \;=\; \frac{\Gamma(h \to SS)}{\Gamma_{\rm tot}^{\rm SM} + \Gamma(h \to SS)}\,, \qquad \Gamma(h \to SS) \propto \lambda_{HS}^2, \tag{19.60}$$

while the *visible* S decays give $c\tau_S \propto \sin^{-2}\theta$ from (19.50). Thus invisible/exotic Higgs limits cap λ_{HS}, whereas LLP searches bound $\sin^2\theta$; their intersection selects a narrow strip.

Dark-photon plane $(m_{A'}, \epsilon^2)$**.** For $m_{A'} < 2m_\mu$, $\Gamma(A' \to e^+e^-)$ in (19.51) implies $c\tau_{A'} \approx (3\alpha\epsilon^2 m_{A'})^{-1}$. Fixed-target reach scales as $N_{\rm dec} \propto \epsilon^4$ at very short decay lengths (acceptance-limited) and transitions to $\propto \epsilon^2$ for decay-in-volume domination.

HNL plane $(m_N, |U|^2)$**.** Eq. (19.53) shows the steep m_N^5 scaling; meson-factory production $\propto |U|^2$ and displaced decays yield characteristic triangular sensitivity.

7. Global profiling with portals

Likelihood blocks. Extend Eq. (19.18) by adding blocks for exotic Higgs, mono-X, LLP, and fixed-target counts:

$$-2\ln\mathcal{L}_{\rm portal} = \sum_{c\in\text{colliders}} (\mathbf{X}_c^{\rm exp}-\mathbf{X}_c^{\rm th})^\top \Sigma_c^{-1} (\mathbf{X}_c^{\rm exp}-\mathbf{X}_c^{\rm th}) - 2\sum_{t\in\text{fixed-target}} \ln \mathrm{Pois}(N_t^{\rm obs} \mid N_t^{\rm th}(\mathbf{p}_{\rm portal})), \tag{19.61}$$

subject to the feasibility gates (19.57)–(19.59) and the master convex set $\mathcal{F}$ of Sec. 18.4.

8. Kill criteria (portal layer)

(K1) Cone LMIs violated after kinetic mixing diagonalization ⇒ exclude any ϵ beyond $\epsilon_{\rm cone}^{\max}$;

(K2) Higgs signal strengths require $\cos^2\theta \sum_f \mathrm{BR}_f^{\rm SM} r_f + \mathrm{BR}_{\rm inv/exo} > 1$ (violates (19.57));

(K3) Forward positivity demands $C_{HB/HW} \geq 0$ but fit prefers $C_{HB/HW} < 0$;

(K4) LLP fit demands lifetimes incompatible with computed $c\tau(\sin\theta, \epsilon, |U|^2)$ from (19.50), (19.51), (1 (19.62)

9. Summary

Spectral portals arise naturally as geometric overlaps of fiber isometries (vector portal), stiffness-induced couplings between the Higgs condensate and spectral scalars (Higgs portal), and overlap-generated Yukawas to spectral fermions (neutrino portal). Their collider and fixed-target phenomenology admits compact analytic control: closed-form widths and lifetimes, production scalings, and convex feasibility enforced by positivity and cone ordering. The parameter planes $(m_S, \sin^2\theta)$, $(m_{A'}, \epsilon^2)$, and $(m_N, |U|^2)$ connect directly to search programs across LHC/HL-LHC, lepton factories, and fixed-target setups, while remaining embedded in the causal and unitary structure dictated by the spectral model.

19.4 Event-generator notes

Aim. This section provides a practical, generator-agnostic pipeline for simulating the spectral model at parton and particle level. We formulate (i) the matching of spectral parameters to SMEFT/Warsaw operators and UFO implementation, (ii) EFT truncation and unitarity/positivity filters at generation time, (iii) parton-shower merging, reweighting, and morphing strategies, (iv) long-lived particle (LLP) handling for spectral portals, and (v) validation and reproducibility checks. Throughout, we enforce the S/G/TQ/D gates and cone LMIs described in Parts 17 and 18.

1. Model files and matching layers

UFO implementation. The collider-facing interface uses a `UFO` model with parameter vector

$$\mathbf{p}_{\rm spec} \mapsto \Big\{ C_i/\Lambda^2 \text{ (Warsaw dim-6)}, C_j^{(8)}/\Lambda^4, \kappa\text{-rescalings, portal params } (\sin\theta, m_S, \epsilon, m_{A'}, |U_{\alpha N}|, m_N) \Big\} \tag{19.63}$$

Matching matrices $\mathsf{M}^{(6,8)}$ from Secs. 19.1–19.3 set $C = \mathsf{M}\,\mathbf{p}_{\rm spec}$. Gauge bosons use the complex-mass scheme for width consistency; R_ξ gauges are supported for Ward tests.

Parameter cards (illustrative).

```
BLOCK SPECTRALINPUTS
  1  1.0e+03   # Lambda (GeV)
  2  0.00e+00  # sin(theta) (Higgs-portal mixing)
  3  1.00e-03  # eps (kinetic mixing)
  4  1.00e+01  # mA' (GeV)
  5  1.00e+01  # mS (GeV)
  6  1.00e-05  # |U_eN|^2
BLOCK WARSOW6
  1  0.0       # C_H
  2  0.0       # C_HD
  3  0.0       # C_HWB
  ...
```

2. EFT truncation, unitarity, and positivity at generation time

Amplitude expansion. For an observable bin b with characteristic partonic energy $\hat{s}$,

$$\mathcal{M} \;=\; \mathcal{M}_{\rm SM} + \sum_i \frac{C_i}{\Lambda^2}\,\mathcal{M}_i + \sum_j \frac{C_j^{(8)}}{\Lambda^4}\,\mathcal{M}_j^{(8)} + \cdots, \qquad \sigma_b \simeq \sigma_b^{\rm SM} + \sum_i \frac{C_i}{\Lambda^2}\sigma_{i,b}^{\rm int} + \Theta_{\rm quad}\sum_{i\le j}\frac{C_iC_j}{\Lambda^4}\sigma_{ij,b}^{\rm quad}. \tag{19.64}$$

Default collider fits use *interference-dominated* settings ($\Theta_{\rm quad} = 0$) unless high-energy tails are explicitly validated.

Validity envelope (unitarity). For channels with energy growth, impose for each generated phase-space point:

$$|\mathrm{Re}\, a_0(\hat{s})| \;\le\; \tfrac{1}{2}, \qquad a_0(\hat{s}) \;=\; \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, \mathcal{M}(\hat{s}, \cos\theta). \tag{19.65}$$

If violated, (i) either veto the phase-space point (*hard* unitarity cut), or (ii) project the parameter vector onto the nearest admissible point under the positivity wedge (see below) before event export.

Forward-positivity and cone LMIs. For $2 \to 2$ sub-amplitudes, enforce the convex inequalities derived in Secs. 17.2–17.3:

$$\mathrm{Hankel}(\{\mu_n\}) \succeq 0, \qquad b \le 0,\; c \le 0,\; bc - \tfrac{1}{4}d^2 \ge 0, \tag{19.66}$$

where (b, c, d) parametrize the transverse-photon co-metric shift. Events violating (19.66) are *killed* irrespective of likelihood weight.

3. Merging, scales, and PDFs

Scale setting. Recommended dynamical scales:

$$\mu_R = \mu_F = \frac{H_T}{2} \quad \text{(inclusive Higgs, mono-}X\text{)}, \qquad \mu_R = \mu_F = \frac{m_{VV}}{2} \ \text{(dibosons)},$$
$$\mu_R = \mu_F = \frac{m_{hh}}{2} \ \text{(di-Higgs)}, \qquad \mu_R = \mu_F = \sqrt{m_{V1} m_{V2}} \ \text{(VBS)}. \tag{19.67}$$

Use NNLO/NNPDF4.0 PDFs for hadron colliders and consistent $\alpha_s(\mu_R)$ evolution.

Shower merging. For `MG5_aMC@NLO`+`Pythia8`:

- **CKKW-L** (LO multi-jet): $Q_{\rm cut} \sim 20$–$30\,\text{GeV}$ at the LHC; 10–15 GeV at e^+e^- Higgs factories with ISR.
- **FxFx** (NLO merging): $Q_{\rm cut} \sim 25\,\text{GeV}$; check stability of differential K-factors vs. $Q_{\rm cut}$.

Ensure SMEFT counterterms are included in NLO runs where available; otherwise remain at LO with conservative theory uncertainties.

4. Reweighting and morphing

Matrix-element reweighting. Given an event with kinematics Φ generated at anchor parameters $\boldsymbol{\theta}_0$, the weight for target $\boldsymbol{\theta}$ is

$$w(\Phi; \boldsymbol{\theta} | \boldsymbol{\theta}_0) \;=\; \frac{|\mathcal{M}_{\rm SM}(\Phi) + \sum_i \theta_i \, \mathcal{M}_i(\Phi)|^2}{|\mathcal{M}_{\rm SM}(\Phi) + \sum_i \theta_{0,i} \, \mathcal{M}_i(\Phi)|^2}, \qquad \theta_i \equiv C_i/\Lambda^2. \tag{19.68}$$

To maintain truncation, set *interference-only* weights:

$$w_{\rm int}(\Phi) \;=\; 1 + 2\,\text{Re}\left[\frac{\sum_i (\theta_i - \theta_{0,i})\, \mathcal{M}_i \mathcal{M}_{\rm SM}^*}{|\mathcal{M}_{\rm SM}|^2}\right]. \tag{19.69}$$

Polynomial morphing. For n active SMEFT directions, choose $n{+}1$ anchor points $\{\boldsymbol{\theta}^{(a)}\}$ spanning the linear subspace. Any distribution admits the *morphing* form

$$\frac{d\sigma}{dX}(\boldsymbol{\theta}) \;\approx\; \sum_{a=0}^{n} \alpha_a(\boldsymbol{\theta}) \left.\frac{d\sigma}{dX}\right|_{\boldsymbol{\theta}^{(a)}}, \qquad \alpha_a(\boldsymbol{\theta}) \text{ linear in } \boldsymbol{\theta}, \tag{19.70}$$

ensuring fast scans without on-the-fly ME calls. Discard quadratic terms unless unitarity-vetted.

5. Long-lived particles (LLPs): portals

Lifetime handling. Use proper lifetimes from Sec. 19.3:

$$c\tau_S = \frac{\hbar c}{\sum_f \Gamma(S \to f)} \propto \sin^{-2}\theta, \quad c\tau_{A'} \simeq \frac{\hbar c}{\sum_f \Gamma(A' \to f)} \propto \epsilon^{-2}, \quad c\tau_N \simeq \frac{\hbar c}{\Gamma_N} \propto \left(\sum |U_{\alpha N}|^2\right)^{-1} m_N^{-5}. \tag{19.71}$$

In `Pythia8` enable particle decays with specified lifetimes and decay tables consistent with Eqs. (19.50), (19.51), (19.53). Ensure *lab-frame* decay lengths $\bar{\ell} = \gamma\beta c\tau$ are propagated to detector simulators.

Displaced acceptances. Yield in a fiducial decay volume $[L_{\min}, L_{\max}]$ follows Eq. (19.56). When reweighting lifetimes, rescale per-event survival/decay probabilities rather than the production cross section.

6. Validation suite

Gauge-parameter (Ward) tests. Generate samples in R_ξ gauges with several ξ values; the *sum* of event weights for IR-safe observables must be ξ-independent within MC errors. Numerical Ward identity: replace an external polarization $\epsilon^\mu(k)$ by k^μ and require the amplitude to vanish (up to FP/BRST contact terms in gauges with ghosts).

IR/UV and truncation checks.

- Stability under (μ_R, μ_F) variations by factors 2.
- Merging-scale scans ($Q_{\rm cut}$) with flat differential K-factors.
- Quadratic contamination: compare $\Theta_{\rm quad} = 0$ vs. 1 within unitarity envelope; large deviations flag EFT breakdown.

Cross-generator cross-checks. Validate key channels (e.g. $pp \to VV$, $pp \to hh$, $e^+e^- \to WW$) with `MG5_aMC`, `POWHEG`, and `Sherpa/Comix`; require agreement within scale/PDF systematics at LO/NLO where available.

7. Recommended benchmark grids

Higgs/EW set.

$$\Big\{\delta\kappa_V,\ \delta\kappa_t,\ c_g,\ c_\gamma,\ C_{HWB}, C_{HD}\Big\}$$

spanned by $\{\boldsymbol{\theta}^{(a)}\}_{a=0}^{6}$ anchors for morphing of h signal strengths and diboson shapes.

VBS set. Two-dimensional grid in $(f_{S,0}, f_{S,1})$ with positivity wedge and partial-wave cap (19.32) enforced bin-by-bin.

Portal sets. Planes $(m_S, \sin^2\theta)$, $(m_{A'}, \epsilon^2)$, $(m_N, |U|^2)$ with lifetimes from (19.71); provide both prompt and LLP cards.

8. Detector interfaces and reproducibility

Detector simulation. Export `LHE` or `HepMC` with truth-level lifetimes and status codes, pass to `Delphes`/`Geant4` configurations including LLP volumes. Use analysis object definitions aligned with experimental selections (jet radius, lepton isolation, b-tag WPs).

Seeds and metadata. Record per-sample: generator version, PDF set and α_s, tune, scales, $Q_{\rm cut}$, kinematic cuts, unitarity/positivity flags, random seeds, and the exact parameter card. Embed the spectral parameter vector $\mathbf{p}_{\rm spec}$ as a comment block inside LHE for full provenance.

9. Quick-reference: kill criteria for MC exports

(K1) $\exists$ phase-space points with $|\mathrm{Re}\, a_0| > \frac{1}{2}$ $\Rightarrow$ veto or project before showering;

(K2) Forward-positivity/Hankel or cone LMIs violated $\Rightarrow$ discard sample (non-causal);

(K3) Gauge-parameter variation changes an IR-safe observable beyond MC error $\Rightarrow$ reject UFO/poin

(K4) Large negative weights in interference-only runs not attributable to NLO subtraction $\Rightarrow$ tighter

(K5) LLP lifetimes used in Pythia not consistent with analytic widths $\Rightarrow$ recompute tables from Eqs

(19.72)

10. Summary

A consistent event-generation pipeline for the spectral model proceeds by: (i) linear matching to SMEFT and portals in a UFO; (ii) enforcing truncation, unitarity, and convex positivity/LMIs at generation time; (iii) adopting stable merging and scale choices; (iv) using interference-based reweighting and morphing; (v) implementing LLP lifetimes from analytic widths; and (vi) validating gauge independence and EFT control. This ensures that simulated samples remain causal, unitary, and directly comparable across LHC/HL-LHC, lepton factories, and fixed-target experiments.

Chapter 20

Low-Energy and Cosmic Probes

20.1 EDMs, β-decay, and $g-2$

Aim. This section develops the low-energy (MeV–GeV) interface of the spectral framework for three precision frontiers: (i) permanent electric dipole moments (EDMs), (ii) nuclear β-decay correlations, and (iii) lepton anomalous magnetic moments. We give a complete operator basis at $\mu \simeq 2$ GeV, the renormalization-group (RG) map from spectral sources to hadronic/atomic observables, and convex feasibility gates compatible with the S/G/TQ/D principles. Where useful, we provide closed analytic expressions and dispersive representations linked to the spectral functions introduced in Parts 18 and 17.

1. CP/TP sources and low-energy operator basis

Spectral sources of CP violation. Within the Quantum–Aether model, CP-odd phases originate from: (i) *spectral phases* of the complex order parameter Φ and parity-odd components of the director sector χ^μ, (ii) Wess–Zumino–Witten (WZW) and Chern–Simons functionals induced by topological textures on $\mathcal{F}$ (Part 5), and (iii) complex overlaps in the Yukawa maps on the fiber (Part 14). At scales below the electroweak threshold, these feed into the standard CP-odd effective operators

$$\mathcal{L}^{\text{low}}_{\not{C}\not{P}} = -\frac{i}{2}\, d_\ell\, \bar{\ell}\, \sigma^{\mu\nu}\gamma_5\, \ell\, F_{\mu\nu} - \frac{i}{2}\, d_q\, \bar{q}\, \sigma^{\mu\nu}\gamma_5\, q\, F_{\mu\nu} - \frac{i}{2}\, \tilde{d}_q\, \bar{q}\, \sigma^{\mu\nu}\gamma_5\, T^A q\, G^A_{\mu\nu} \tag{20.1}$$

$$+ \frac{1}{6}\, w\, f^{ABC}\, G^A_{\mu\rho}\, G^{B\,\rho}{}_{\nu}\, \tilde{G}^{C\,\mu\nu} + \sum_i C_i^{(4f)}\, \mathcal{O}_i^{(4f)}\,, \tag{20.2}$$

where d_ℓ and d_q are lepton and quark EDMs, $\tilde{d}_q$ the quark chromo-EDMs (CEDMs), w the Weinberg three-gluon operator coefficient, and $\mathcal{O}^{(4f)}$ denote CP-odd four-fermion semileptonic and purely hadronic operators (scalar, pseudoscalar, and tensor types). Matching to (**??**) occurs at the electroweak scale from spectral SMEFT coefficients (e.g. $C_{e\gamma}$, C_{uG}, C_{Hud}, $C^{(1,3)}_{\ell equ}$), followed by QCD/QED running.

2. Spectral→SMEFT→low matching and RG

Matching at the EW scale. Let $\mathbf{C}^{(6)}(\Lambda)$ denote the SMEFT Wilson vector (Warsaw basis) generated by spectral microphysics at scale Λ (Part 19). The one-step tree/one-loop

matching to (**??**) at $\mu = m_W$ yields, schematically,

$$d_\ell(m_W) = \frac{\sqrt{2}v}{\Lambda^2}\,\mathrm{Im}\,C_{e\gamma}^{\ell\ell} \;+\; \frac{e y_\ell}{16\pi^2}\,\frac{v^2}{\Lambda^2}\,\sum_a \kappa_a\,\mathrm{Im}\,C_a \;+\; \cdots\,, \tag{20.3}$$

$$\tilde{d}_q(m_W) = \frac{\sqrt{2}v}{\Lambda^2}\,\mathrm{Im}\,C_{uG,\,dG}^{qq} \;+\; \frac{g_s y_q}{16\pi^2}\,\frac{v^2}{\Lambda^2}\,\sum_b \tilde{\kappa}_b\,\mathrm{Im}\,C_b \;+\; \cdots\,, \tag{20.4}$$

$$w(m_W) = \frac{\alpha_s}{4\pi}\,\frac{v^2}{\Lambda^2}\,\sum_c \omega_c\,\mathrm{Im}\,C_c \;+\; \cdots\,, \tag{20.5}$$

with calculable coefficients $(\kappa_a, \tilde{\kappa}_b, \omega_c)$ (Barr–Zee and heavy-quark loops included). Spectral phases enter through $\mathrm{Im}\,C$; geometric alignment (custodial and fiber isometries) can suppress certain combinations.

QCD/QED running. Between m_W and $\mu \sim 2$ GeV,

$$\frac{d}{d\ln\mu}\begin{pmatrix} \vec{d}_q \\ \vec{\tilde{d}}_q \\ w \\ \vec{C}^{(4f)} \end{pmatrix} = \begin{pmatrix} \gamma_{dd} & \gamma_{d\tilde{d}} & \gamma_{dw} & \gamma_{d4f} \\ 0 & \gamma_{\tilde{d}\tilde{d}} & \gamma_{\tilde{d}w} & \gamma_{\tilde{d}4f} \\ 0 & 0 & \gamma_{ww} & \gamma_{w4f} \\ 0 & 0 & 0 & \gamma_{4f\,4f} \end{pmatrix} \begin{pmatrix} \vec{d}_q \\ \vec{\tilde{d}}_q \\ w \\ \vec{C}^{(4f)} \end{pmatrix}, \tag{20.6}$$

with γ the anomalous-dimension blocks (CEDM mixes into EDM; both mix into w at higher orders). For leptons, d_ℓ runs slowly under QED: $d_\ell(\mu) = \big(\alpha(\mu)/\alpha(m_W)\big)^{\gamma_e/\beta_e} d_\ell(m_W)$.

3. EDM observables: hadronic and atomic layers

Neutron EDM. At $\mu = 2$ GeV one finds the standard linear decomposition

$$d_n \;=\; \eta_u d_u + \eta_d d_d + \eta_s d_s + \tilde{\eta}_u \tilde{d}_u + \tilde{\eta}_d \tilde{d}_d + \tilde{\eta}_s \tilde{d}_s + \eta_w\, w, \tag{20.7}$$

with hadronic coefficients $(\eta, \tilde{\eta}, \eta_w)$ obtained from QCD sum rules or lattice. The Weinberg operator w often provides the leading sensitivity to CP phases in the colored spectral sector.

Diamagnetic atoms (Schiff moments). For ^{199}Hg, ^{225}Ra etc.,

$$d_A \;\simeq\; \kappa_S^A\, S_{\mathrm{nuc}} \;+\; \kappa_e^A\, d_e \;+\; \kappa_C^A\, C_S \;+\; \cdots\,, \tag{20.8}$$

where the nuclear Schiff moment S_{nuc} depends on CP-odd pion–nucleon couplings $\bar{g}_{0,1}$ generated by $(\tilde{d}_q, w)$ and hadronic four-fermion operators. The semileptonic scalar $C_S(\bar{e} i\gamma_5 e)(\bar{N}N)$ can be induced by spectral scalar exchange (Higgs portal).

Paramagnetic systems. For ThO, YbF, HfF^+, the observable is dominated by the *effective electron EDM*,

$$\omega_{\mathrm{eff}} \;\propto\; W_d\, d_e \;+\; W_S\, C_S, \tag{20.9}$$

with large molecular enhancement factors $W_{d,S}$. Combining paramagnetic and diamagnetic systems separates $(d_e,\, C_S)$ from hadronic sources.

Global EDM map. Define the vector of low-energy Wilsons

$$\mathbf{W}_{\rm CP} \;:=\; (d_e,\; d_{u,d,s},\; \tilde{d}_{u,d,s},\; w,\; C_S,\; C_T, \ldots)^{\top}. \tag{20.10}$$

Each EDM dataset $\{d_n, d_A, \omega_{\rm eff}\}$ provides linear measurements

$$\mathbf{D}_{\rm EDM} \;=\; \mathsf{E}\,\mathbf{W}_{\rm CP} \quad + \quad \text{nuisance (hadronic/atomic)}. \tag{20.11}$$

The spectral model predicts $\mathbf{W}_{\rm CP}$ in terms of spectral phases via (20.3)–(20.5) and the RG (20.6).

4. β-decay: correlations, Fierz term, and D coefficient

Low-energy Hamiltonian. For $d \to u\, e^- \bar{\nu}$ transitions,

$$\begin{aligned}\mathcal{H}_\beta \;=\; \frac{G_F V_{ud}}{\sqrt{2}} \Big[&(1+\epsilon_V)\, \bar{e}\gamma^\mu(1-\gamma_5)\nu\; \bar{u}\gamma_\mu(1-\gamma_5)d + \epsilon_S\, \bar{e}(1-\gamma_5)\nu\; \bar{u}d \\ &+ \epsilon_T\, \bar{e}\sigma^{\mu\nu}(1-\gamma_5)\nu\; \bar{u}\sigma_{\mu\nu}(1-\gamma_5)d + \epsilon_P\, \bar{e}(1-\gamma_5)\nu\; \bar{u}\gamma_5 d + \text{h.c.}\Big],\end{aligned} \tag{20.12}$$

where $(\epsilon_V, \epsilon_S, \epsilon_T, \epsilon_P)$ encode BSM deformations. In SMEFT, to leading order,

$$\epsilon_S \propto \frac{v^2}{\Lambda^2} C^{(1)}_{\ell equ}, \qquad \epsilon_T \propto \frac{v^2}{\Lambda^2} C^{(3)}_{\ell equ}. \tag{20.13}$$

Spectral CP phases generate $\mathrm{Im}\,\epsilon_{S,T}$ through complex overlap integrals on $\mathcal{F}$ (Part 14).

Observables. For allowed decays, the electron energy–angle distribution reads (neglecting recoil)

$$\frac{d\Gamma}{dE_e d\Omega_e} \;\propto\; p_e E_e (E_0 - E_e)^2 \left[1 + a\,\frac{\mathbf{p}_e\cdot\mathbf{p}_\nu}{E_e E_\nu} + b\,\frac{m_e}{E_e} + \mathbf{P}\cdot\left(A\,\frac{\mathbf{p}_e}{E_e} + B\,\frac{\mathbf{p}_\nu}{E_\nu} + D\,\frac{\mathbf{p}_e\times\mathbf{p}_\nu}{E_e E_\nu}\right)\right], \tag{20.14}$$

with polarization $\mathbf{P}$ and correlation coefficients

$$b \;\simeq\; \frac{2\,\mathrm{Re}(\epsilon_S)}{1+3\lambda^2}\,\frac{g_S}{g_V} + \frac{6\,\mathrm{Re}(\epsilon_T)}{1+3\lambda^2}\,\lambda\,\frac{g_T}{g_A}, \tag{20.15}$$

$$D \;\simeq\; \frac{2\,\mathrm{Im}(\epsilon_S)}{1+3\lambda^2}\,\frac{g_S}{g_V} + \frac{6\,\mathrm{Im}(\epsilon_T)}{1+3\lambda^2}\,\lambda\,\frac{g_T}{g_A}, \tag{20.16}$$

where $\lambda \equiv g_A/g_V$, and (g_S, g_T) are nucleon form factors. Thus b and D provide linear probes of Re and Im parts of $(\epsilon_{S,T})$. EDM and β-decay observables are complementary: EDMs scale as loop and hadronic enhanced combinations of Im Wilsons, while D is a direct *tree-level* T-odd probe.

5. Lepton $g-2$: dispersive and portal contributions

Vacuum polarization (VP). As in Sec. 18.1,

$$\Delta a_\ell^{\rm VP} \;=\; \frac{\alpha^2}{3\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}\,K_\ell(s)\,R_{\rm spec}(s), \qquad K_\ell(s) = \int_0^1 dx\,\frac{x^2(1-x)}{x^2 + (1-x)\,s/m_\ell^2}, \tag{20.17}$$

with $R_{\rm spec}(s)$ the spectral ratio including any spectral-sector hadrons/hidden states that couple to the photon.

Light-by-light (LbL) and Euler–Heisenberg. Nonlinear spectral optics contributes via the EH-like operators of Sec. 18.2:

$$\Delta a_\ell^{\rm LbL} \;\simeq\; \beta_1 \frac{\alpha^3}{\pi^2}\frac{m_\ell^2}{m_0^2} + \beta_2 \frac{\alpha^3}{\pi^2}\frac{m_\ell^2}{m_0^2}\ln\frac{m_0^2}{m_\ell^2} + \cdots\,, \tag{20.18}$$

where $(\beta_{1,2}) \geq 0$ by forward-positivity (Part 17), and m_0 is the onset scale of spectral nonlinearities.

Portal contributions. For a kinetically mixed dark photon A',

$$\Delta a_\ell^{A'} \;=\; \frac{\alpha\,\epsilon^2}{2\pi}\, f\!\left(\frac{m_{A'}}{m_\ell}\right), \qquad f(r) = \int_0^1 dx\, \frac{2x(1-x)^2}{(1-x)^2 + r^2 x}\,, \tag{20.19}$$

and for a mixed scalar S,

$$\Delta a_\ell^{S} \;=\; \frac{\sin^2\theta}{8\pi^2}\,\frac{m_\ell^2}{v^2}\, F_S\!\left(\frac{m_S}{m_\ell}\right), \tag{20.20}$$

with known loop functions F_S. Both (20.19)–(20.20) enter the global portal analysis of Sec. 19.3.

6. Convex feasibility gates and null tests

Positivity/analyticity. The forward-limit Hankel positivity of Sec. 18.2 implies

$$\Delta a_\ell^{\rm VP} \;\geq\; 0, \qquad \Delta a_\ell^{\rm LbL} \;\geq\; 0, \tag{20.21}$$

and the EH coefficients satisfy $\beta_1 \geq 0$, $\beta_1 + \beta_2 \geq 0$. Any fit requiring negative contributions from these pieces violates dispersion relations and is *killed*.

Cone ordering. Photon co-metric shifts induced by kinetic mixing or anisotropy obey the cone LMIs (Part 17.3), which cap superluminal deformations and thereby bound $|\epsilon|$ at the purely geometric level (independent of data).

EDM–β consistency. From (20.15)–(20.16), a large $\mathrm{Im}\,\epsilon_T$ that would generate observable D also induces, via loops and RG, nonzero $(\tilde{d}_q, w)$ and thus d_n. Schematically,

$$|D| \;\lesssim\; \mathcal{K}_T\,\frac{g_T}{g_A}\,\mathrm{Im}\,\epsilon_T \quad\Rightarrow\quad d_n \;\gtrsim\; \mathcal{C}_T\,\mathrm{Im}\,\epsilon_T, \tag{20.22}$$

for calculable $(\mathcal{K}_T, \mathcal{C}_T)$. Infeasible combinations (D, d_n) are excluded by RG consistency alone.

7. Worked likelihood and kill criteria

Block likelihood. Collect data vectors

$$\mathbf{X}_{\rm EDM} = (d_n,\; d_{\rm Hg},\; \omega_{\rm ThO}, \ldots), \quad \mathbf{X}_\beta = (b,\; D,\; a,\; A,\; B, \ldots), \quad \mathbf{X}_{g-2} = (\Delta a_e,\; \Delta a_\mu). \tag{20.23}$$

Predictions $\mathbf{X}_{\rm th}(\mathbf{W}_{\rm CP}, \epsilon, \sin\theta, \ldots)$ derive from (20.11), (20.15)–(20.16), and (20.17)–(20.20). The Gaussian/Poisson mixed likelihood is

$$-2\ln\mathcal{L}_{\rm lowE} \;=\; \sum_{Y\in\{\rm EDM,\beta,g-2\}} \Big(\mathbf{X}_Y^{\rm exp} - \mathbf{X}_Y^{\rm th}\Big)^{\top} \Sigma_Y^{-1} \Big(\mathbf{X}_Y^{\rm exp} - \mathbf{X}_Y^{\rm th}\Big) \tag{20.24}$$

subject to the convex feasibility set $\mathcal{F}$ (Hankel positivity, cone LMIs, RG consistency) defined in Sec. 18.4.

Kill criteria (low-energy layer).

(K1) $\Delta a_\ell^{\rm VP/LbL} < 0$ (violates (20.21));

(K2) Cone LMIs violated by kinetic mixing or anisotropy (superluminal photon cones);

(K3) EDM vs. β mismatch beyond RG envelopes, e.g. D large while d_n impossibly small per (20.22);

(K4) Hadronic sign patterns (d_n, d_A) incompatible with any ($\tilde{d}_q$, w, C_S) within the RG/positivity co (20.25)

8. Summary

We have established a complete and admissible map from spectral CP/TP sources to the low-energy observables probed by EDMs, nuclear β-decay, and lepton g–2: spectral phases feed SMEFT dipoles and four-fermion operators, which run into $(d_\ell, d_q, \tilde{d}_q, w, C_S, \ldots)$; EDMs provide loop- and hadronic-enhanced probes of Im Wilsons, while β correlations (b, D) isolate Re/Im parts at tree level; $g-2$ receives dispersive (VP/LbL) and portal-induced positive contributions bounded by forward positivity and cone ordering. A single convex feasibility layer (Hankel + LMIs + RG consistency) supplies robust null tests and kill criteria that are independent of experimental central values, closing the precision loop of the spectral model at low energies.

20.2 Neutrino facilities and long baseline

Aim. We build the long-baseline (LBL) neutrino interface of the spectral framework, from microphysical overlaps on the internal fiber $\mathcal{F}$ to oscillation Hamiltonians in terrestrial matter, to analytic appearance/disappearance probabilities and facility-level event predictions. Our treatment (i) derives the effective propagation Hamiltonian including spectral nonstandard interactions (NSI) and tiny anisotropy-induced CPT-odd terms; (ii) provides controlled series for $P_{\mu e}$, $P_{\mu\mu}$ up to $\mathcal{O}(\alpha, s_{13}^2, \varepsilon)$ with $\alpha \equiv \Delta m_{21}^2/\Delta m_{31}^2$; (iii) maps spectral parameters to NSI Wilsons; (iv) states convex feasibility gates (probability unitarity, forward-positivity, cone LMIs); and (v) formulates an experiment-agnostic likelihood suitable for present and future LBL facilities.

1. Propagation Hamiltonian and spectral deformations

Vacuum term. Let U be the PMNS matrix (possibly non-unitary at $\mathcal{O}(v^2/\Lambda^2)$, see Part 14) and $M^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$ the Majorana mass eigenvalues supplied by the spectral see-saw (Sec. 14.3). In the flavor basis $\nu_\alpha = (\nu_e, \nu_\mu, \nu_\tau)$ the vacuum Hamiltonian

is

$$H_{\rm vac} \;=\; \frac{1}{2E}\,U\,M^2\,U^\dagger \;=\; \frac{\Delta m^2_{31}}{2E}\left[U\begin{pmatrix}0 & & \\ & \alpha & \\ & & 1\end{pmatrix}U^\dagger\right], \qquad \alpha \;\equiv\; \frac{\Delta m^2_{21}}{\Delta m^2_{31}}. \tag{20.26}$$

Matter potential with NSI. Coherent forward scattering in ordinary matter adds

$$H_{\rm mat} \;=\; V_{CC}\begin{pmatrix}1+\varepsilon_{ee} & \varepsilon_{e\mu} & \varepsilon_{e\tau}\\ \varepsilon^*_{e\mu} & \varepsilon_{\mu\mu} & \varepsilon_{\mu\tau}\\ \varepsilon^*_{e\tau} & \varepsilon^*_{\mu\tau} & \varepsilon_{\tau\tau}\end{pmatrix}, \qquad V_{CC} \;\equiv\; \sqrt{2}\,G_F\,n_e, \tag{20.27}$$

where the Hermitian $\varepsilon_{\alpha\beta}$ encode NSI with electrons, up quarks, and down quarks via $n_e \to n_e + n_u\,Y_u + n_d\,Y_d$ if desired. In our spectral model, these entries arise from integrating out fiber-borne mediators (vector/scalar portals of Sec. 19.3); see the map below.

Tiny CPT-odd spectral anisotropy. Couplings of neutrino currents to the director background χ^μ (Part I) induce CPT-odd, energy-independent potentials

$$(H_{\rm aniso})_{\alpha\beta} \;=\; a_{\alpha\beta,\mu}\,u^\mu \;\approx\; a_{\alpha\beta,0}\,, \tag{20.28}$$

with u^μ the aether rest frame and $a_{\alpha\beta,0} = a^*_{\alpha\beta,0}$ Hermitian. Cone LMIs (Sec. 17.3) and atomic-clock bounds (Sec. 18.3) enforce $|a_{\alpha\beta,0}| \ll V_{CC}$ for terrestrial LBL; we retain (20.28) only at linear order for degeneracy diagnostics.

Total Hamiltonian.

$$H \;=\; H_{\rm vac} \;+\; H_{\rm mat} \;+\; H_{\rm aniso} \quad\Longrightarrow\quad i\frac{d}{dx}\,\nu \;=\; H\,\nu, \qquad \bar\nu:\; H \;\to\; H|_{\delta\to-\delta,\;\varepsilon\to\varepsilon^*,\;a\to-a}. \tag{20.29}$$

For layered Earth profiles, we use piecewise-constant n_e with exact evolution via Cayley–Hamilton exponentiation per layer.

2. Spectral→NSI map and admissible sets

Vector portal (dark Z'). A fiber isometry current J^μ_D mixed with hypercharge (Sec. 19.3) generates at low energies

$$\mathcal{L}^{(V)}_{\rm NSI} \;=\; -\frac{g_D^2}{m^2_{Z'}}\,(\bar\nu_\alpha\gamma_\mu P_L\nu_\beta)\sum_{f=e,u,d}\left(q_f\,\bar f\gamma^\mu f\right), \qquad \varepsilon^{(f)}_{\alpha\beta} \;=\; \frac{g_D^2\,q_f}{2\sqrt{2}\,G_F\,m^2_{Z'}}\,, \tag{20.30}$$

with q_f set by the embedding of f in $G_{\rm dark}$. Forward-positivity of $\nu f \to \nu f$ scattering requires the Hermitian matrix $\varepsilon_{\alpha\beta}$ to be positive semidefinite in the forward limit for purely vector exchange.

Scalar portal. A mixed spectral scalar S (Sec. 19.3) coupled as $y^{\alpha\beta}_\nu \bar L_\alpha H \nu_{\beta R} + y^S_f \bar f f S$ yields, after integrating out S,

$$\mathcal{L}^{(S)}_{\rm NSI} \;=\; -\frac{y^{\alpha\beta}_\nu\,y^S_f}{m^2_S}\,(\bar\nu_\alpha\nu_\beta)\,(\bar f f), \qquad \varepsilon^{(f)}_{\alpha\beta} \;=\; \frac{y^{\alpha\beta}_\nu\,y^S_f}{2\sqrt{2}\,G_F\,m^2_S}\,\frac{\langle\bar f f\rangle}{n_e}\,, \tag{20.31}$$

which inherits sign constraints from forward-positivity in the scalar channel (Part 17); the effective ε are suppressed in ordinary matter unless y^S_f couples to electrons.

Admissible set.

$$\mathcal{A}_{\rm NSI} := \Big\{ \varepsilon \text{ Hermitian} \,\Big|\, \varepsilon \succeq 0 \text{ for pure vector exchange, } \|\varepsilon\| \ll 1, \text{ cone LMIs satisfied for } a_{\alpha\beta,0} \Big\}. \tag{20.32}$$

3. Analytic appearance/disappearance probabilities

Notation. Define $s_{ij}=\sin\theta_{ij}$, $c_{ij}=\cos\theta_{ij}$, Dirac phase δ,

$$\Delta := \frac{\Delta m^2_{31} L}{4E}, \qquad A := \frac{2EV_{CC}}{\Delta m^2_{31}}, \qquad \hat{\varepsilon}_{\alpha\beta} := \frac{\varepsilon_{\alpha\beta} V_{CC}}{\Delta m^2_{31}/(2E)} = A\,\varepsilon_{\alpha\beta}. \tag{20.33}$$

Golden channel $P_{\mu e}$ with NSI and tiny $a_{\alpha\beta,0}$. Keeping $\mathcal{O}(\alpha, s^2_{13}, \hat{\varepsilon})$ and linear a_0 terms,

$$\begin{aligned} P_{\mu e} \;\simeq\; & 4\, s^2_{13} s^2_{23} \frac{\sin^2[(1-A)\Delta]}{(1-A)^2} \;+\; \alpha^2 \sin^2 2\theta_{12}\, c^2_{23} \frac{\sin^2(A\Delta)}{A^2} \\ & +\; 2\alpha s_{13} \sin 2\theta_{12} \sin 2\theta_{23} \frac{\sin(A\Delta)}{A} \frac{\sin[(1-A)\Delta]}{(1-A)} \cos\!\big(\Delta+\delta\big) \\ & +\; 4\, s_{13} s_{23} \frac{\sin[(1-A)\Delta]}{(1-A)} \Big[c_{23}\, \mathrm{Re}\,\hat{\varepsilon}_{e\mu} \sin(A\Delta) - s_{23}\, \mathrm{Re}\,\hat{\varepsilon}_{e\tau} \sin(A\Delta) \Big] \\ & -\; 4\, s_{13} s_{23} \frac{\sin[(1-A)\Delta]}{(1-A)} \Big[c_{23}\, \mathrm{Im}\,\hat{\varepsilon}_{e\mu} \big(1-\cos(A\Delta)\big) - s_{23}\, \mathrm{Im}\,\hat{\varepsilon}_{e\tau} \big(1-\cos(A\Delta)\big) \Big] \\ & +\; 2\,\Delta \Big[c_{23}\, a_{e\mu,0} - s_{23}\, a_{e\tau,0} \Big] \frac{\sin[(1-A)\Delta]}{(1-A)} \;+\; \mathcal{O}(\hat{\varepsilon}^2, a_0^2). \end{aligned} \tag{20.34}$$

For antineutrinos: $A \to -A$, $\delta \to -\delta$, $\hat{\varepsilon} \to \hat{\varepsilon}^*$, $a_0 \to -a_0$.

Survival $P_{\mu\mu}$ (leading terms).

$$\begin{aligned} P_{\mu\mu} \;\simeq\; & 1 - \sin^2 2\theta_{23} \sin^2\Delta - 4\, s^2_{13} s^2_{23} \frac{A}{1-A} \sin^2\Delta \;-\; 2\,\mathrm{Re}\,\hat{\varepsilon}_{\mu\tau} \sin 4\theta_{23} \sin^2\Delta \\ & -\; \mathrm{Im}\,\hat{\varepsilon}_{\mu\tau} \sin 2\theta_{23} \sin 2\Delta \;+\; 2\,\Delta\, a_{\mu\mu,0} \sin 2\Delta \;+\; \cdots . \end{aligned} \tag{20.35}$$

Equations (20.34)–(20.35) suffice to expose parameter degeneracies and baseline/energy optimization.

4. Baseline/energy optimization and degeneracies

Magic baseline. Terms proportional to α vanish when $\sin(A\Delta) = 0$, giving the *magic* condition

$$A\Delta = n\pi \;\Rightarrow\; L_{\rm magic} \simeq \frac{n\pi}{\sqrt{2}\, G_F\, n_e} \approx (7.5\text{–}8.0) \times 10^3 \text{ km } (n=1), \tag{20.36}$$

weakly dependent on energy through n_e. At $L_{\rm magic}$, CP sensitivity from α-interference is suppressed, isolating mass ordering and NSI.

Octant and hierarchy–δ degeneracies. The pairs $(\theta_{23}, \pi/2 - \theta_{23})$ and $(\mathrm{NH}, \delta) \leftrightarrow (\mathrm{IH}, \pi - \delta)$ generate near-identical probabilities at fixed (L, E) in the absence of NSI. A second baseline and the $\nu/\bar{\nu}$ comparison break these degeneracies; the $\hat{\varepsilon}_{e\mu}, \hat{\varepsilon}_{e\tau}$ terms shift the effective CP-phase by

$$\delta_{\mathrm{eff}} \;\approx\; \delta \;+\; \arg\Big[s_{23}\hat{\varepsilon}_{e\mu} - c_{23}\hat{\varepsilon}_{e\tau}\Big], \tag{20.37}$$

emphasizing the need for joint fits including NSI directions.

5. Event predictions and likelihood

Event rates. For a detector mass M, live time T, flux Φ_{ν_μ}, efficiency ϵ_b, and cross section $\sigma^{\mathrm{CC}}_{\nu_e}$ in bin b of $(E, \cos\theta)$,

$$N_b(\mathbf{p}) \;=\; MT \int_b dE\, d\Omega\; \Phi_{\nu_\mu}(E,\Omega)\; \sigma^{\mathrm{CC}}_{\nu_e}(E)\; \epsilon_b(E,\Omega)\; P_{\mu e}(E, L; \mathbf{p}), \tag{20.38}$$

where $\mathbf{p}$ collects $\{\theta_{ij}, \delta, \Delta m^2_{ij}, \varepsilon_{\alpha\beta}, a_{\alpha\beta,0}\}$. Analogous expressions hold for $\bar{\nu}$ and other channels.

Layered matter profile. For K layers with densities $n_e^{(k)}$ and lengths L_k, the evolution operator is

$$S(L) \;=\; \prod_{k=1}^{K} \exp\Big[-i\, H(n_e^{(k)})\, L_k\Big], \qquad P_{\alpha\beta} = \Big|[S(L)]_{\beta\alpha}\Big|^2, \tag{20.39}$$

computed exactly per layer by diagonalization or via the Cayley–Hamilton form.

Likelihood. With Poisson counts and Gaussian systematics,

$$-2\ln\mathcal{L}_{\mathrm{LBL}} \;=\; \sum_b 2\Big[N_b^{\mathrm{th}} - N_b^{\mathrm{obs}} + N_b^{\mathrm{obs}} \ln\frac{N_b^{\mathrm{obs}}}{N_b^{\mathrm{th}}}\Big] \;+\; \sum_j \frac{\xi_j^2}{\sigma_{\xi_j}^2}, \tag{20.40}$$

where ξ_j encode flux, cross section, and detector uncertainties (pulled in the prediction).

6. Feasibility gates and convex constraints

Probability unitarity and positivity. For each (E, L) layer,

$$0 \;\le\; P_{\alpha\beta} \;\le\; 1, \qquad \sum_\beta P_{\alpha\beta} = 1, \qquad H = H^\dagger, \tag{20.41}$$

violated if non-Hermitian ε or complex a_0 are (incorrectly) introduced.

Forward-positivity (matter NSI). The $\nu f \to \nu f$ forward amplitude implies positivity of the 3×3 Hermitian ε matrix for purely vector exchange (Sec. 17.2); scalar-induced NSI must satisfy the sign/size relations inherited from their spectral origin (20.31). We therefore enforce

$$\varepsilon \in \mathcal{A}_{\mathrm{NSI}} \quad (\text{Eq. (20.32)}), \tag{20.42}$$

as a convex feasibility set in fits.

Cone LMIs for a_0. Tiny CPT-odd $a_{\alpha\beta,0}$ must respect cone-ordering LMIs (Sec. 17.3) to prevent superluminal group velocities in the electromagnetic sector through kinetic mixing loops. We bound $|a_0|$ at the geometric level before statistical inference.

7. Worked limits and diagnostics

Near detector (ND) constraints. At $L \to 0$ the vacuum phases vanish and

$$P^{\mathrm{ND}}_{\alpha\beta} \;\simeq\; \delta_{\alpha\beta} \;-\; i\,L\,(H_{\mathrm{mat}} + H_{\mathrm{aniso}})_{\beta\alpha} \;+\; \mathcal{O}(L^2), \tag{20.43}$$

so ND spectra directly constrain *flavor off-diagonal* $\varepsilon_{e\mu}, \varepsilon_{e\tau}, \varepsilon_{\mu\tau}$ and $a_{\alpha\beta,0}$ without oscillation degeneracies.

Ordering diagnostic. The matter-resonant term in (20.34) peaks when $A \simeq \cos 2\theta_{13}$, producing opposite shifts for normal and inverted ordering; NSI phases emulate part of this effect, but the combined $(\nu, \bar{\nu})$ program and a second baseline de-correlate δ from ε via the $A \to -A$ flip.

Magic-baseline program. At L_{magic}, Eq. (20.36), CP-odd vacuum interference is suppressed, magnifying sensitivity to $(\varepsilon_{e\mu}, \varepsilon_{e\tau})$ and $a_{e\mu,0}, a_{e\tau,0}$ from the last two lines of (20.34).

8. Kill criteria (LBL layer)

(K1) $\exists\,(E, L) : \sum_{\beta} P_{\alpha\beta} \neq 1$ or $P_{\alpha\beta} \notin [0, 1] \;\Rightarrow$ reject (non-Hermitian or ill-posed H);

(K2) $\varepsilon \notin \mathcal{A}_{\mathrm{NSI}}$ (violates forward-positivity or convex bounds) $\;\Rightarrow$ exclude;

(K3) Cone LMIs broken by $a_{\alpha\beta,0} \;\Rightarrow$ exclude independently of fit quality;

(K4) ND vs. FD tension: parameters that fit FD spectra but violate ND linear constraints (20.43). (20.44)

9. Summary

The spectral framework yields a closed LBL chain: spectral overlaps on $\mathcal{F}$ generate vector/scalar portals $\Rightarrow$ NSI matrices ε and minuscule CPT-odd a_0; these enter the Hermitian propagation Hamiltonian (20.29), whose appearance/disappearance probabilities admit controlled expansions (20.34)–(20.35). Facility predictions follow from (20.38) with layered matter (20.39). A convex feasibility layer (unitarity, forward-positivity, cone LMIs) provides robust kill criteria (20.44) that are independent of experimental central values, enabling clean inference on spectral parameters in the global program of Part VII.

20.3 Astroparticle links and cosmic messengers

Aim. We develop the *cosmic interface* of the spectral framework by mapping geometric/portal parameters to observables in gamma rays, neutrinos, cosmic rays, CMB polarization, gravitational waves, supernovae/stellar cooling, and early-Universe probes. The analysis is formulated in terms of (i) propagation on spectral cones (Part 17), (ii)

parity-odd WZW/CS functionals (Part 5), (iii) vector/scalar portals (Sec. 19.3), and (iv) the convex feasibility gates (Hankel positivity, cone LMIs, unitarity) used throughout the volume.

1. Propagation on spectral cones and dispersive media

Photon and neutrino dispersion. In an isotropic cosmic medium (CMB/EBL/plasma) the effective quadratic action for the photon reads

$$\mathcal{L}_\gamma^{(2)} \;=\; -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;-\; \frac{1}{4}\,\delta\kappa^{\mu\nu}{}_{\alpha\beta}\,F_{\mu\nu}F^{\alpha\beta} \;+\; \frac{1}{2}\,\Pi(\omega,\mathbf{k})\,A_\mu A^\mu, \tag{20.45}$$

where $\delta\kappa$ is the (tiny) spectral co-metric deformation (Sec. 17.3) and Π the medium polarization (e.g. $\Pi \simeq \omega_p^2$ in a cold plasma with frequency ω_p). To leading order the group velocity is

$$v_\gamma(\omega) \;\simeq\; 1 - \frac{\omega_p^2}{2\omega^2} \;-\; \frac{1}{2}\,\hat{\mathbf{n}}^i\hat{\mathbf{n}}^j\,\delta c_{ij}, \qquad \delta c_{ij} \;\equiv\; \delta\kappa_{0i0j}, \tag{20.46}$$

with propagation unit vector $\hat{\mathbf{n}}$. Cone LMIs enforce $b_\gamma \le 0$, $c_\gamma \le 0$, $b_\gamma c_\gamma - \frac{1}{4}d_\gamma^2 \ge 0$ (Sec. 17.3), ensuring $v_\gamma \le 1$ and forbidding superluminal fronts.

For neutrinos, the flavor-basis Hamiltonian along a baseline L in a homogeneous medium is

$$H \;=\; \frac{1}{2E}UM^2U^\dagger \;+\; V_{CC}\,(1+\varepsilon) \;+\; a_0, \tag{20.47}$$

with NSI matrix ε and CPT-odd a_0 from director couplings (Sec. 20.2). Cosmological propagation uses $V_{CC}\to 0$ and redshifts $E\to E(1+z)$ in the optical-depth integrals below.

Cosmic optical depths. For messengers $X=\gamma,\nu$ emitted at redshift z_s with rest-frame energy $E_0 = E(1+z)$, the survival probability is

$$P_X(E,z_s) \;=\; \exp[-\tau_X(E,z_s)]\,, \qquad \tau_X(E,z_s) \;=\; \int_0^{z_s}\frac{dz}{(1+z)H(z)}\,n_{\rm targ}(z)\,\sigma_X(E_0;z)\,, \tag{20.48}$$

with $n_{\rm targ}$ the target density (EBL/CνB/plasma) and σ_X the relevant cross section possibly deformed by portal interactions.

2. Parity-odd optics and cosmic birefringence

Phason-induced $F\tilde{F}$ term. A spectral phason a (phase of Φ) generates

$$\mathcal{L}_{a\gamma\gamma} \;=\; \frac{1}{4}\,g_{a\gamma}\,a\,F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad \tilde{F}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{20.49}$$

For a homogeneous $a(t)$ the linear polarization angle of CMB/AGN photons rotates by

$$\Delta\alpha \;=\; \frac{1}{2}\,g_{a\gamma}\Big[a(t_0)-a(t_{\rm em})\Big] \;=\; \frac{1}{2}\,g_{a\gamma}\int_{t_{\rm em}}^{t_0}dt\,\dot{a}(t). \tag{20.50}$$

Frequency independence of $\Delta\alpha$ distinguishes it from Faraday rotation. Forward-positivity implies $g_{a\gamma}^2 \ge 0$; cone LMIs bound any accompanying parity-even δc distortions.

3. Photon–scalar mixing and TeV transparency

Mixing in cosmic magnetic domains. In a transverse magnetic field B_T over a domain of size $L_{\rm dom}$, the $(\gamma_\parallel, a)$ system evolves with mixing matrix

$$\mathsf{M} \;=\; \begin{pmatrix} \Delta_\gamma & \Delta_{a\gamma} \\ \Delta_{a\gamma} & \Delta_a \end{pmatrix}, \qquad \Delta_{a\gamma} = \frac{1}{2} g_{a\gamma} B_T, \quad \Delta_\gamma = -\frac{\omega_p^2}{2E}, \quad \Delta_a = -\frac{m_a^2}{2E}. \tag{20.51}$$

The single-domain conversion probability is

$$P_{\gamma\to a} \;=\; \sin^2(2\theta)\,\sin^2\!\big(\Delta_{\rm osc} L_{\rm dom}\big), \qquad \tan 2\theta = \frac{2\Delta_{a\gamma}}{\Delta_a - \Delta_\gamma}, \quad \Delta_{\rm osc} = \frac{1}{2}\sqrt{(\Delta_a-\Delta_\gamma)^2 + 4\Delta_{a\gamma}^2}. \tag{20.52}$$

The *critical energy*

$$E_c \;=\; \frac{|m_a^2 - \omega_p^2|}{2 g_{a\gamma} B_T}, \tag{20.53}$$

marks the onset of strong mixing ($E \gg E_c$). For N uncorrelated domains the mean survival probability of TeV photons subject to EBL attenuation τ_γ is

$$\big\langle P_{\gamma\to\gamma} \big\rangle \;\simeq\; \frac{1}{3} + \frac{2}{3}\,\exp\!\Big[-\frac{3}{2}\,\tau_\gamma(E, z_s)\Big] \quad \text{(maximal mixing, large } N\text{)}, \tag{20.54}$$

softening the exponential EBL cutoff. Positivity and LMIs place no obstruction to (20.52) provided $g_{a\gamma}$ originates from a causal UV completion (Part 17).

4. High-energy neutrinos and secret spectral interactions

Resonant scattering on the CνB. A light dark vector Z' coupled to ν via the vector portal (Sec. 19.3) induces $\nu\bar\nu \to \nu\bar\nu$ resonance at

$$E_{\rm res} \;\simeq\; \frac{m_{Z'}^2}{2 m_\nu (1+z)}\,, \tag{20.55}$$

leading to a dip in the diffuse spectrum. The optical depth for a flavor eigenstate ν_α is

$$\tau_{\nu_\alpha}(E, z_s) \;=\; \int_0^{z_s} \frac{dz}{(1+z) H(z)} \sum_\beta n_{\nu_\beta}(z) \int d\Pi\, \sigma_{\alpha\beta}^{Z'}\!\big(s = 2 m_\nu E (1+z)\big), \tag{20.56}$$

with a Breit–Wigner cross section

$$\sigma_{\alpha\beta}^{Z'}(s) \;=\; \frac{4\pi}{m_{Z'}^2}\,{\rm Br}(Z'\to\nu_\alpha\bar\nu_\alpha)\,{\rm Br}(Z'\to\nu_\beta\bar\nu_\beta)\,\frac{\Gamma_{Z'}^2/4}{(s-m_{Z'}^2)^2 + \Gamma_{Z'}^2 m_{Z'}^2}\,. \tag{20.57}$$

Forward-positivity of $\nu\nu$ scattering ensures the absence of pathologies; cone LMIs limit kinetic-mixing backreaction on the photon sector.

5. Stellar/SN cooling and longitudinal dark photons

Plasma production. In a star or supernova core of temperature T and plasma frequency ω_p, kinetically mixed dark photons A' are produced at a rate controlled by the *in-medium* mixing parameter

$$\epsilon_{\rm eff}^2(\omega) \;=\; \epsilon^2\,\frac{m_{A'}^4}{[(m_{A'}^2 - {\rm Re}\,\Pi_T)^2 + ({\rm Im}\,\Pi_T)^2]}\,, \tag{20.58}$$

for transverse modes (replace $T \to L$ for longitudinal). The volume emissivity is

$$Q_{A'} \;=\; \int \frac{d^3k}{(2\pi)^3}\, \epsilon_{\rm eff}^2(\omega)\, \Gamma_\gamma(\omega)\, \omega\, n_B(\omega;T), \tag{20.59}$$

with Γ_γ the photon absorption rate. Cooling constraints follow from $L_{A'} = \int Q_{A'}\, dV \lesssim L_{\rm cool}$ of the object. These bounds map to $(\epsilon, m_{A'})$ and, by Eq. (19.47), to spectral overlaps on $\mathcal{F}$.

6. Early-Universe probes: $\Delta N_{\rm eff}$ and spectral decays

Dark radiation. A relativistic spectral sector x decoupled at temperature T_D contributes

$$\Delta N_{\rm eff} \;=\; \frac{8}{7}\left(\frac{11}{4}\right)^{4/3} \left(\frac{g_{*s}(T_0)}{g_{*s}(T_D)}\right)^{4/3} \frac{\rho_x}{\rho_\gamma}\,, \tag{20.60}$$

with entropy factor g_{*s}. Vector/scalar portal strengths determine T_D through the freeze-out condition $n\langle\sigma v\rangle \simeq H$.

Spectral decays and CMB distortions. An unstable spectral particle X injecting energy $\dot{Q}/\rho_\gamma$ at redshift z yields μ- and y-type distortions

$$\mu \;\simeq\; 1.4 \int_{z_y}^{z_\mu} dz\, \frac{\dot{Q}}{\rho_\gamma}\, \mathcal{J}_\mu(z), \qquad y \;\simeq\; \frac{1}{4}\int_0^{z_y} dz\, \frac{\dot{Q}}{\rho_\gamma}\, \mathcal{J}_y(z), \tag{20.61}$$

with window functions $\mathcal{J}_{\mu,y}(z)$. Portal parameters $(\epsilon, \sin\theta, |U_{\alpha N}|)$ fix $\dot{Q}$ from $X \to \gamma\gamma, e^+e^-, \nu\bar{\nu}$.

7. Gravitational waves and topological textures

Chiral GW propagation. A gravitational CS term from phason backgrounds,

$$\mathcal{L}_{\rm gCS} \;=\; \frac{\alpha_{\rm g}}{4}\, a\, R\tilde{R}, \tag{20.62}$$

induces circular-polarization asymmetries and birefringent phase shifts for GWs:

$$\Delta\phi_{\rm GW} \;=\; \alpha_{\rm g} \int_{t_{\rm em}}^{t_0} dt\, \dot{a}(t)\, k\,, \tag{20.63}$$

with k the comoving wavenumber. Forward-positivity constrains $\alpha_{\rm g}^2 \geq 0$ while unitarity requires adiabatic evolution (no ghost crossing).

Defect-induced stochastic background. Global strings/walls from spectral phase textures produce a stochastic background

$$\Omega_{\rm GW}(f) \;\simeq\; \mathcal{A}\,(G\mu)^2\, \mathcal{S}\!\left(\frac{f}{f_*}\right), \tag{20.64}$$

with tension $G\mu$ set by the spectral stiffness (Part **??**) and shape $\mathcal{S}$ from loop/cusp statistics. Compatibility with PTA/LIGO/Virgo/KAGRA analyses imposes integral bounds on $\Omega_{\rm GW}$ over frequency bands.

8. Cosmic-ray LIV thresholds and cone gates

Vacuum Cherenkov and photon decay. If spectral cones yielded $c_{\rm matter} > c_\gamma$, a charged particle would radiate vacuum Cherenkov photons above

$$E_{\rm th}^{p\to p\gamma} \simeq \frac{m_p}{\sqrt{2\delta c}}, \qquad \delta c \equiv c_{\rm matter} - c_\gamma > 0, \tag{20.65}$$

while $c_\gamma > c_e$ would allow $\gamma \to e^+e^-$ above

$$E_{\rm th}^{\gamma\to e^+e^-} \simeq \frac{m_e}{\sqrt{\delta c_{\gamma e}}}, \qquad \delta c_{\gamma e} \equiv c_\gamma - c_e > 0. \tag{20.66}$$

Observation of photons up to $E_{\rm obs}$ and of UHECRs up to $E_{\rm UHE}$ then imposes the *geometric* bounds

$$\delta c_{\gamma e} \lesssim \frac{m_e^2}{E_{\rm obs}^2}, \qquad \delta c \lesssim \frac{m_p^2}{2E_{\rm UHE}^2}, \tag{20.67}$$

which our cone LMIs automatically satisfy when $|\delta c| \ll 1$ (Sec. 18.3).

9. Global cosmic likelihood

Datasets and forward maps. Define a block-diagonal likelihood combining independent cosmic probes:

$$-2\ln\mathcal{L}_{\rm cosmic} = \sum_{Y\in\{\gamma,\nu,{\rm SN/SC,CMB\ pol,GW,CMB\ spec}\}} \Big(\mathbf{X}_Y^{\rm exp} - \mathbf{X}_Y^{\rm th}(\mathbf{p}_{\rm spec})\Big)^\top \Sigma_Y^{-1} \Big(\mathbf{X}_Y^{\rm exp} - \mathbf{X}_Y^{\rm th}(\mathbf{p}_{\rm spec})\Big), \tag{20.68}$$

where $\mathbf{p}_{\rm spec}$ includes portal parameters $(g_{a\gamma}, m_a, \epsilon, m_{A'}, |U_{\alpha N}|, m_N)$, cone deformations (δc_{ij}), and topological inputs $(G\mu, \alpha_{\rm g})$. The forward maps are given by Eqs. (20.48), (20.52)–(20.54), (20.56)–(20.57), (20.59), (20.60)–(20.61), and (20.64).

10. Kill criteria (cosmic layer)

(K1) Cone LMIs violated $\Rightarrow$ exclude irrespective of data fits (prevents superluminal fronts).

(K2) LIV thresholds: if $\delta c_{\gamma e}, \delta c$ implied by a parameter point exceed bounds (20.67), kill.

(K3) Photon survival: $P_{\gamma\to\gamma}(E, z_s) > 1$ or negative optical depth in (20.48), kill (violates unitarity/pos

(K4) SN/stellar cooling: $L_{A'}(\epsilon, m_{A'}) > L_{\rm cool}$ from (20.59), kill.

(K5) Early Universe: $\Delta N_{\rm eff}$ or $\{\mu, y\}$ outside admissible envelopes (20.60)–(20.61), kill.

(K6) GW birefringence/CS: non-adiabatic evolution (sign-flip in kinetic term) or negative energy flux

(20.69)

11. Summary

Spectral microphysics leaves correlated fingerprints across cosmic messengers: parity-odd phason gradients rotate polarization and mix photons with scalars; vector portals reshape gamma-ray and neutrino optical depths (resonant ν absorption on the CνB); kinetically mixed dark photons drain stellar/SN cores; light spectral sectors alter $\Delta N_{\rm eff}$ and CMB spectral distortions; topological textures source a stochastic GW background; and cone LMIs protect causal propagation and forbid LIV pathologies. A single convex

feasibility layer—Hankel positivity, cone ordering, unitarity—combined with the block likelihood (20.68) enables robust, end-to-end inference of spectral parameters from the multi-messenger sky.

20.4 Joint likelihoods and inference

Aim. We formulate an end-to-end statistical framework that fuses the collider (Part 19), low-energy (Sec. 20.1), long-baseline neutrino (Sec. 20.2), and astroparticle/cosmic (Sec. 20.3) layers into a *single* constrained inference engine for the spectral model. The construction (i) defines a global parameter vector with hierarchical nuisances, (ii) specifies composite likelihoods and convex feasibility gates (S/G/TQ/D, Hankel positivity, cone LMIs, unitarity), (iii) derives analytic Fisher objects and profiling formulas, (iv) details constrained sampling/optimization, and (v) states discovery/limit procedures with proper coverage.

1. Global parameter vector and forward model

Spectral parameters. Let

$$\mathbf{p}_{\rm spec} := \Big(\underbrace{\mathbf{c}^{(6)}, \mathbf{c}^{(8)}}_{\text{SMEFT (Warsaw) at } \mu_0} ; \underbrace{\sin\theta,\ m_S,\ \epsilon,\ m_{A'},\ |U_{\alpha N}|,\ m_N,\ g_{a\gamma}, m_a}_{\text{portals}} ; \underbrace{\delta c_{ij}}_{\text{cone shifts}} ; \underbrace{\varepsilon_{\alpha\beta}, a_{\alpha\beta,0}}_{\text{NSI/CPT-odd}} \Big)^{\top}, \tag{20.70}$$

defined at a reference scale μ_0 (typically m_Z or 1 TeV). The *forward model* maps $\mathbf{p}_{\rm spec}$ to predicted data vectors:

$$\mathbf{X}_Y^{\rm th}(\mathbf{p}_{\rm spec}) \;=\; \mathcal{F}_Y\big(\mathbf{p}_{\rm spec}\big), \qquad Y \in \{\text{coll}, \text{lowE}, \text{LBL}, \text{cosmic}\}, \tag{20.71}$$

using the RG/matching chains of Parts 3, 18, 5, and Secs. 19.1–19.4, 20.1–20.3.

Hierarchical nuisances. Collect shared theory/experimental systematics as $\boldsymbol{\nu}$:

$$\boldsymbol{\nu} := \Big(\alpha_s,\ \text{PDF eigs},\ m_t,\ m_h,\ \text{lattice } g_{S,T},\ \text{flux norms},\ \text{detector scales},\ \ldots\Big)^{\top}. \tag{20.72}$$

They enter (20.71) and carry Gaussian (or GP) hyper-priors.

2. Composite likelihood and feasibility gates

Block likelihood. Define for each sector Y a (typically) Gaussian/Poisson likelihood,

$$-2\ln\mathcal{L}_Y \;=\; \big(\mathbf{X}_Y^{\rm exp} - \mathbf{X}_Y^{\rm th}\big)^{\top} \Sigma_Y^{-1} \big(\mathbf{X}_Y^{\rm exp} - \mathbf{X}_Y^{\rm th}\big) \quad \text{or} \quad 2\sum_b \left[N_b^{\rm th} - N_b^{\rm obs} + N_b^{\rm obs} \ln \frac{N_b^{\rm obs}}{N_b^{\rm th}} \right], \tag{20.73}$$

with Σ_Y the covariance (including correlations across observables within Y). The *global* likelihood multiplies all Y and the nuisance priors:

$$\mathcal{L}_{\rm glob}(\mathbf{p}_{\rm spec}, \boldsymbol{\nu}) \;=\; \prod_Y \mathcal{L}_Y\Big(\mathbf{X}_Y^{\rm exp} \,|\, \mathbf{X}_Y^{\rm th}(\mathbf{p}_{\rm spec}, \boldsymbol{\nu})\Big) \;\times\; \pi(\boldsymbol{\nu}) \;\times\; \mathbf{1}_{\mathcal{F}}(\mathbf{p}_{\rm spec}), \tag{20.74}$$

where $\mathbf{1}_{\mathcal{F}}$ is the *indicator* of the admissible convex set $\mathcal{F}$ enforcing S/G/TQ/D gates, forward Hankel positivity, cone LMIs, EFT/partial-wave unitarity, and RG consistency (Secs. 17.2–17.3, 18.4, 19.1, 19.3, 20.1–20.3).

Barrier representation. For gradient-based samplers/optimizers we replace the hard indicator by a logarithmic barrier:

$$\mathbf{1}_{\mathcal{F}}(\mathbf{p}) \;\rightsquigarrow\; \exp\Big[-\lambda_{\mathrm{bar}}\,\Phi(\mathbf{p})\Big], \quad \Phi(\mathbf{p}) \;:=\; -\sum_j \ln s_j(\mathbf{p}) \;-\; \sum_\ell \ln\det \mathsf{H}_\ell(\mathbf{p}), \tag{20.75}$$

where $\{s_j \geq 0\}$ are scalar slack variables (e.g. $(b, c, bc - \frac{1}{4}d^2)$ for cones) and $\{\mathsf{H}_\ell \succeq 0\}$ are Hankel matrices for forward-positivity; $\lambda_{\mathrm{bar}} > 0$ is annealed to large values.

3. Frequentist profiling and Bayesian posteriors

Profile likelihood. The test statistic for a subset $\boldsymbol{\theta}$ of parameters is

$$q(\boldsymbol{\theta}) \;:=\; -2\ln \frac{\max_{\mathbf{p}\setminus\boldsymbol{\theta},\,\boldsymbol{\nu}\,\in\,\mathcal{F}}\,\mathcal{L}_{\mathrm{glob}}(\mathbf{p},\boldsymbol{\nu})}{\max_{\mathbf{p},\,\boldsymbol{\nu}\,\in\,\mathcal{F}}\,\mathcal{L}_{\mathrm{glob}}(\mathbf{p},\boldsymbol{\nu})}\,. \tag{20.76}$$

Under regularity (interior point, identifiable directions) q follows a χ^2_k law (Wilks) with $k = \dim\boldsymbol{\theta}$; otherwise we calibrate by toys drawn from (20.74).

Posterior. With prior $\pi(\mathbf{p}_{\mathrm{spec}})$,

$$p(\mathbf{p}_{\mathrm{spec}}, \boldsymbol{\nu}\,|\,\mathrm{data}) \;\propto\; \mathcal{L}_{\mathrm{glob}}(\mathbf{p}_{\mathrm{spec}}, \boldsymbol{\nu})\;\pi(\mathbf{p}_{\mathrm{spec}})\;\pi(\boldsymbol{\nu}), \tag{20.77}$$

and the evidence $Z = \int d\mathbf{p}\,d\boldsymbol{\nu}\;\mathcal{L}_{\mathrm{glob}}\pi(\mathbf{p})\pi(\boldsymbol{\nu})$ enables Bayes factors for model comparison (e.g. portals on/off, NSI subsets).

4. Linearized sensitivity and Fisher geometry

Jacobian and Fisher. Near a point $(\mathbf{p}_\star, \boldsymbol{\nu}_\star)$ with Gaussian blocks,

$$\mathsf{J}_Y \;:=\; \frac{\partial \mathbf{X}_Y^{\mathrm{th}}}{\partial \mathbf{p}}\bigg|_\star, \qquad \mathsf{F} \;=\; \sum_Y \mathsf{J}_Y^\top\,\Sigma_Y^{-1}\,\mathsf{J}_Y \;+\; \mathsf{F}_{\mathrm{prior}}, \tag{20.78}$$

yields the Cramér–Rao bound $\mathrm{Cov}(\mathbf{p}) \simeq \mathsf{F}^{-1}$ subject to the tangent cone of $\mathcal{F}$ at $\mathbf{p}_\star$. Eigenvectors of F furnish *principal spectral directions* (most/least constrained combinations).

Asimov significance and CLs. For a nested hypothesis $H_1 \supset H_0$ with parameter shift $\Delta\mathbf{p}$,

$$Z_{\mathrm{A}} \;\simeq\; \sqrt{\Delta\mathbf{p}^\top\;\mathsf{F}\;\Delta\mathbf{p}}\,, \qquad \mathrm{CL_s}\ \text{via toys if boundaries of } \mathcal{F}\ \text{are active.} \tag{20.79}$$

5. Nuisance treatment, theory errors, and correlations

Gaussian pulls and shared nuisances. We parameterize systematics by *pulls* ξ_j entering predictions linearly or logarithmically, with quadratic penalties $\xi_j^2/\sigma_{\xi_j}^2$ in each block (cf. Eqs. (19.18), (20.40)). Shared nuisances (e.g. α_s, PDFs, $g_{S,T}$) are *global* and thus correlate $\mathcal{L}_{\mathrm{coll}}$, $\mathcal{L}_{\mathrm{lowE}}$, and $\mathcal{L}_{\mathrm{cosmic}}$.

Theory-uncertainty GPs. For smooth truncation/scale uncertainties, introduce Gaussian processes $\Delta_Y(X)$ with kernels K_Y over kinematic variables X, promoting $\mathbf{X}_Y^{\rm th} \to \mathbf{X}_Y^{\rm th} + \Delta_Y$ and adding the GP prior to $\pi(\boldsymbol{\nu})$. This captures shape systematics without double-counting bin-to-bin errors.

MC statistical uncertainties. Use Beeston–Barlow lite: per-bin Poisson constraints on MC template counts propagate finite-sample errors into the fit.

6. Constrained optimization and sampling

Unconstrained reparameterizations. For positivity variables (rates, widths, lifetimes) use log or `softplus`; for LMIs use a Cholesky factorization $\mathsf{H} = LL^\top$ to parametrize $\mathsf{H} \succeq 0$.

Solvers and samplers.

- **Profiling:** interior-point Newton with barrier (20.75), line-search safeguarded by feasibility checks (unitarity, EFT envelopes).
- **Sampling:** HMC/NUTS on whitened parameters with barrier terms; for multimodality (e.g. hierarchy degeneracies) employ parallel tempering.

RG/EFT validity loop. At each step, run the RG map and veto points that violate: partial-wave unitarity (Eq. (19.32)), Hankel positivity, or cone LMIs; if near a boundary, shrink step size or increase $\lambda_{\rm bar}$.

7. Joint posteriors, projections, and tension metrics

Joint contours. Report marginalized posteriors or profiled contours for parameter planes of interest, e.g. $(m_S, \sin^2\theta)$, $(m_{A'}, \epsilon^2)$, $(m_N, |U|^2)$, (C_{HWB}, C_{HD}), and NSI planes $(\varepsilon_{e\mu}, \varepsilon_{e\tau})$, always annotated with feasibility wedges from positivity/LMIs.

Synergy quantifiers. Define the *information gain* from adding a dataset block Y:

$$\mathcal{I}_Y \;:=\; \tfrac{1}{2}\ln\frac{\det \mathsf{F}_{\rm prior}}{\det \mathsf{F}_{\rm prior} + \sum_{Y'\le Y}\mathsf{F}_{Y'}}\,, \qquad \mathsf{F}_{Y'} = \mathsf{J}_{Y'}^\top \Sigma_{Y'}^{-1}\mathsf{J}_{Y'}. \tag{20.80}$$

Large $\mathcal{I}_Y$ signals true complementarity rather than redundancy.

Tension metrics. For two blocks A, B, the parameter-shift statistic

$$T^2 \;:=\; (\hat{\mathbf{p}}_A - \hat{\mathbf{p}}_B)^\top \Big(\mathrm{Cov}_A + \mathrm{Cov}_B\Big)^{-1}(\hat{\mathbf{p}}_A - \hat{\mathbf{p}}_B) \tag{20.81}$$

diagnoses hidden inconsistencies (e.g. EDM vs. β-decay, or LBL vs. cosmic ν).

8. Discovery, limits, and model comparison

Discovery. Use the profiled likelihood ratio (20.76) for signal-strength or portal-turn-on hypotheses; quote Z with toys if boundaries are active. Look-elsewhere corrections apply to broad mass scans (e.g. $m_{A'}$).

Limits. Provide one-sided CLs or Bayesian credible intervals under the feasibility prior $\mathbf{1}_{\mathcal{F}}$. Report both *expected* (Asimov) and *observed* limits, including nuisance impacts.

Model comparison. Compute evidences Z (nested sampling/Laplace) for alternative spectral sub-models (e.g. scalar-only vs. vector-only portals; NSI textures). Bayes factors quantify preference while penalizing superfluous directions (Occam).

9. Reproducibility checklist

(R1) Exact data vectors $\mathbf{X}_Y^{\mathrm{exp}}$ and covariances Σ_Y.
(R2) Full parameter card for $\mathbf{p}_{\mathrm{spec}}$, nuisance priors $\pi(\boldsymbol{\nu})$.
(R3) RG/matching scheme and scales; EFT truncation choices.
(R4) Feasibility gate definitions (Hankel moments, cone matrices, unitarity caps).
(R5) Solver/sampler versions, seeds, barrier schedule, convergence diagnostics.
(R6) Public notebooks/cards to reproduce all figures/tables.

(20.82)

10. Summary

We have cast the spectral program into a single constrained-inference problem: a composite likelihood over collider, low-energy, LBL, and cosmic datasets, multiplied by hierarchical nuisance priors and restricted to a convex feasibility set encoding S/G/TQ/D, positivity, cone ordering, unitarity, and RG consistency. Linearized Fisher geometry quantifies sensitivities and synergies; constrained profilers and HMC samplers deliver contours and discovery metrics with proper coverage near physical boundaries. This backbone supports both present constraints and timeline-aligned forecasts, ensuring that any claimed signal of spectral microphysics is simultaneously *statistically significant* and *theoretically admissible*.

Chapter 21

Numerical Protocols and Reproducibility

21.1 Spectral-lattice for gauge sectors

Aim. We construct a lattice regularization tailored to the spectral framework, enabling nonperturbative studies of the emergent gauge sector and its coupling to the internal fiber map $y : M \to \mathcal{F}$. Our goals are: (i) a gauge- and fiber-covariant discretization faithful to the continuum constructions of Part I; (ii) anisotropic support for the effective metric g_{eff} and director-induced co-metric deformations; (iii) HMC/heatbath algorithms with provable reversibility, area-preservation, and detailed balance; (iv) observables (Wilson loops, Polyakov lines, glueball correlators, spectral string tension) with controlled continuum extrapolations; (v) internal consistency gates (reflection positivity, unitarity of links, Ward checks) and a full reproducibility checklist.

1. Discretizing spacetime and the internal fiber

Anisotropic lattice. Discretize (M, g_{eff}) by a hyper-rectangular lattice $\Lambda = \{x\}$ with spacings a_μ (no sum) along coordinate axes $\mu = 0, 1, 2, 3$:

$$x \;=\; (n_0 a_0,\; n_1 a_1,\; n_2 a_2,\; n_3 a_3), \qquad n_\mu \in \mathbb{Z}, \qquad L_\mu = N_\mu a_\mu. \tag{21.1}$$

Anisotropy $\xi \equiv a_s/a_0$ (with $a_s = a_1 = a_2 = a_3$) will be nonperturbatively tuned below.

Fiber field and local frames. At each site x store a point $y(x) \in \mathcal{F}$ and (optionally) an orthonormal frame ${e_A}^a(y)$ for $(\mathcal{F}, h)$ so that $h_{AB} = {e_A}^a {e_B}^b \delta_{ab}$. Killing vectors $\xi_I^A(y)$ are supplied either analytically (for homogeneous $\mathcal{F}$) or tabulated/interpolated.

2. Link variables, plaquettes, and spectral weights

Links and plaquettes. The emergent gauge connection is represented by link variables $U_\mu(x) \in G$ on oriented bonds (x, μ), with plaquette

$$U_{\mu\nu}(x) \;=\; U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x). \tag{21.2}$$

We write tr for a fixed faithful representation (fundamental for $SU(N)$, unit charge for $U(1)$).

Spectral weights from $g_{\rm eff}$ and χ^μ. In the continuum, the gauge kinetic term is $\propto \kappa_G\, g_{\rm eff}^{\mu\alpha} g_{\rm eff}^{\nu\beta} \mathrm{Tr}\, \mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}$ with small director-induced perturbations $\delta\kappa^{\mu\nu}{}_{\alpha\beta}(\chi)$ (cf. Sec. 1.1). We discretize this by direction-dependent couplings $\beta_{\mu\nu}^{IJ}$:

$$S_G[U] \;=\; \sum_x \sum_{\mu<\nu} \sum_{I,J} \beta_{\mu\nu}^{IJ} \left(1 - \frac{1}{d_R}\,\mathrm{Re}\,\mathrm{tr}_R\Big[T_I U_{\mu\nu}(x)\Big]\right)\delta^J{}_I, \tag{21.3}$$

with d_R the rep dimension and

$$\beta_{\mu\nu}^{IJ} \;=\; \frac{2\,\kappa_G(\mathcal{S})}{g_{IJ}^2(\mu_0)}\,\frac{a_\mu a_\nu}{a_0 a_1 a_2 a_3}\left(g_{\rm eff}^{\mu\mu} g_{\rm eff}^{\nu\nu} + \delta\kappa^{\mu\nu}{}_{\mu\nu}\right) + \mathcal{O}(a^2). \tag{21.4}$$

For simple groups with diagonal gauge metric $g_{IJ}^2 = g_I^2 \delta_{IJ}$ the last Kronecker becomes implicit; mixed terms $I \neq J$ are available for kinetic mixing (e.g. $U(1)_Y$–$SU(2)$ via C_{HWB} matching).

3. Fiber-covariant differences and y–U coupling

Parallel transport on $\mathcal{F}$. To realize the covariant derivative $\mathcal{D}_\mu y^A = \partial_\mu y^A - \xi_I^A(y)\mathcal{A}_\mu^I$, define the *gauge-rotated neighbor*

$$y_\parallel(x+\hat\mu) \;:=\; U_\mu^{-1}(x)\cdot y(x+\hat\mu), \tag{21.5}$$

where "$\cdot$" is the G-action on $\mathcal{F}$. A fiber-covariant forward difference is then

$$(\mathcal{D}_\mu y)^A(x) \;:=\; \frac{1}{a_\mu}\,\Big[\,\exp_{y(x)}^{-1}\Big(y_\parallel(x+\hat\mu)\Big)\,\Big]^A, \tag{21.6}$$

using the exponential map on $(\mathcal{F}, h)$ to bring differences to $T_{y(x)}\mathcal{F}$. The leading-order lattice fiber action reads

$$S_y[y,U] \;=\; \frac{\kappa_y}{2}\sum_{x,\mu} a_0 a_1 a_2 a_3\; g_{\rm eff}^{\mu\mu}(x)\; h_{AB}\Big(y(x)\Big)\,(\mathcal{D}_\mu y)^A (\mathcal{D}_\mu y)^B + \sum_x a^4\, V_{\rm int}\Big(y;\mathcal{S}\Big). \tag{21.7}$$

Equations (21.3) and (21.7) together implement the minimal-coupling template (*cf.* Sec. 1.1) at finite a.

4. Algorithms: pure gauge and dynamical fiber

Pure gauge updates. For $G = SU(N)$ we use Cabibbo–Marinari heatbath on embedded $SU(2)$ subgroups with overrelaxation sweeps; for $U(1)$, exact angle heatbath. These satisfy detailed balance with respect to S_G in (21.3) for direction-dependent $\beta_{\mu\nu}$.

HMC for (y,U). Introduce momenta $\pi_\mu(x) \in \mathfrak{g}$ and $p_A(x) \in T^*_{y(x)}\mathcal{F}$ with Hamiltonian

$$\mathcal{H} \;=\; \frac{1}{2}\sum_{x,\mu}\mathrm{Tr}\,\pi_\mu^2 \;+\; \frac{1}{2}\sum_x h^{AB}(y)\,p_A p_B \;+\; S_G[U] \;+\; S_y[y,U]. \tag{21.8}$$

Leapfrog (or Omelyan) integration updates (U,π) via left-invariant flows ($\dot U = i\pi U$) and (y,p) via geodesic steps on $\mathcal{F}$ using the Levi–Civita connection of h. Forces require the

variations

$$\frac{\partial S_G}{\partial U_\mu} = -\sum_{\nu\neq\mu} \beta_{\mu\nu}\,\mathrm{Proj}_{\mathfrak{g}}\Big[U_\mu(x)U_\nu(x+\hat{\mu})\cdots - \text{h.c.}\Big], \tag{21.9}$$

$$\frac{\delta S_y}{\delta U_\mu} = -\kappa_y\, a^4\, g_{\text{eff}}^{\mu\mu}\,\mathrm{Proj}_{\mathfrak{g}}\Big[\xi_I(y)\, h_{AB}(y)\,(\mathcal{D}_\mu y)^B\Big]T^I, \qquad \frac{\delta S_y}{\delta y^A} = -\kappa_y a^4\,\nabla_\mu\Big(g_{\text{eff}}^{\mu\mu}(\mathcal{D}_\mu y)_A\Big) + \partial_A V_{\text{int}}, \tag{21.10}$$

where $\mathrm{Proj}_{\mathfrak{g}}$ denotes the traceless anti-Hermitian projection. A final Metropolis accept/reject restores exactness.

Gauge fixing (optional). Landau gauge for spectroscopic observables uses Fourier-accelerated steepest descent on $\mathcal{F}_{\text{gf}}[g] = -\sum_{x,\mu} \mathrm{Re\ tr}\,[g(x)U_\mu(x)g^\dagger(x+\hat{\mu})]$ with stopping tolerance $\|\partial_\mu A_\mu\| < \epsilon$.

5. Observables and scale setting

Wilson loops and string tension. Rectangular loops $W(R,T) = \langle \mathrm{tr}\,\mathcal{P}\prod U\rangle$ yield the static potential $V(R)$ via

$$V(R) \;=\; -\lim_{T\to\infty}\frac{1}{Ta_0}\ln\frac{W(R,T)}{W(R,T-a_0)} \;\simeq\; \sigma R + \mu + \frac{\gamma}{R} + \cdots, \tag{21.11}$$

with string tension σ estimated by Creutz ratios $\chi(R,T) = -\ln\frac{W(R,T)W(R-1,T-1)}{W(R,T-1)W(R-1,T)} \xrightarrow{R,T\gg 1} \sigma a_s^2$.

Polyakov loop and deconfinement. Temporal Polyakov loops $P(\mathbf{x})$ diagnose finite-T transitions on $N_0 \times N_s^3$ lattices with $T = 1/(N_0 a_0)$.

Glueball correlators and GEVP. Gauge-invariant operators $\mathcal{O}_i$ (Ape/Teper-smeared) produce matrices $C_{ij}(t) = \langle \mathcal{O}_i(t)\mathcal{O}_j(0)\rangle$. Solve the generalized eigenvalue problem $C(t)\,v_n = \lambda_n(t,t_0)C(t_0)\,v_n$ and extract masses from $\lambda_n(t,t_0) \sim e^{-m_n(t-t_0)}$.

Gradient flow scales. Evolve gauge links by Wilson flow to flow time t and define t_0 via $t^2\langle E(t)\rangle|_{t=t_0} = 0.3$. Use w_0 or t_0 to set the scale and monitor lattice artefacts.

Anisotropy tuning. Match temporal/spatial Wilson-loop ratios or meson dispersion relations to determine the renormalized anisotropy ξ_R and tune bare (a_0, a_s) to reach $\xi_R = \xi$.

6. Continuum, infinite-volume, and autocorrelations

Extrapolation strategy. Simulate at (a/L) triplets with fixed ξ, extrapolate

$$\mathcal{O}(a,L) \;=\; \mathcal{O}^{\text{cont}} \;+\; c_1 a^2 \;+\; c_2\frac{e^{-m_\pi L}}{(m_\pi L)^{3/2}} \;+\; \cdots, \tag{21.12}$$

using $O(a)$-improved actions where applicable.

Autocorrelations and errors. Integrated autocorrelation time

$$\tau_{\rm int}[\mathcal{O}] \;=\; \frac{1}{2}+\sum_{t=1}^{W}\rho_{\mathcal{O}}(t), \qquad \rho_{\mathcal{O}}(t)=\frac{\Gamma_{\mathcal{O}}(t)}{\Gamma_{\mathcal{O}}(0)}, \tag{21.13}$$

with window W chosen self-consistently. Use $\sigma^2_{\mathcal{O}} = 2\tau_{\rm int}\,\Gamma_{\mathcal{O}}(0)/N_{\rm cfg}$ and binning/jackknife to propagate to derived quantities. Mitigate topological freezing at small a via open boundaries, metadynamics, or tempering.

7. Consistency gates and kill criteria (numerics)

Reflection positivity and link unitarity. Reject runs where (i) average plaquette leaves $(0,1)$ bounds or (ii) $\langle\|U_\mu^\dagger U_\mu - \mathbb{1}\|\rangle > \varepsilon_{\rm unit}$ (floating-point drift).

Ward/Slavnov checks. In Landau gauge, verify lattice Ward identity $\langle\partial_\mu A_\mu\rangle \approx 0$ and BRST-inspired transversality of two-point functions up to discretization errors.

Anisotropy/positivity gates. Direction-dependent couplings must obey the cone LMIs of Sec. 17.3 mapped to the lattice: the symmetric matrix built from $\{\beta_{\mu\nu}\}$ has to be positive semidefinite; negativity flags acausal cones $\Rightarrow$ kill.

HMC stability. Monitor acceptance $P_{\rm acc}$ (target $\gtrsim 0.7$), Hamiltonian violations $\Delta\mathcal{H}$ (Gaussian with $\langle e^{-\Delta\mathcal{H}}\rangle = 1$), and reversibility. Failures $\Rightarrow$ decrease step size or increase integration order.

8. Matching to continuum couplings

Gradient-flow coupling. Define $g^2_{\rm GF}(L)$ by

$$g^2_{\rm GF}(L) \;:=\; \mathcal{N}\,t^2\langle E(t)\rangle\Big|_{t=c^2L^2/8}, \qquad \beta_{\rm step}(u,s) \;=\; \frac{g^2_{\rm GF}(sL)-g^2_{\rm GF}(L)}{\ln s^2}, \tag{21.14}$$

and determine the nonperturbative step-scaling function to match $\kappa_G/g_I^2(\mu)$ in Eq. (21.4).

Schrödinger functional (SF). With Dirichlet temporal boundaries, compute $g^2_{\rm SF}(L)$ and match to perturbation theory at small L for an absolute normalization (cross-check to $g^2_{\rm GF}$).

9. Minimal reproducible example (MRE)

Model card (YAML).

```
lattice:
  Ns: [32, 32, 32]; Nt: 96; as: 0.06; a0: 0.03  # fm
gauge:
  group: SU(3)
  beta_spatial: 6.10
  beta_temporal: 6.35
fiber:
```

```
  manifold: S3
  kappa_y: 1.0
  potential: "V = m^2 Tr(1 - y4) + lambda (1 - y4)^2"
hmc:
  traj_length: 1.0
  n_steps: 20
  integrator: Omelyan-2MN
meas:
  wilson_loops: [[R=2..10],[T=2..12]]
  polyakov: true
  glueball: ["A1++","E++","T2++"]
```

Outputs. Store configurations and measurements in HDF5 with metadata:

`/config/U`, `/fiber/y`, `/obs/plaquette`, `/obs/Wilson[R,T]`, `/obs/GEVP/eigvals`, `/meta/{beta,as,`

Provide checksums, random seeds, code commit hash, compiler flags, and container digest.

10. Summary

We presented a fiber- and gauge-covariant lattice formulation: anisotropic links with spectral weights (21.4), fiber-parallel differences (21.6), and robust HMC/heatbath algorithms (21.8)–(21.10). A standard spectroscopic toolkit (Wilson loops, Polyakov lines, glueball GEVP, gradient flow) enables precise determinations of the spectral string tension, glueball spectra, and nonperturbative running, with continuum and volume control (21.12). Consistency gates (reflection positivity, LMIs for anisotropic couplings, Ward checks) and an explicit MRE ensure reproducibility and theoretical admissibility, forming the numerical backbone for all gauge-sector claims of the spectral model.

21.2 RG solvers and matching tools

Aim. We specify the renormalization-group (RG) and matching machinery that transports spectral parameters from the UV (spectral scale Λ) to all phenomenological layers:

$$\text{Spectral UV} \xrightarrow{\text{match}} \text{SMEFT at } \mu=\Lambda \xrightarrow{\text{RG}} \text{SMEFT at } \mu\sim m_W \xrightarrow{\text{match}} \text{LEFT at } \mu\sim m_W \xrightarrow{\text{QCD/}}$$

We give (i) a state-vector formulation with sparse anomalous-dimension blocks and stiff, event-driven ODE integrators; (ii) threshold decoupling as linear maps with optional smooth profiles; (iii) one-loop (and key two-loop) matching formulae for gauge/Yukawa/self couplings and Wilson operators; (iv) scheme conversions and background-field normalization that preserve Ward identities; (v) error propagation, invariants, and kill criteria.

1. State vector and RG equations

State vector. Collect running parameters at scale μ into

$$\mathbf{y}(\mu) \;:=\; \Big(\underbrace{g_1,g_2,g_3}_{\text{gauge}}\;;\;\underbrace{Y_u,Y_d,Y_e}_{\text{Yukawa}}\;;\;\underbrace{\lambda,v,m^2}_{\text{Higgs}}\;;\;\underbrace{\mathbf{C}^{(6)},\mathbf{C}^{(8)}}_{\text{SMEFT Wilsons}}\Big)^{\top}, \tag{21.15}$$

with $\mathbf{C}^{(d)}$ stacked in a fixed basis (Warsaw for $d=6$, a curated subset for $d=8$ used in this volume).

RG form. Let $t \equiv \ln \mu$. The coupled RGEs are

$$\frac{d\mathbf{y}}{dt} \;=\; \boldsymbol{\beta}(\mathbf{y}) \;=\; \begin{pmatrix} \boldsymbol{\beta}_g \\ \boldsymbol{\beta}_Y \\ \boldsymbol{\beta}_H \\ \Gamma^{(6)}(\mathbf{y})\,\mathbf{C}^{(6)} + \mathbf{S}^{(6)}(\mathbf{y}) \\ \Gamma^{(8)}(\mathbf{y})\,\mathbf{C}^{(8)} + \mathbf{S}^{(8)}(\mathbf{y}, \mathbf{C}^{(6)}) \end{pmatrix}. \tag{21.16}$$

Here $\Gamma^{(d)}$ are the (block-sparse) anomalous-dimension matrices at the chosen loop order; $\mathbf{S}^{(d)}$ collect inhomogeneous pieces (e.g. running induced by products $C^{(6)}\times$SM couplings into $d=8$). For gauge couplings we use two-loop β's; for Yukawa and λ at least one-loop (two-loop optionally).

2. Solvers: stiffness, events, and sparsity

Event-driven integration. Thresholds $\{\mu_k\}$ (particle masses or user-defined decoupling scales) are treated as *events*. We integrate $d\mathbf{y}/dt = \boldsymbol{\beta}(\mathbf{y})$ piecewise on intervals (μ_{k+1}, μ_k) and apply matching maps at each μ_k (Sec. 21.2).

Stiff ODE integrators. The SMEFT Wilson sector can be stiff (hierarchical eigenvalues of Γ). We recommend L-stable linearly implicit Rosenbrock–W methods (ROS3/ROS4) or SDIRK(2/3):

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \sum_{i=1}^{s} b_i \mathbf{k}_i, \qquad \Big[\mathbb{1} - \gamma_i h \mathsf{J}_n\Big]\mathbf{k}_i = h\,\boldsymbol{\beta}\left(\mathbf{y}_n + \sum_{j<i} a_{ij}\mathbf{k}_j\right), \tag{21.17}$$
$$\mathsf{J}_n \equiv \left.\frac{\partial \boldsymbol{\beta}}{\partial \mathbf{y}}\right|_{\mathbf{y}_n}, \quad h = \Delta t.$$

Automatic differentiation (AD) supplies J_n exactly; sparse direct solvers (e.g. LDL$^\top$) exploit the block structure.

Step control and invariants. Adapt h by embedded error estimators; reject steps if feasibility (Sec. 21.2) is violated. Monitor RG invariants, e.g. $d(\det Y_u Y_u^\dagger)/dt = \mathrm{Tr}(\cdots)\det(\cdots)$, or the background-field identity for gauge couplings (below).

3. Background-field scheme and gauge normalization

Background-field method (BFM). Using BFM-MS$\overline{\text{MS}}$ ensures Ward identities that tie gauge coupling renormalization to the background two-point function:

$$\beta_g \;=\; \frac{g}{2}\,\gamma_{\hat{A}}, \qquad \gamma_{\hat{A}} \equiv -\mu\frac{d}{d\mu}\ln Z_{\hat{A}}, \tag{21.18}$$

where $\hat{A}$ is the background gauge field. This choice stabilizes the numerical flow and simplifies matching across thresholds (gauge coupling renormalization follows directly from vacuum polarization).

Scheme conversions. We provide converters between $\overline{\text{MS}}$ and on-shell inputs:

$$\Big\{m_Z^{\text{pole}}, m_W^{\text{pole}}, G_F, \alpha_{\text{Thomson}}\Big\} \;\longleftrightarrow\; \Big\{g_1(\mu), g_2(\mu), v(\mu), \lambda(\mu)\Big\}, \tag{21.19}$$

including one-loop finite parts and, optionally, known two-loop corrections for precision EW fits.

4. Thresholds, decoupling, and matching

Linear matching maps. At a threshold $\mu = \mu_k$ relating EFT$_+$ (above) to EFT$_-$ (below),

$$\mathbf{y}_-(\mu_k) \;=\; \mathsf{M}_k\, \mathbf{y}_+(\mu_k) \;+\; \mathbf{\Delta}_k, \tag{21.20}$$

with block structure:

$$\mathsf{M}_k = \text{diag}\Big(\mathsf{M}_g,\ \mathsf{M}_Y,\ \mathsf{M}_H,\ \mathsf{M}_{C^{(6)}},\ \mathsf{M}_{C^{(8)}}\Big), \quad \mathbf{\Delta}_k = (\Delta_g, \Delta_Y, \Delta_H, \Delta_{C^{(6)}}, \Delta_{C^{(8)}})^\top.$$

Gauge couplings (one-loop, heavy field h). For a heavy particle h in representation R of gauge factor G_i with Dynkin index $T(R)$,

$$\alpha_i^{-1}\Big|_- \;=\; \alpha_i^{-1}\Big|_+ \;-\; \frac{b_i^{(h)}}{2\pi}\,\ln\frac{\mu_k}{m_h}\,, \qquad b_i^{(h)} = \begin{cases} -\frac{4}{3}T(R) & \text{Dirac fermion,} \\ -\frac{1}{6}T(R) & \text{real scalar,} \\ \frac{11}{3}C_A & \text{(remove gauge adjoint).} \end{cases} \tag{21.21}$$

The background-field choice makes (21.21) sufficient at one loop; two-loop pieces can be added if needed.

Yukawa and λ thresholds. Integrating out a heavy scalar S mixing with H shifts λ and m^2:

$$\lambda_- \;=\; \lambda_+ \;-\; \frac{\kappa_{HS}^2}{2m_S^2} \;+\; \cdots, \qquad m_-^2 \;=\; m_+^2 \;-\; \frac{\kappa_{HS}\,\mu_{HS}}{m_S^2}\,v^2 \;+\; \cdots, \tag{21.22}$$

with portal couplings κ_{HS}, μ_{HS}. Fermionic thresholds (e.g. top) use the standard decoupling relations in $\overline{\text{MS}}$.

SMEFT $\to$ LEFT at $\mu{=}m_W$. Tree/one-loop matching yields LEFT Wilsons (dipoles and four-fermion) in terms of SMEFT:

$$C_{e\gamma}^{\text{LEFT}}(\mu_W) = \frac{v}{\sqrt{2}}\,C_{e\gamma}^{\text{SMEFT}}(\mu_W) + \frac{e\,y_\ell}{16\pi^2}\,v\,\Big(a_1 C_{H\ell}^{(1)} + a_2 C_{H\ell}^{(3)} + \cdots\Big), \tag{21.23}$$

$$C_{ud\ell\nu}^{V,LL}(\mu_W) = 2\sqrt{2}\,G_F V_{ud}\,(1+\epsilon_V), \qquad \epsilon_{S,T} \;\propto\; \frac{v^2}{\Lambda^2}\,\Big(C_{\ell equ}^{(1)}, C_{\ell equ}^{(3)}\Big), \tag{21.24}$$

consistent with Secs. 20.1–20.2.

Smooth decoupling (optional). To avoid solver chattering near dense thresholds, replace sharp maps by profiles

$$w_h(\mu) \;=\; \frac{1}{1+(\mu/m_h)^{\pm n}}, \qquad b_i \to b_i - w_h\, b_i^{(h)}, \tag{21.25}$$

and compute envelope uncertainties by comparing sharp vs. smooth schemes.

5. Spectral → SMEFT matching at $\mu = \Lambda$

Tree level (illustrative). A heavy vector V_μ^a (fiber isometry excitation) with coupling $g_* J_a^\mu V_\mu^a$ and mass M_V induces

$$\mathcal{L}_{\text{eff}} \supset -\frac{g_*^2}{2M_V^2} J_a^\mu J_\mu^a \longrightarrow \{C_{H\ell}^{(1,3)}, C_{Hq}^{(1,3)}, C_{\ell\ell}, C_{qq}, \ldots\} \propto \frac{g_*^2}{M_V^2}, \tag{21.26}$$

with group factors fixed by the embedding of SM currents in the fiber isometries (Part 7).

One loop (Barr–Zee class). Spectral scalars S with CP phases contribute to dipoles

$$C_{f\gamma}(\Lambda) \simeq \frac{Q_f\, y_f}{16\pi^2} \frac{\kappa_{HS}\, \sin\phi_S}{M_S^2} \mathcal{B}\left(\frac{m_t^2}{M_S^2}\right), \tag{21.27}$$

feeding EDMs and $g-2$ per Sec. 20.1. Similar loop dictionaries exist for C_{HW}, C_{HB}, C_{HWB}.

6. QCD/QED running to hadronic scales

Gauge couplings. Run α_s at four-loop (optionally) with decoupling at $\mu = \{m_t, m_b, m_c\}$. For QED, evolve α with leptonic and hadronic contributions (via $R(s)$ or Adler function tabulations if high precision is needed).

LEFT Wilsons. The quark EDM/CEDM/Weinberg system evolves as

$$\frac{d}{dt}\begin{pmatrix} \vec{d}_q \\ \vec{\tilde{d}}_q \\ w \end{pmatrix} = \begin{pmatrix} \gamma_{dd} & \gamma_{d\tilde{d}} & \gamma_{dw} \\ 0 & \gamma_{\tilde{d}\tilde{d}} & \gamma_{\tilde{d}w} \\ 0 & 0 & \gamma_{ww} \end{pmatrix} \begin{pmatrix} \vec{d}_q \\ \vec{\tilde{d}}_q \\ w \end{pmatrix}, \tag{21.28}$$

and mixes into hadronic four-fermion operators at subleading order.

7. Error propagation and sensitivities

Linear covariance flow. For small uncertainties and linear response,

$$\dot{\mathsf{C}} = \mathsf{J}\,\mathsf{C} + \mathsf{C}\,\mathsf{J}^\top + \mathsf{Q}, \qquad \mathsf{J} = \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{y}}, \quad \mathsf{Q} = \text{process noise (e.g. scheme truncation model)}, \tag{21.29}$$

solved alongside (21.16). For non-linear cases, sample hypercube priors and propagate with the same solver (common random numbers for variance reduction).

Automatic differentiation. Provide $\partial \mathbf{X}^{\text{th}}/\partial \mathbf{y}$ via AD to build Fisher matrices and gradients for global fits (Sec. 20.4).

8. Feasibility gates and kill criteria

EFT validity and Landau poles. Kill points where any coupling exceeds a unitarity cap or hits a Landau pole before the target scale:

$$\max\{g_i^2,\ \|Y_f\|^2,\ |\lambda|\} > g_{\max}^2 \quad \Rightarrow \quad \text{reject}, \qquad \alpha_i^{-1}(\mu^*) \to 0^+ \text{ with } \mu^* < \mu_{\text{target}} \Rightarrow \text{reject}. \tag{21.30}$$

Positivity and cone LMIs. Running must preserve forward-positivity bounds (Hankel matrices $\succeq 0$ for EH-like operators) and cone ordering for co-metric deformations; violation at any μ kills the trajectory.

Scheme/threshold robustness. Repeat the flow with (i) sharp vs. smooth decoupling (21.25), (ii) μ_k varied by a factor two, and (iii) $\overline{\text{MS}}$ vs. on-shell input sets; large spread beyond declared truncation error $\Rightarrow$ reject (ill-conditioned).

9. Interfaces, caching, and reproducibility

YAML configuration.

```
rg:
  mu0: 1.0e3          # GeV
  mu_min: 2.0         # GeV
  scheme: MSbar-BFM
  loops: {beta_g: 2, beta_Y: 1, gamma6: 1}
  thresholds:
    - {name: top, mu: 172.5, type: fermion, rep: [SU3:3, SU2:2, U1:1/6]}
    - {name: WZ,  mu: 91.1876, type: gauge}
    - {name: Higgs, mu: 125.1, type: scalar}
  decoupling: sharp  # or smooth:n=4
solver:
  method: ROS4
  rtol: 1.0e-6
  atol: 1.0e-9
  jacobian: AD
```

Caching and checks. Cache precomputed sparse patterns of $\Gamma^{(d)}$ and AD tapes. Emit logs of thresholds hit, step rejections, invariant drifts, and feasibility flags. Save *flow traces* $\{\mu_i, \mathbf{y}(\mu_i)\}$ in HDF5 with schema and code commit hashes.

10. Summary

We formulated RG as a sparse, stiff, event-driven flow with AD-powered Jacobians and background-field normalization that preserves Ward identities. Linear matching maps connect EFT layers at thresholds, with optional smooth decoupling to stabilize numerics, and complete dictionaries (tree and one-loop) translate spectral mediators into SMEFT (and then LEFT) Wilsons. Feasibility gates (EFT validity, positivity, cone LMIs) are enforced *during* integration, and uncertainty propagation is handled via covariance flows or sampling with common randomness. Standardized YAML cards, deterministic caching, and HDF5 traces guarantee reproducibility and seamless interoperability with the global inference engine of Sec. 20.4.

21.3 Synthetic datasets and validation

Aim. We define a principled pipeline to generate *synthetic datasets* (Asimov and toy pseudo-experiments) across the collider, low-energy, long-baseline, and cosmic layers, and

to validate the full inference chain of Sec. 20.4. The construction provides (i) generative models for counts, spectra, and event-level samples; (ii) detector-response emulation with response matrices and smearing kernels; (iii) correlated systematics via hierarchical nuisances; (iv) closure, coverage, and goodness-of-fit tests; (v) adversarial stress tests and blinding; and (vi) a reproducibility-ready on-disk format.

1. Generative models and Asimov datasets

Binned Poisson processes. For a channel partitioned into bins $b = 1, \dots, B$, the prediction at parameter point $\mathbf{p}$ and nuisance pulls $\boldsymbol{\xi}$ is

$$\nu_b(\mathbf{p}, \boldsymbol{\xi}) \;=\; \theta_{\mathrm{lumi}} \int_{\Delta_b} d\Phi \; \epsilon(\Phi; \boldsymbol{\xi}) \; A(\Phi; \boldsymbol{\xi}) \; \frac{d\sigma(\Phi \,|\, \mathbf{p})}{d\Phi} \;+\; B_b(\boldsymbol{\xi}), \tag{21.31}$$

with integrated luminosity scale θ_{lumi}, acceptance A, efficiency ϵ, and background template B_b. A *toy* dataset draws $N_b \sim \mathrm{Poisson}(\nu_b)$ independently across bins:

$$P(\mathbf{N} \,|\, \mathbf{p}, \boldsymbol{\xi}) \;=\; \prod_{b=1}^{B} \frac{e^{-\nu_b} \nu_b^{N_b}}{N_b!}. \tag{21.32}$$

The *Asimov* dataset is defined by $N_b^{\mathrm{A}} = \nu_b(\mathbf{p}_\star, \boldsymbol{\xi}_\star)$ at a reference point $(\mathbf{p}_\star, \boldsymbol{\xi}_\star)$.

Correlated Gaussian measurements. For vector-valued observables $\mathbf{X}$ (e.g. EW precision inputs, atomic shifts),

$$\mathbf{X}^{\mathrm{toy}} \;\sim\; \mathcal{N}\Big(\boldsymbol{\mu}(\mathbf{p}, \boldsymbol{\xi}),\; \Sigma(\boldsymbol{\xi})\Big), \qquad \mathcal{L} \;\propto\; \exp\Big[-\tfrac{1}{2} (\mathbf{X}^{\mathrm{exp}} - \boldsymbol{\mu})^{\top} \Sigma^{-1} (\mathbf{X}^{\mathrm{exp}} - \boldsymbol{\mu})\Big]. \tag{21.33}$$

Event-level generators with reweighting. Given a baseline event set $\{\Phi_i, w_i^{(0)}\}$ sampled from $d\sigma_0$, differential predictions for a new point $\mathbf{p}$ are obtained by matrix-element reweighting

$$w_i(\mathbf{p}) \;=\; w_i^{(0)} \; \frac{|\mathcal{M}(\Phi_i \,|\, \mathbf{p})|^2}{|\mathcal{M}(\Phi_i \,|\, \mathbf{p}_0)|^2} \; \frac{f_a(x_a, \mu; \mathbf{p}) \, f_b(x_b, \mu; \mathbf{p})}{f_a(x_a, \mu; \mathbf{p}_0) \, f_b(x_b, \mu; \mathbf{p}_0)}, \tag{21.34}$$

then histogrammed after detector-response emulation (below). This enables fast scans in global fits without regenerating events.

2. Detector response and unfolding-safe emulation

Response matrices and smearing kernels. Truth-level binned yields $\mathbf{T}$ map to reconstructed yields $\mathbf{R}$ via a non-negative response matrix R_{ij} and additive noise $\boldsymbol{\eta}$:

$$R_i \;=\; \sum_{j=1}^{B_{\mathrm{truth}}} R_{ij} \, T_j \;+\; \eta_i, \qquad R_{ij} \geq 0, \;\; \sum_i R_{ij} \leq 1. \tag{21.35}$$

For continuous variables we use smearing kernels $K(y|x)$ (e.g. Gaussian with heteroscedastic width) and efficiencies $\epsilon(x)$,

$$\frac{dN_{\mathrm{reco}}}{dy} \;=\; \int dx \; K(y|x) \, \epsilon(x) \, \frac{dN_{\mathrm{truth}}}{dx}. \tag{21.36}$$

Synthetic datasets are produced at *reco*-level by construction, avoiding ill-posed unfolding in validation loops.

Nuisance morphing. Template deformations under pulls ξ_k are realized by piecewise polynomial (cubic) morphing or log-normal scalings,

$$\nu_b(\boldsymbol{\xi}) \;\approx\; \nu_b(0)\,\prod_k \exp\!\Big(\alpha_{bk}\,\xi_k\Big) \;\times\; \text{shape-morph}(\xi_k), \tag{21.37}$$

with $\xi_k \sim \mathcal{N}(0,1)$ (or GP priors for smooth shape systematics).

3. Correlations, hierarchies, and priors

Hierarchical nuisance model. Let $\boldsymbol{\xi}$ split into block-shared and channel-specific components,

$$\boldsymbol{\xi} \;=\; (\boldsymbol{\xi}^{\text{shared}},\; \boldsymbol{\xi}^{(1)},\ldots,\boldsymbol{\xi}^{(Y)}), \qquad \pi(\boldsymbol{\xi}) \;=\; \mathcal{N}\Big(0,\; \text{diag}(\mathsf{C}_{\text{shared}},\mathsf{C}_1,\ldots,\mathsf{C}_Y)\Big). \tag{21.38}$$

Shared pulls (e.g. α_s, PDFs, flux normalizations) induce cross-dataset correlations in toys and must be drawn *once* per pseudo-experiment.

Theory-error Gaussian processes. For theory shapes $\mathcal{T}(x)$ with truncation errors, introduce a zero-mean GP $\Delta\mathcal{T} \sim \mathcal{GP}(0,K)$ and sample it jointly across bins/kinematics; then set $\mathcal{T} \to \mathcal{T} + \Delta\mathcal{T}$ when generating toys. This avoids double counting of bin-wise uncorrelated errors.

4. Validation tests: closure, coverage, goodness-of-fit

Level-A closure (template). Generate toys with $\nu_b(\mathbf{p}_\star,\boldsymbol{\xi}_\star)$ and fit them using the same model (same R_{ij}, same morphing). The MLE $\hat{\mathbf{p}}$ must satisfy

$$\mathbb{E}[\hat{\mathbf{p}}] \;=\; \mathbf{p}_\star \quad \text{within} \quad \mathcal{O}(1/\sqrt{N})\ \text{bias}, \qquad \text{Cov}(\hat{\mathbf{p}}) \;\approx\; \mathsf{F}^{-1}(\mathbf{p}_\star). \tag{21.39}$$

Level-B closure (detector). Generate with R_{ij}^{true} and fit with a perturbed $\tilde{R}_{ij} = R_{ij}^{\text{true}} + \delta R_{ij}$. Bias vs. $\|\delta R\|$ quantifies detector-model robustness; acceptable if smaller than declared systematics.

Level-C closure (theory). Generate from an *alternative* theory within the admissible set $\mathcal{F}$ (e.g. different higher-order terms) and fit with the baseline; inflated χ^2 identifies model incompleteness.

Frequentist coverage. For a nominal $1-\alpha$ interval construction $\mathcal{I}_{1-\alpha}(\text{data})$, the empirical coverage

$$\widehat{\text{Cov}}_{1-\alpha} \;=\; \frac{1}{N_{\text{toys}}}\sum_t \mathbf{1}\{\mathbf{p}_\star \in \mathcal{I}_{1-\alpha}(\text{data}_t)\} \tag{21.40}$$

must lie within the Clopper–Pearson band around $1-\alpha$; undercoverage $\Rightarrow$ expand intervals (e.g. Feldman–Cousins) or refine likelihood approximations.

Goodness-of-fit. Use saturated Poisson deviance for count data,

$$D \;=\; 2\sum_b \Big[\nu_b - N_b + N_b\ln(N_b/\nu_b)\Big], \tag{21.41}$$

and Mahalanobis distance for Gaussian blocks; p-values should be uniform on toys.

5. Adversarial stress tests and blinding

Adversarial perturbations. Inject bounded, structured mismodellings $\delta\nu_b$ (e.g. coherent tilts, localized bumps, non-factorizing nuisances) and quantify estimator drift, coverage, and discovery-spillover rates. Report *stress envelopes* as functions of $\|\delta\nu\|/\nu$.

Blinding. Replace truth $\mathbf{p}_{\text{data}}$ by $\mathbf{p}_{\text{data}} \oplus \Delta$ with hidden Δ or scramble seeds in a way that preserves covariances and test-statistic distributions. Unblind only after the validation checklist passes.

6. Synthetic & Asimov cards and on-disk format

Dataset card (YAML).

```
dataset:
  name: "V4-synthetic-001"
  seed: 123456
  layers: [collider, lowE, LBL, cosmic]
model:
  p_star: {C_HWB: 0.0, C_HD: 0.0, epsilon: 1e-3, mAp: 0.04}
  nuis_shared:
    alpha_s: {mean: 0.118, sigma: 0.001}
    lumi:    {mean: 1.0,   sigma: 0.02}
  response:
    collider: {matrix_file: R_hjets.h5}
    LBL:      {layers: [crust,mantle], densities: [2.9, 4.5]}
generator:
  type: "Asimov+Poisson"
  toys: 10000
  morphing: "cubic"
```

HDF5 schema. Store truth and reco in a versioned HDF5:

`/meta/{card,commit,seed,version}`, `/truth/{bins,nu,labels}`, `/reco/{R,nu,obs,cov}`, `/n`

All arrays carry units and bin edges; checksums ensure integrity.

7. Worked primitives

Poisson toy with nuisances. Draw shared pulls once, then per-channel toys:

$$\xi_j \sim \mathcal{N}(0,1), \qquad N_b \sim \text{Poisson}(\nu_b(\mathbf{p}_\star, \boldsymbol{\xi}))\,. \tag{21.42}$$

Gaussian toy with GP theory error. Sample $\Delta\mathcal{T} \sim \mathcal{GP}(0, K)$, set $\boldsymbol{\mu} \to \boldsymbol{\mu} + \Delta\mathcal{T}$, then

$$\mathbf{X}^{\text{toy}} \;=\; \boldsymbol{\mu}(\mathbf{p}_\star) + \Delta\mathcal{T} \;+\; L\,\mathbf{z}, \quad \mathbf{z} \sim \mathcal{N}(0, \mathbb{1}), \quad \Sigma = LL^\top. \tag{21.43}$$

8. Kill criteria (validation layer)

(K1) Undercoverage: $\widehat{\mathrm{Cov}}_{0.95} < 0.90$ after $N_{\mathrm{toys}} \geq 10^4 \;\Rightarrow$ reject model/interval rule.
(K2) GoF non-uniformity: KS test on toy p-values rejects uniformity at $p < 10^{-3} \;\Rightarrow$ revise likelihoo
(K3) Closure failure: $|\mathbb{E}[\hat{\mathbf{p}}] - \mathbf{p}_\star| > 2\sigma \;\Rightarrow$ diagnose model or response.
(K4) Stress fragility: small $\|\delta\nu\|/\nu < 1\%$ induces $|\Delta\hat{\mathbf{p}}| > 1\sigma \;\Rightarrow$ inflate systematics or refine model.
(K5) Reproducibility: run-to-run hashes or seeds mismatch outputs $\Rightarrow$ invalidate dataset release.

(21.44)

9. Summary

Synthetic datasets are the backbone of the spectral program's validation: they connect first-principles predictions to analysis-ready reco-level inputs with faithful detector emulation, correlated nuisances, and standardized storage. Asimov sets provide deterministic benchmarks; toys probe statistical properties, coverage, and robustness; adversarial stress tests quantify model fragility. A strict kill-criteria layer (21.44), together with blinding and a public, checksummed HDF5 schema, ensures that all numerical claims are reproducible, well-calibrated, and theoretically admissible within the positivity/causality gates of Part VI.

21.4 Open-source package structure

Aim. We specify a modular, reproducible, and community-friendly *open-source* stack that implements the algorithms of Chapters 21–17: spectral-lattice HMC, RG/matching flows, synthetic data generation, and global inference with feasibility gates (S/G/TQ/D, Hankel positivity, cone LMIs, unitarity). The design emphasizes: (i) clean module boundaries; (ii) language interop (C++/CUDA kernels with Python orchestration); (iii) deterministic builds and data provenance; (iv) continuous integration (CI) and golden-file tests; (v) documentation-first releases.

1. Repository layout and module boundaries

Top-level tree.

```
spectral-model/
|-- src/
|   |-- spectralx/                  # Python front-end (orchestration)
|   |   |-- api/                    # Stable user-facing API facades
|   |   |-- cli/                    # Command-line entry points
|   |   |-- io/                     # HDF5, YAML, schema validators
|   |   |-- rg/                     # RG solvers & matching (Sec. V4-21.2)
|   |   |-- lattice/                # HMC, heatbath, observables (Sec. V4-21.1)
|   |   |-- inference/              # likelihoods, samplers (Sec. V4-20.4)
|   |   |-- positivity/             # Hankel, cone LMIs, unitarity gates
|   |   |-- generators/             # synthetic/Asimov datasets (Sec. V4-21.3)
|   |   `-- viz/                    # plotting, tables (publication-ready)
|   `-- cpp/                        # High-performance kernels (pybind11)
|       |-- lattice/
```

```
|       |-- algebra/
|       |-- positivity/              # SDP primitives (Eigen + LDL^T)
|       `-- utils/
|-- include/                         # C++ headers (public)
|-- tests/
|   |-- unit/                        # fast, deterministic (pytest, gtest)
|   |-- integ/                       # cross-module scenarios
|   `-- golden/                      # HDF5 golden baselines (hash-checked)
|-- examples/                        # MREs, notebooks, cards (YAML)
|-- configs/                         # validated cards for pipelines
|-- docs/                            # sphinx-book + API/autodoc + theory notes
|-- docker/                          # containers (CPU/GPU) + lockfiles
|-- .github/workflows/               # CI (build, tests, linters, docs)
|-- pyproject.toml                   # Python packaging (PEP 517)
|-- CMakeLists.txt                   # C++ build (targets, options)
|-- CITATION.cff                     # citation metadata
|-- LICENSE                          # SPDX license (e.g., Apache-2.0)
`-- RELEASE.md                       # changelog + semver policy
```

Dependency arrows.

`cpp/*` $\rightarrow$ `spectralx/*` $\rightarrow$ `cli`, `examples`, `docs`.

No reverse imports (acyclic). The `positivity` core is dependency-free (only `algebra`); `rg` does not import `lattice`; `inference` depends only on `io`, `positivity`, and `rg`.

2. Public API and CLI entry points

Stable Python API (facade).

```
from spectralx.api import (
    run_rg,                # RG/matching flow (Sec. V4-21.2)
    run_hmc,               # lattice HMC trajectory (Sec. V4-21.1)
    make_synthetic,        # synthetic datasets (Sec. V4-21.3)
    fit_global,            # joint likelihood (Sec. V4-20.4)
    check_feasibility,     # S/G/TQ/D, Hankel, cone LMIs
)
```

CLI commands.

```
$ spectralx rg run        --card configs/rg.yml           --out flows.h5
$ spectralx lattice hmc   --card configs/lattice.yml      --out cfgs.h5
$ spectralx data synth    --card configs/synthetic.yml    --out toys.h5
$ spectralx fit global    --card configs/global-fit.yml   --out fit.h5
$ spectralx check gates   --in fit.h5
$ spectralx viz report    --in fit.h5 --out report.pdf
```

3. Configuration cards and schema validation

YAML with JSON-Schema. All pipelines are driven by validated cards; example:

```
rg:
  mu0: 1e3
  scheme: MSbar-BFM
  loops: {beta_g: 2, beta_Y: 1, gamma6: 1}
  thresholds:
    - {name: top, mu: 172.5, type: fermion, rep: {SU3: 3, SU2: 2, U1: 1/6}}
positivity:
  hankel: {moments: [m0,m2,m4], tol: 1e-10}
  cones:  {metric: geff, tol: 1e-12}
inference:
  sampler: {type: NUTS, steps: 1000, warmup: 500, target_accept: 0.9}
io:
  out: flows.h5
```

Cards are validated at runtime (fast fail) and embedded in HDF5 outputs for provenance.

4. Data formats, provenance, and determinism

HDF5 schema (versioned).

/meta/{version, git, build_env, card}, /rg/trace/{μ, y(μ)}, /lattice/{U, y, obs}, /fit/{po

Each group stores units, shapes, and checksums; random seeds and PRNG types are recorded per run.

Determinism. All kernels support fixed seeds; HMC includes *reversible integrators* and logs acceptance, step sizes, and leapfrog counts. FP math uses `-ffp-contract=off` (or CUDA deterministic flags) for bitwise reproducibility where feasible.

5. C++ kernels and Python bindings

Numerical cores.

- **lattice/**: SU(N) links (SOA layout), plaquettes, staples, Cabibbo–Marinari heat-bath, Omelyan HMC; optional CUDA backends.
- **positivity/**: Hankel builders, LDL$^\top$ factorizations, cone LMI checks.
- **algebra/**: Lie-algebra ops, exponentials (scaling–squaring + Padé), log/exp maps on $\mathcal{F}$.

Bindings.

```
// CMake + pybind11 target
pybind11_add_module(_spectralx cpp/bindings.cpp)
target_link_libraries(_spectralx PRIVATE spectralx_core)
```

Python `spectralx.lattice.core` exposes zero-copy views (py::array / DLPack) for fast IO.

6. Testing strategy and golden baselines

Unit tests.

- Lattice: plaquette forces match analytic derivatives (finite-diff check).
- RG: closed-form flows (e.g. pure one-loop) match solver trajectories.
- Positivity: synthetic amplitudes produce PSD Hankel matrices.

Golden files. Integration tests compare HDF5 outputs to versioned baselines (hash + tolerance). Any drift ⇒ CI failure unless changelog marks an intentional change and `golden` is updated.

7. Continuous integration, docs, and packaging

CI matrix (GitHub Actions).

```
jobs:
  build-test:
    runs-on: ubuntu-latest
    strategy: {matrix: {py: [3.10, 3.11], backend: [cpu]}}
    steps:
      - uses: actions/checkout@v4
      - uses: actions/setup-python@v5
      - run: pip install .[dev]
      - run: pytest -q
      - run: sphinx-build -b html docs/ docs/_build/html
      - run: spectralx rg run --card examples/rg.yml --out /tmp/flow.h5
```

Docs. Sphinx-book with math, API autodoc, and *theory notes* that cross-reference this volume; examples rendered as executable notebooks (nbclient). Versioned docs per release tag.

Packaging. PEP 517/518 via `pyproject.toml`; wheels for Linux/macOS; optional `conda` recipe. C++ builds through CMake; CUDA builds behind `-DENABLE_CUDA=ON`.

8. Reproducible containers and HPC

Containers. Dockerfiles pin OS, compilers, BLAS, CUDA/cuDNN. Images published with digests; CI asserts byte-identical builds for release tags.

HPC. Slurm scripts in `docker/hpc/` (or `scripts/slurm/`) orchestrate array jobs for HMC trajectories and toy ensembles; environment modules load MPI where needed.

9. Interfaces to external ecosystems

Event generators and PDFs. Optional adapters to `MadGraph`/`Pythia` (reweighting hooks, Eq. (21.34)) and LHAPDF; results ingested via `io` with unit checks.

SDP backends. For large LMIs, optional bindings to `MOSEK/SDPA` (guarded imports); the default Eigen/LDL$^\top$ path remains license-free.

10. Versioning, releases, and governance

Semantic versioning. `MAJOR.MINOR.PATCH` with:

- **MAJOR**: breaking API/HDF5 schema changes.
- **MINOR**: new features, backward-compatible schema extensions.
- **PATCH**: fixes and performance; golden files unchanged.

`CITATION.cff` and Zenodo DOI minted per release; `RELEASE.md` documents changes and migration notes.

Governance. CODE_OF_CONDUCT, CONTRIBUTING, issue templates, and `good first issue` labels. All new features require: design doc, tests, docs, and at least one review.

11. Minimal reproducible example (MRE)

End-to-end card.

```
pipeline:
  steps: [rg, gates, synth, fit]
rg: {card: configs/rg.yml}
gates: {cone: true, hankel: true}
synth: {card: configs/synthetic.yml, toys: 1000, mode: Asimov}
fit: {card: configs/global-fit.yml, sampler: NUTS}
```

Run:

```
$ spectralx pipeline run --card examples/pipeline.yml --out out.h5
```

12. Summary

The proposed package structure separates concerns (numerical kernels vs. orchestration), guarantees determinism and provenance (versioned HDF5 + embedded cards), and automates correctness (golden baselines, CI, validated schemas). It exposes a stable API/CLI for practitioners while remaining performant through C++/CUDA backends. By integrating feasibility gates at the API boundary and encoding theory/data interfaces natively, the repository serves as a faithful, reproducible reference implementation for the spectral program described in this volume.

Part VIII

Synthesis, Gates, and Decisive Tests

Chapter 22

Global Consistency Gates (S/G/TQ/D) for Particle Physics

22.1 Stability/ghost-free and hyperbolicity

Aim. We implement the **S-gate**: absence of ghosts and gradient instabilities together with **well-posedness** (hyperbolicity) of the field equations on the emergent background (M, g_{eff}). The gate is stated and checked at the level of the quadratic action around an arbitrary slowly varying background configuration of the spectral medium $\mathcal{S} = \{\Phi, \rho, \eta, \chi^\mu, T\}$ and of the internal map $y : M \to \mathcal{F}$ (Part I). The outcome is a set of *coordinate-free* inequalities and principal-symbol conditions that are sufficient (and in the cases below necessary) for ghost-freeness and hyperbolicity.

1. Quadratic action and kinetic matrices

Background and fluctuations. Let φ denote a generic bosonic multiplet and ψ a fermionic multiplet that include the SM composites and the emergent gauge fields:

$$\varphi \ \in \ \{h,\ y^A,\ \mathcal{A}^I_\mu\}, \qquad \psi \ \in \ \{\Psi_{\mathrm{quark}}, \Psi_{\mathrm{lepton}}\}.$$

Decompose $\varphi = \bar{\varphi} + \delta\varphi$, $\psi = \bar{\psi} + \delta\psi$ around a background obeying the classical equations up to $\mathcal{O}(\partial^2)$ gradients. The quadratic action reads

$$S^{(2)} = \frac{1}{2}\int d^4x\,\sqrt{-g_{\mathrm{eff}}}\,\Big[\ (\nabla_\mu\delta\varphi)^T \mathsf{K}^{\mu\nu}(\bar{\Phi})\,(\nabla_\nu\delta\varphi) - \delta\varphi^T \mathsf{M}^2(\bar{\Phi})\,\delta\varphi\ \Big] + \int d^4x\,\sqrt{-g_{\mathrm{eff}}}\ \bar{\psi}\,\not{\mathsf{D}}\psi, \tag{22.1}$$

where $\mathsf{K}^{\mu\nu}$ is a positive type tensor-valued bilinear form and $\not{\mathsf{D}} = i\gamma^\mu \mathsf{Z}_\mu{}^\nu(\bar{\Phi})\, D_\nu + \cdots$ is the fermion kinetic operator. Gauge fields are treated in a BRST gauge-fixed form with parameter ξ (Part 2).

Rest-frame kinetic matrices. Fix a unit timelike covector n_μ (e.g. $n_\mu u^\mu = 1$ with u^μ the aether rest frame) and project

$$\mathsf{K}_{tt} := \mathsf{K}^{\mu\nu} n_\mu n_\nu, \qquad \mathsf{K}_\perp := \mathsf{K}^{\mu\nu}\Pi_\mu{}^\alpha \Pi_\nu{}^\beta, \qquad \Pi_\mu{}^\alpha = \delta^\alpha_\mu - n_\mu u^\alpha. \tag{22.2}$$

Ghost-freeness demands $\mathsf{K}_{tt} \succ 0$ on the physical (gauge-invariant) fluctuations; **no gradient instabilities** requires the spatial symbol $\mathsf{K}_\perp$ to be non-negative in the sense detailed below.

2. Sector-by-sector ghost-freeness

Scalars (Higgs h, fiber coordinates y^A). From Parts 8 and 1, the kinetic terms are

$$\mathcal{L}_{\rm kin}^{(h,y)} = \frac{1}{2} Z_h \, g_{\rm eff}^{\mu\nu} \, \partial_\mu h \, \partial_\nu h + \frac{\kappa_y}{2} \, g_{\rm eff}^{\mu\nu} h_{AB}(y) \, \mathcal{D}_\mu y^A \mathcal{D}_\nu y^B + \Delta\mathcal{L}_\chi, \tag{22.3}$$

with $\Delta\mathcal{L}_\chi$ small anisotropic deformations linear or quadratic in χ^μ.

$$\boxed{Z_h > 0, \qquad \kappa_y > 0, \qquad h_{AB} \text{ positive definite}} \tag{22.4}$$

suffice for ghost-freeness. Small χ^μ-induced tensors must satisfy the cone inclusion (Sec. 22.1) so that the principal symbol remains hyperbolic.

Gauge bosons. In R_ξ gauge the quadratic Lagrangian is

$$\mathcal{L}_{\rm YM}^{(2)} = -\frac{\kappa_G}{4} \mathcal{F}^I_{\mu\nu} \mathcal{F}_I^{\mu\nu} - \frac{\kappa_G}{2\xi} \left(D_\mu \mathcal{A}^{\mu I} \right)^2 + \frac{1}{2} \, \mathcal{A}^I_\mu \, \Delta\kappa_{IJ}{}^{\mu\nu} \, \mathcal{A}^J_\nu, \tag{22.5}$$

with $\Delta\kappa$ small, symmetric in $(I\mu) \leftrightarrow (J\nu)$.

$$\boxed{\kappa_G > 0 \quad \text{and} \quad \Delta\kappa \text{ respects cone inclusion w.r.t. } g_{\rm eff}} \tag{22.6}$$

guarantee positive energy for the transverse modes; the ξ-sector is BRST-exact and does not introduce physical ghosts (Part 2).

Fermions. The leading operator $i\bar\psi\gamma^\mu \mathsf{Z}_\mu{}^\nu D_\nu \psi$ is ghost-free if the co-metric

$$G^{\mu\nu} := \mathsf{Z}_\alpha{}^\mu \, \eta^{\alpha\beta} \, \mathsf{Z}_\beta{}^\nu \tag{22.7}$$

remains in the cone of $g_{\rm eff}$ (see Sec. 22.1). At leading order $\mathsf{Z}_\mu{}^\nu = \delta_\mu^\nu + \mathcal{O}(\chi)$, hence standard Dirac hyperbolicity follows.

3. Hyperbolicity: principal symbol and cone inclusion

Principal symbol. Let $\mathscr{P}(x,\xi)$ be the principal symbol of the Euler–Lagrange operator acting on $\delta\varphi$:

$$\mathscr{P}(x,\xi) = \mathsf{K}^{\mu\nu}(x) \, \xi_\mu \xi_\nu. \tag{22.8}$$

Gårding hyperbolicity holds at x iff there exists a covector n_μ such that for all ξ the polynomial $\lambda \mapsto \det \mathscr{P}(x, \xi + \lambda n)$ has only real roots. Sufficient condition (used operationally): existence of n_μ with

$$\mathsf{K}^{\mu\nu} n_\mu n_\nu \succ 0 \quad \text{and} \quad \mathsf{K}^{\mu\nu} \text{ has Lorentzian signature on the physical subspace.} \tag{22.9}$$

Cone inclusion (CI). Deformations by the director field χ^μ modify the co-metric to

$$G^{\mu\nu} = g_{\rm eff}^{\mu\nu} + \epsilon \, C^{\mu\nu}(\chi; \mathcal{S}) + \mathcal{O}(\epsilon^2). \tag{22.10}$$

We impose the *CI inequality*

$$\boxed{\forall \xi \neq 0 \quad g_{\rm eff}^{\mu\nu} \xi_\mu \xi_\nu \geq 0 \;\Rightarrow\; G^{\mu\nu} \xi_\mu \xi_\nu \geq 0} \tag{22.11}$$

(i.e. the null cone of G lies inside or coincides with that of $g_{\rm eff}$). This guarantees hyperbolicity and prevents microcausal conflicts at the S-gate level; stronger causality/positivity gates are enforced in Sec. 22.2.

Symmetric-hyperbolic form. There exists a symmetrizer $\mathsf{H}(\bar{\Phi}) \succ 0$ such that the first-order reduction

$$\mathsf{H}\,\partial_t \mathbf{U} \;+\; \sum_{i=1}^{3} \mathsf{A}^i(\bar{\Phi})\,\partial_i \mathbf{U} \;=\; \mathsf{B}(\bar{\Phi})\,\mathbf{U} \tag{22.12}$$

is symmetric-hyperbolic. Here $\mathbf{U}$ collects canonical variables (fields and selected first derivatives). Energy estimates then yield existence, uniqueness, and continuous dependence on data in Sobolev spaces H^s.

4. Higher-derivative operators and Ostrogradsky control

Auxiliary-field demotion. Any $\mathcal{O}(\partial^4)$ correction allowed by the positivity gates (Part 17) is represented by auxiliary fields so that time derivatives stay at most second order. Example: a gauge-sector operator

$$\Delta\mathcal{L} = \frac{\alpha_1}{\Lambda^2}\,(D_\mu \mathcal{F}^{\mu\nu})^2 \tag{22.13}$$

is rewritten introducing B_ν as

$$\Delta\mathcal{L} \;\widehat{=}\; -\,\frac{1}{4\alpha_1}\,B_\nu B^\nu \;+\; B_\nu\, D_\mu \mathcal{F}^{\mu\nu}, \tag{22.14}$$

which is manifestly second order and subject to the same CI test. This eliminates Ostrogradsky ghosts within the EFT domain $E \ll \Lambda$.

5. BRST gauge fixing and gauge-parameter independence

Physical sector. With the R_ξ gauge-fixing functional and Faddeev–Popov ghosts (Part 2), the BRST cohomology identifies the physical subspace. On that subspace the kinetic operator reduces to the transverse block with

$$\mathscr{P}^{\mathrm{phys}}_{\mu\nu}(\xi) = \kappa_G\, g^{\alpha\beta}_{\mathrm{eff}}\,\Pi_{\mu\alpha}\Pi_{\nu\beta}\,\xi_\alpha\xi_\beta + \mathcal{O}(\Delta\kappa), \tag{22.15}$$

independent of ξ. Hence ghost-freeness and hyperbolicity are *gauge-parameter independent*.

6. Energy functionals and Källén–Lehmann test

Conserved energy. For time-like n^μ the canonical (or Belinfante) energy density of bosons is

$$\mathcal{E} = \frac{1}{2}\,(\partial_t\delta\varphi)^T \mathsf{K}_{tt}(\partial_t\delta\varphi) + \frac{1}{2}\,(\nabla_\perp\delta\varphi)^T \mathsf{K}_\perp(\nabla_\perp\delta\varphi) + \frac{1}{2}\,\delta\varphi^T \mathsf{M}^2\delta\varphi, \tag{22.16}$$

which is positive if (22.4)–(22.6) and (22.11) hold.

Spectral density. At the 2-point level the Källén–Lehmann representation implies a positive spectral density $\rho(\mu^2) \geq 0$ for all physical fields. This is automatic when the quadratic form is positive and the propagator poles lie on the real axis with positive residues (a direct corollary of (22.9) plus BRST).

7. Algorithmic checklist (LMIs at a point)

At any $x \in M$, given background $\bar{\Phi}(x)$ and n_μ:

(S1) $\mathsf{K}_{tt}(\bar{\Phi}; n) \succ 0$ on physical subspace; (22.17)

(S2) All eigenvalues of $\mathsf{K}^{\mu\nu}$ have one $+$ and three $-$ signatures (Lorentzian block); (22.18)

(S3) Cone inclusion $G^{\mu\nu} \preceq_{\text{cone}} g_{\text{eff}}^{\mu\nu}$ as in (22.11); (22.19)

(S4) Higher-derivative terms demoted to auxiliary fields and tested by (S1)–(S3); (22.20)

(S5) BRST-physical block of $\mathscr{P}$ is ξ-independent and hyperbolic. (22.21)

Conditions (22.17)–(22.21) are linear matrix inequalities (LMIs) in the small-deformation regime and are enforced numerically in the package of Sec. 21.4.

8. Worked illustrations

U(1) with mild anisotropy. Let

$$\mathcal{L} = -\frac{\kappa_G}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\,\delta\kappa\, u_\mu u_\nu F^{\mu\alpha}F^{\nu}{}_{\alpha}, \qquad u^\mu u_\mu = 1, \quad |\delta\kappa| \ll \kappa_G. \tag{22.22}$$

The dispersion relation is $\omega^2 = \mathbf{k}^2\,(1 + \delta\kappa\,\sin^2\theta)/\kappa_G + \mathcal{O}(\delta\kappa^2)$, hence ghost-free if $\kappa_G > 0$ and hyperbolic if $1 + \delta\kappa\,\sin^2\theta > 0$ for all angles θ; i.e.

$$\boxed{\kappa_G > 0, \qquad \delta\kappa > -1} \tag{22.23}$$

which is a special case of the CI inequality.

Fiber scalar with director coupling. For $\mathcal{L} = \frac{\kappa_y}{2}\left[g_{\text{eff}}^{\mu\nu} + \alpha\, u^\mu u^\nu\right] h_{AB}\,\partial_\mu y^A \partial_\nu y^B$, ghost-freeness and hyperbolicity demand $\kappa_y > 0$ and $1 + \alpha > 0$.

9. Summary

The **S-gate** consists of: (i) positive-definite time-kinetic matrices for all physical bosons, (ii) cone inclusion for all co-metrics generated by spectral anisotropies, (iii) symmetric-hyperbolic first-order reduction ensuring a well-posed Cauchy problem, (iv) auxiliary-field demotion of higher-derivative operators within EFT validity, and (v) BRST-parameter independence of the physical principal symbol. These conditions provide a minimal yet robust foundation on which the stronger causality/analyticity and positivity gates of Sec. 22.2 are erected.

22.2 Causality/analyticity and positivity

Aim. We put on firm ground the **C/A/TQ-gates**: microscopic *causality* ⇒ analyticity of retarded/forward amplitudes ⇒ *positivity* of suitable dispersive moments. Within the spectral framework, microcausality is defined with respect to the emergent cone of (M, g_{eff}) (Part VI); the cone-inclusion condition of Sec. 22.1 then guarantees that spectral anisotropies do not enlarge support outside the g_{eff}-light cone. We derive forward

dispersion relations, state polynomial-boundedness hypotheses (Froissart–Martin type) and produce *convex* linear-matrix-inequality (LMI) tests on EFT Wilson coefficients that will be used globally in Sec. 22.4.

1. Microcausality ⇒ analyticity of retarded/forward objects

Retarded objects. For any (Heisenberg) operator $\mathcal{O}(x)$ built from fields in the BRST-physical sector,

$$\Delta_R(x) \;:=\; i\,\Theta(x^0)\,\langle[\mathcal{O}(x),\mathcal{O}(0)]\rangle \quad\Rightarrow\quad \operatorname{supp}\Delta_R \;\subseteq\; \overline{\mathcal{J}^+_{g_{\rm eff}}}, \tag{22.24}$$

(*microcausality*), where $\mathcal{J}^+_{g_{\rm eff}}$ is the $g_{\rm eff}$-future. By the Paley–Wiener–Schwartz theorem, the Fourier transform $\tilde\Delta_R(p)$ is the boundary value of a function analytic for $\operatorname{Im} p^0 > 0$ (*upper half-plane*) and tempered in $|\mathbf{p}|$. The same holds for time-ordered two-point functions and, by LSZ reduction (Sec. 16.1), for fixed-t scattering amplitudes $\mathcal{M}(s,t)$ in a Lehmann ellipse of t and a cut s-plane.

Optical theorem (OT). For forward kinematics ($t=0$) and physical $s > s_{\rm th}$,

$$\operatorname{Im}\mathcal{M}_{ii}(s,0) \;=\; s\,\sigma_{i\to\rm all}(s) \;\geq 0, \tag{22.25}$$

where i labels the external channel (including spin/helicity). The non-negativity on the cut furnishes positivity of dispersive *moments* below.

2. Forward dispersion relations and polynomial boundedness

Polynomial boundedness and subtractions. Assume for some $N \in \mathbb{N}$ (typically $N \leq 2$ in gapped, local QFTs) that

$$\big|\mathcal{M}_{ii}(s,0)\big| \;\leq\; C\,|s|^N\,\log^2|s| \qquad (|s|\to\infty \text{ in the physical sheet}), \tag{22.26}$$

compatible with Froissart–Martin for massive exchanges. Then the $(N{+}1)$-th derivative admits an *unsubtracted* dispersion relation:

$$\frac{d^{N+1}}{ds^{N+1}}\mathcal{M}_{ii}(s,0) = \frac{(N{+}1)!}{\pi}\int_{s_{\rm th}}^{\infty}\frac{ds'}{(s'-s)^{N+2}}\operatorname{Im}\mathcal{M}_{ii}(s',0)\,. \tag{22.27}$$

Evaluating at $s = s_0$ in the analyticity domain and using (22.25) yields sign-definite combinations of derivatives.

Crossing and even/odd parts. For identical bosons or CPT-invariant channels, the forward amplitude is even in s around $s=0$ (after subtracting t-channel poles if needed). Then for N odd one obtains particularly simple positive moments at $s_0 = 0$:

$$\underbrace{\frac{d^{2m}}{ds^{2m}}\mathcal{M}^{(+)}_{ii}(0,0)}_{\equiv\; M^{(2m)}_{ii}} = \frac{(2m)!}{\pi}\int_{s_{\rm th}}^{\infty}\frac{ds'}{s'^{\,2m+1}}\operatorname{Im}\mathcal{M}_{ii}(s',0) \;\geq 0, \qquad m=1,2,\dots \tag{22.28}$$

These are the classic forward-limit *positivity bounds*.

3. EFT mapping: positive Hankel moments ⇒ constraints on Wilsons

Wilson expansion at low energy. For an EFT with cutoff Λ and a mass gap below Λ, the forward amplitude admits

$$\mathcal{M}_{ii}(s,0) \;=\; \sum_{n\geq 0} a_{2n}^{(i)}\, s^{2n} \;+\; \mathcal{O}(s^{2n_*}\,\log s)\,, \tag{22.29}$$

where $a_{2n}^{(i)}$ are polynomials in Wilson coefficients of higher-derivative operators. From (22.28) we read

$$a_2^{(i)} \;\geq 0, \qquad \begin{pmatrix} a_2^{(i)} & a_4^{(i)} & a_6^{(i)} & \cdots \\ a_4^{(i)} & a_6^{(i)} & a_8^{(i)} & \cdots \\ a_6^{(i)} & a_8^{(i)} & a_{10}^{(i)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \;\succeq\; 0, \tag{22.30}$$

i.e. the *Hankel matrix* of even coefficients is positive semidefinite for each forward channel i. Eq. (22.30) is precisely the LMI enforced numerically in Sec. 21.4.

Canonical examples.

- **Single real scalar with $\Delta\mathcal{L} = \frac{c_{(1)}}{\Lambda^4}(\partial\phi)^4 + \cdots$.** Then $a_2 \propto c_{(1)}$ and (22.30) gives

$$\boxed{c_{(1)} \;\geq 0} \qquad \text{(any forward polarization).} \tag{22.31}$$

- **Photon (or Abelian gauge) EFT with $\Delta\mathcal{L} = \frac{\alpha_1}{\Lambda^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{\alpha_2}{\Lambda^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \cdots$.** Forward $\gamma\gamma \to \gamma\gamma$ in linear polarizations yields

$$\boxed{\alpha_1 \;\geq 0, \qquad \alpha_1 + \alpha_2 \;\geq 0} \qquad \text{(scanned over helicities/polarizations).} \tag{22.32}$$

- **Non-Abelian gauge bosons**: for color-singlet forward combinations, the same analysis applies channel by channel after projecting onto definite external polarizations/color singlets; bounds reduce to positive sums over intermediate spectral states by OT.

4. Subtractions, IR poles, and $t \neq 0$ improvements

IR poles and subtractions. Massless exchanges (photons/gluons/gravitons) generate t-channel singularities that must be subtracted before applying (22.28). Let $\widehat{\mathcal{M}}(s,0) = \mathcal{M}(s,0) - \mathcal{M}_{\text{pole}}(s,0)$; then $\widehat{\mathcal{M}}$ satisfies the same analyticity and boundedness, and positivity applies to its even derivatives.

$t < 0$ improved bounds. Analyticity in a Lehmann ellipse allows t-derivatives at $t<0$:

$$\partial_s^{2m}\partial_t^n\, \widehat{\mathcal{M}}_{ii}(0,t) \;\geq\; 0 \qquad (t<0), \tag{22.33}$$

which sharpen (22.30) by correlating operators with different tensor structures (e.g. in the electroweak/helicity basis).

5. Causality cones, anisotropy, and analyticity domain

Cone inclusion ⇒ unchanged support. Let $G^{\mu\nu}$ be a co-metric deformation as in (22.10). If the *cone-inclusion* inequality (22.11) holds, the retarded commutator support stays inside $\overline{\mathcal{J}^+_{g_{\rm eff}}}$; hence the analyticity domain of forward amplitudes is *unchanged*, and the same dispersion/positivity machinery applies. Violating CI would enlarge support and spoil the half-plane analyticity of retarded objects, disqualifying the background by the S-gate already.

Polynomial boundedness with anisotropy. Small anisotropy affects high-energy growth only at subleading order in the Regge–Eikonal regime; we therefore adopt the same $N \le 2$ hypothesis as (22.26) and verify it a posteriori by stress tests (Sec. 17.1).

6. Sum rules and convex moment cones

Moment integrals. Define non-negative moments

$$\mu_k^{(i)} \;:=\; \frac{1}{\pi}\int_{s_{\rm th}}^{\infty} \frac{ds'}{s'^{\,k+1}}\,\mathrm{Im}\,\mathcal{M}_{ii}(s',0) \;\ge 0. \tag{22.34}$$

Then $a_{2m}^{(i)} = (2m)!\,\mu_{2m-1}^{(i)}$ up to known subtractions, and the Hankel PSD condition in (22.30) is equivalent to total positivity of the moment sequence $\{\mu_k^{(i)}\}$ (Hamburger moment cone). In practice we enforce a truncated LMI on $(a_2,\dots,a_{2n})$.

Example: Baldin-type sum rules. In Compton-like channels the a_2 coefficient is proportional to a positive integral of total photoabsorption (electric+magnetic polarizabilities). Our EFT maps these to (α_1,α_2) in (22.32), guaranteeing their positivity without recourse to model-dependent fits.

7. Algorithmic gate (implementation)

Inputs. For each forward channel i: EFT Wilson vector $\mathbf{c}$, the analytic subtraction set (IR poles removed), and a polarization basis.

LMIs. Build the truncated Hankel matrix $H_{mn} = \partial_s^{2(m+n)}\widehat{\mathcal{M}}_i(0,0)$ as an affine function of $\mathbf{c}$:

$$H(\mathbf{c}) \;=\; H_0 + \sum_r c_r\,H_r \;\succeq\; 0, \qquad \text{for all polarizations and channels } i \text{ in the library.} \tag{22.35}$$

Optionally add $t{<}0$ blocks from (22.33). Feasibility of (22.35) defines the *positivity cone* $\mathcal{C}_{\rm pos}$ in Wilson space.

Kill criteria. If any $H(\mathbf{c}) \not\succeq 0$ for a physical channel or polarization, the parameter point is rejected (TQ-gate fail), independently of data fits (cf. Sec. 22.4).

8. Worked illustrations (EFT to bounds)

Scalar EFT. With $\Delta\mathcal{L} = \sum_I \frac{c_I}{\Lambda^{d_I-4}}\mathcal{O}_I$ including $(\partial\phi)^4$ and $(\partial\phi)^2(\Box\phi)^2/\Lambda^2$, remove the IR pole from $(\Box\phi)^2$ exchange, then obtain

$$c_{(\partial\phi)^4} \;\geq\; 0, \qquad \begin{pmatrix} c_{(\partial\phi)^4} & \tilde{c} \\ \tilde{c} & \cdots \end{pmatrix} \succeq 0, \tag{22.36}$$

where $\tilde{c}$ mixes derivative operators (basis-dependent but LMI-invariant).

Electroweak gauge EFT. Project forward helicity channels for $W^+W^- \to W^+W^-$ and $\gamma\gamma \to \gamma\gamma$; translate to Warsaw or Higgs basis coefficients (e.g. C_{HW}, C_{HB}, C_{HWB}). Sampling polarizations gives linear maps $\{C\} \mapsto a_{2m}^{(i)}$ and produces LMIs analogous to (22.32) with mixing from custodial breaking (Sec. 8.2).

9. Summary

Microcausality on (M, g_{eff}) yields upper-half-plane analyticity of retarded objects and forward analyticity of $\mathcal{M}(s,0)$; polynomial boundedness fixes the number of subtractions. The optical theorem provides non-negative discontinuities, which translate into *positive* dispersive moments. At the EFT level this becomes a family of *convex LMIs* (Hankel PSD and t<0 improvements) on Wilson coefficients, robust to small spectral anisotropies thanks to the cone-inclusion property. These positivity cones, combined with the S-gate conditions of Sec. 22.1, define the admissible theory region used in the global synthesis of Sec. 22.4.

22.3 Thermodynamic/dissipative constraints

Aim. We formalize the **D-gate**: all transport and dissipative responses emergent from the spectral medium must be *passive* (no negative work extraction), satisfy *KMS* (thermal) analyticity, and obey an *entropy-production inequality*. These principles yield convex, basis-independent *positivity constraints* on conductivities, viscosities, and mixed transport; they can be implemented as linear-matrix inequalities (LMIs) on the tensors that couple thermodynamic forces to fluxes, including anisotropies controlled by the director field χ^μ.

1. Linear response, passivity, and KMS

Retarded response and work. For a (conserved) operator $\mathcal{J}_a^\mu$ (charge, momentum, heat) coupled to a source A_μ^a, the induced work in a finite-time protocol is

$$W \;=\; \int d^4x\, A_\mu^a(x)\,\langle \mathcal{J}_a^\mu(x)\rangle \;=\; \int \frac{d\omega}{2\pi}\, A_a^*(\omega)\left[-i\,\omega\,\chi_{ab}^R(\omega)\right] A_b(\omega), \tag{22.37}$$

with $\chi_{ab}^R(\omega)$ the retarded susceptibility (Fourier in the local rest frame). *Passivity* (no net work extraction by cyclic sources) demands, for any real A,

$$\boxed{\omega\;\operatorname{Im}\chi_{ab}^R(\omega) \;\succeq\; 0 \qquad (\omega>0)\;,} \tag{22.38}$$

i.e. the Hermitian matrix $\omega\,\mathrm{Im}\,\chi^R(\omega)$ is positive semidefinite (PSD). KMS analyticity at temperature T relates the Keldysh correlator to χ^R:

$$G^K_{ab}(\omega) \;=\; \coth\!\left(\frac{\omega}{2T}\right) 2\,\mathrm{Im}\,\chi^R_{ab}(\omega), \tag{22.39}$$

so that noise PSDs are non-negative automatically when (22.38) holds.

Kubo formulas (DC limits). For conserved charges and stress,

$$\sigma_{ij} = \lim_{\omega\to 0^+}\frac{1}{\omega}\,\mathrm{Im}\,G^R_{J_iJ_j}(\omega,\mathbf{k}{=}0) \;\succeq\; 0, \tag{22.40}$$

$$\eta_{ijkl} = \lim_{\omega\to 0^+}\frac{1}{\omega}\,\mathrm{Im}\,G^R_{T_{ij}T_{kl}}(\omega,\mathbf{k}{=}0)\,, \quad \zeta = \tfrac{1}{9}\,\eta_{iijj}, \tag{22.41}$$

and (22.38) implies that the *symmetric* parts of σ and η are PSD in the appropriate index pairings. Antisymmetric (Hall) pieces are non-dissipative and unconstrained by (22.38) but subject to C/A gates (Sec. 22.2).

2. Entropy current and Onsager structure

Constitutive relations. In the aether rest frame u^μ, write

$$J^\mu = nu^\mu + \nu^\mu, \qquad T^{\mu\nu} = (\varepsilon{+}p)u^\mu u^\nu + p\,g^{\mu\nu}_{\rm eff} + \pi^{\mu\nu}, \qquad q^\mu \equiv T^{\mu\nu}u_\nu - \varepsilon u^\mu, \tag{22.42}$$

with $u_\mu\nu^\mu = u_\mu q^\mu = u_\mu\pi^{\mu\nu} = 0$. To first order in gradients (Landau frame),

$$\nu^\mu = \sigma^{\mu\nu}\left(E_\nu - T\nabla_\nu\frac{\mu}{T}\right), \qquad q^\mu = -\kappa^{\mu\nu}\,\nabla_\nu T, \tag{22.43}$$

$$\pi^{\mu\nu} = -2\,\eta^{\mu\nu\alpha\beta}\,\nabla_\alpha u_\beta - \zeta\,\Delta^{\mu\nu}\,\nabla{\cdot}\,u, \qquad \Delta^{\mu\nu} = g^{\mu\nu}_{\rm eff} + u^\mu u^\nu, \tag{22.44}$$

where $\eta^{\mu\nu\alpha\beta}$ has the symmetries of a rank-4 viscosity tensor.

Entropy production. The canonical entropy current

$$J^\mu_S = su^\mu - \frac{\mu}{T}\,\nu^\mu - \frac{1}{T}\,q^\mu \tag{22.45}$$

obeys

$$\partial_\mu J^\mu_S = \underbrace{\left(E_\mu - T\nabla_\mu\tfrac{\mu}{T}\right)\sigma^{\mu\nu}\left(E_\nu - T\nabla_\nu\tfrac{\mu}{T}\right)}_{\geq 0} + \underbrace{\frac{1}{T}\,\nabla_\mu T\,\kappa^{\mu\nu}\,\nabla_\nu T}_{\geq 0} + \underbrace{\frac{1}{T}\,\nabla_\alpha u_\beta\,\eta^{\alpha\beta\gamma\delta}\,\nabla_\gamma u_\delta + \frac{\zeta}{T}\,(\nabla{\cdot}\,u)^2}_{\geq 0}. \tag{22.46}$$

Hence the *D-gate* requires the symmetric parts (in appropriate indices) to be PSD:

$$\boxed{\sigma^{(\mu\nu)} \succeq 0, \qquad \kappa^{(\mu\nu)} \succeq 0, \qquad \eta^{(\mu\nu)(\alpha\beta)} \succeq 0, \qquad \zeta \geq 0} \tag{22.47}$$

and Onsager reciprocity (with time-reversal signatures τ_a)

$$L_{ab} = \tau_a\,\tau_b\,L_{ba}, \qquad \mathbf{J} = L\,\mathbf{X}, \quad \mathbf{X} = \{\nabla T,\; E - \nabla(\mu/T),\; \nabla u\}. \tag{22.48}$$

3. Director-induced anisotropy and tensor decomposition

Projectors. Let u^μ be the rest frame and $\hat{\chi}^\mu$ a unit spatial director ($u_\mu \hat{\chi}^\mu = 0$). Define projectors on the spatial slice $\Delta^{\mu\nu}$:

$$P_\parallel^{\mu\nu} = \hat{\chi}^\mu \hat{\chi}^\nu, \qquad P_\perp^{\mu\nu} = \Delta^{\mu\nu} - \hat{\chi}^\mu \hat{\chi}^\nu, \qquad \mathcal{R}^{\mu\nu\alpha\beta} = \tfrac{1}{2}\Big(P_\perp^{\mu\alpha} P_\perp^{\nu\beta} + P_\perp^{\mu\beta} P_\perp^{\nu\alpha}\Big) - \tfrac{1}{d_\perp} P_\perp^{\mu\nu} P_\perp^{\alpha\beta}, \tag{22.49}$$

with $d_\perp = 2$ in 3+1 dimensions.

Anisotropic transport matrices. Decompose, e.g., the electric conductivity and viscosity as

$$\sigma^{\mu\nu} = \sigma_\parallel \, P_\parallel^{\mu\nu} + \sigma_\perp \, P_\perp^{\mu\nu} + \sigma_H \, \epsilon^{\mu\nu\rho\sigma} u_\rho \hat{\chi}_\sigma, \tag{22.50}$$

$$\eta^{\mu\nu\alpha\beta} = \eta_\perp \, \mathcal{R}^{\mu\nu\alpha\beta} + \eta_\parallel \, P_\parallel^{\mu\langle\alpha} P_\parallel^{\beta\rangle\nu} + \eta_\times \Big(P_\parallel^{\mu\langle\alpha} P_\perp^{\beta\rangle\nu} + P_\perp^{\mu\langle\alpha} P_\parallel^{\beta\rangle\nu} \Big) + \eta_H \, \epsilon^{\mu\nu\rho(\alpha} u_\rho \hat{\chi}^{\beta)}, \tag{22.51}$$

where $\langle\cdot\rangle$ denotes symmetrization and traceless projection on the indicated pair. The D-gate enforces

$$\boxed{\sigma_\parallel, \ \sigma_\perp \ \geq 0, \qquad \begin{pmatrix} \eta_\perp & \eta_\times \\ \eta_\times & \eta_\parallel \end{pmatrix} \succeq \ 0, \qquad \zeta \geq 0} \tag{22.52}$$

while σ_H, η_H are non-dissipative (odd under time reversal) and unconstrained by (22.52).

4. Hydrodynamic modes and attenuation bounds

Diffusion. Charge/heat diffusion modes $\omega(k) = -iDk^2 + \cdots$ require

$$D_\parallel = \frac{\sigma_\parallel}{\chi_\parallel} \geq 0, \qquad D_\perp = \frac{\sigma_\perp}{\chi_\perp} \geq 0, \tag{22.53}$$

with static susceptibilities $\chi_{\parallel,\perp} > 0$ from equilibrium thermodynamics.

Sound. Longitudinal sound $\omega(k) = \pm c_s k - i\Gamma_s k^2 + \cdots$ with

$$\Gamma_s = \frac{1}{2(\varepsilon + p)} \Big(\tfrac{4}{3}\eta_{\mathrm{L}} + \zeta \Big) \ + \ \cdots \ \geq 0, \tag{22.54}$$

where η_{L} is the longitudinal combination of η set by $\hat{\chi}$; (22.52) ensures $\Gamma_s \geq 0$.

5. Memory kernels, causality windows, and sum rules

Causal memory. For finite relaxation times the constitutive relations take the convolution form

$$J_a(t) = \int_{-\infty}^{t} dt' \ K_{ab}(t - t') \, X_b(t'), \qquad K_{ab}(t < 0) = 0, \tag{22.55}$$

with X_b forces. Passivity implies, for $\omega > 0$, $\omega \, \mathrm{Im}\, \tilde{K}(\omega) \succeq 0$; Kramers–Kronig then provides *sum rules*, e.g.

$$\int_0^\infty \frac{d\omega}{\pi} \ \frac{\mathrm{Im}\, \chi_{ab}^R(\omega)}{\omega} = \tfrac{1}{2} \, \chi_{ab}^{\mathrm{stat}} \ \succeq \ 0, \tag{22.56}$$

relating dissipative weight to static susceptibilities.

6. Schwinger–Keldysh (SK) effective action and entropy

SK structure. Introduce doubled fields $\phi_{1,2}$ and r/a-basis $\phi_r = \frac{1}{2}(\phi_1+\phi_2)$, $\phi_a = \phi_1-\phi_2$. An SK action $\mathcal{I}_{\rm SK}[\phi_r, \phi_a]$ consistent with unitarity and KMS can be organized as

$$\operatorname{Im}\mathcal{I}_{\rm SK} \;=\; \tfrac{1}{2}\int d^4x\, \phi_a\, \mathbb{N}\, \phi_a \;+\; \cdots, \qquad \mathbb{N} \succeq 0, \tag{22.57}$$

so that Gaussian noise covariance $\mathbb{N}$ is PSD. KMS then fixes $\mathbb{N}$ in terms of the dissipative part of the r-a kernel, reproducing (22.39) and ensuring $\partial_\mu J^\mu_S \geq 0$ at the level of the effective action.

7. Algorithmic D-gate (implementation) and kill criteria

Inputs. Hydrodynamic/EFT tensors $(\sigma^{\mu\nu}, \kappa^{\mu\nu}, \eta^{\mu\nu\alpha\beta}, \zeta)$ possibly anisotropic via $\hat{\chi}^\mu$; static susceptibilities; retarded correlators (or their low-frequency expansions) from the spectral model.

LMIs (frequency domain). For a finite grid $\omega_n > 0$,

$$\forall n: \quad \omega_n\, \operatorname{Im}\chi^R_{ab}(\omega_n) \;\succeq\; 0 \quad\Rightarrow\quad \text{PSD tests on } \{\sigma(\omega_n),\ \eta(\omega_n),\ \kappa(\omega_n)\}. \tag{22.58}$$

DC limits give (22.47) and (22.52). In the time domain, discretize $K_{ab}(t)$ with $K(t<0)=0$ and enforce $\sum_{m,n} A_m K_{mn} A_n \geq 0$ for all test vectors A (Toeplitz PSD).

Kill criteria.

(D1) $\exists\, \omega > 0:\ \omega\, \operatorname{Im}\chi^R(\omega) \not\succeq 0 \ \Rightarrow$ reject (passivity violation);

(D2) $\sigma^{(\mu\nu)} \not\succeq 0$ or $\left(\begin{smallmatrix} \eta_\perp & \eta_\times \\ \eta_\times & \eta_\parallel \end{smallmatrix}\right) \not\succeq 0$ or $\zeta < 0 \ \Rightarrow$ reject (negative entropy production);

(D3) Diffusion/sound with $D<0$ or $\Gamma_s<0 \ \Rightarrow$ reject (ill-posed attenuation);

(D4) KMS/noise mismatch in SK fit $(\mathbb{N} \not\succeq \coth(\omega/2T)\, 2\, \operatorname{Im}\chi^R) \ \Rightarrow$ reject. (22.59)

8. Worked illustrations

EM conductivity with director. Using (22.50), DC Joule heating rate is $Q = \mathbf{E}\cdot\boldsymbol{\sigma}\,\mathbf{E} = \sigma_\parallel(\mathbf{E}\cdot\hat{\chi})^2 + \sigma_\perp|\mathbf{E}_\perp|^2 \geq 0$, hence $\sigma_{\parallel,\perp} \geq 0$.

Shear viscosity block. For a planar shear in the $\perp$-subspace, energy dissipation rate is $Q = \eta_\perp\,(\partial_{(x}u_{y)})^2 + 2\eta_\times(\partial_{(x}u_{\parallel)})(\partial_{(x}u_{y)}) + \eta_\parallel(\partial_{(\parallel}u_{\parallel)})^2 \geq 0$, which is equivalent to the 2×2 PSD condition in (22.52).

9. Summary

The D-gate is the thermodynamic pillar of the spectral framework: (i) passivity $\Rightarrow$ $\omega\, \operatorname{Im}\chi^R(\omega) \succeq 0$; (ii) KMS $\Rightarrow$ consistent fluctuation–dissipation; (iii) a canonical entropy current with $\partial_\mu J^\mu_S \geq 0$ fixes the PSD character of transport tensors; (iv) director-induced anisotropies are tested by PSD blocks in adapted projectors; (v) hydrodynamic attenuations (D, Γ_s) are non-negative. Together with the S- and C/A-gates, these constraints carve out a convex, algorithmically testable admissible region in transport space that all spectral realizations must inhabit.

22.4 Summary of admissible volumes

Aim. We synthesize the **global admissible region** of the spectral framework by intersecting the consistency gates established in Secs. 22.1–22.3 across sectors (gauge, scalar/Higgs, fermions, transport) and across scales along the RG trajectory (Part 3). We give a precise set-theoretic and convex-analytic description, state structural properties (closure, convexity, gauge-parameter invariance, monotonicity under coarse graining), formulate computable relaxations as LMIs, and define *decisive tests* as boundary-saturation problems.

1. Parameterization and sectoral cones

Wilson/transport parameter vector. Let

$$\mathbf{p}(\mu) \;:=\; \Big(\; \mathbf{c}(\mu),\; \boldsymbol{\kappa}(\mu),\; \boldsymbol{\alpha}(\mu),\; \delta\boldsymbol{\kappa}(\mu);\; \boldsymbol{\sigma}(\mu),\; \boldsymbol{\eta}(\mu),\; \zeta(\mu),\; \boldsymbol{\kappa}_{\mathrm{th}}(\mu)\;;\; G^{\mu\nu}(\mu) \;\Big), \tag{22.60}$$

where $\mathbf{c}$ collects EFT Wilsons (Warsaw/Higgs/gauge bases), $\boldsymbol{\kappa}$ the canonical kinetic normalizations (e.g. κ_G, κ_y, Z_h), $\boldsymbol{\alpha}$ higher-derivative coefficients, $\delta\boldsymbol{\kappa}$ anisotropy tensors (Sec. 22.1), and $(\boldsymbol{\sigma}, \boldsymbol{\eta}, \zeta, \boldsymbol{\kappa}_{\mathrm{th}})$ the transport matrices (Sec. 22.3). The co-metric $G^{\mu\nu}$ encodes cone deformations relative to g_{eff}.

Sectoral feasibility sets at fixed scale μ.

$$\mathcal{S}_{\mathrm{S}}(\mu) \;:=\; \Big\{\mathbf{p}:\; \text{ghost-free \& hyperbolic} \;\Leftrightarrow\; \mathsf{K}_{tt} \succ 0,\; \text{Lorentzian principal symbol, CI}\Big\}, \tag{22.61}$$

$$\mathcal{S}_{\mathrm{CA/TQ}}(\mu) \;:=\; \Big\{\mathbf{p}:\; \text{forward analyticity \& positivity} \;\Leftrightarrow\; H_i(\mathbf{c}) \succeq 0 \text{ (Hankel)},\; \widehat{\mathcal{M}}\; t{<}0 \text{ improvements}\Big\} \tag{22.62}$$

$$\mathcal{S}_{\mathrm{D}}(\mu) \;:=\; \Big\{\mathbf{p}:\; \text{passivity/KMS/entropy} \;\Leftrightarrow\; \omega\,\mathrm{Im}\,\chi^{R}(\omega) \succeq 0,\; \sigma^{(\cdot)} \succeq 0,\; \left(\begin{smallmatrix}\eta_\perp & \eta_\times\\ \eta_\times & \eta_\parallel\end{smallmatrix}\right) \succeq 0,\; \zeta \geq 0\Big\}. \tag{22.63}$$

Each set is the solution of *linear matrix inequalities* (LMIs) plus linear equalities (normalizations, gauge-parameter independence of physical blocks).

Admissible set at fixed scale.

$$\boxed{\mathcal{V}_{\mathrm{adm}}(\mu) \;:=\; \mathcal{S}_{\mathrm{S}}(\mu) \;\cap\; \mathcal{S}_{\mathrm{CA/TQ}}(\mu) \;\cap\; \mathcal{S}_{\mathrm{D}}(\mu)} \tag{22.64}$$

By construction $\mathcal{V}_{\mathrm{adm}}(\mu)$ is a closed, convex, semi-algebraic cone (up to affine normalizations).

2. RG flow tubes and global admissibility

Flow map and tube. Let $\Phi_{\mu}^{\ \mu'}$ denote the RG flow map (solution of the beta system with thresholds, Sec. 3.2) sending $\mathbf{p}(\mu)$ to $\mathbf{p}(\mu')$. Define the *admissible tube*

$$\mathcal{T}_{\mathrm{adm}}[\mu_{\mathrm{IR}} \to \mu_{\mathrm{UV}}] \;:=\; \Big\{\; \mathbf{p}(\cdot)\ \text{trajectory} \;\Big|\; \forall \mu \in [\mu_{\mathrm{IR}}, \mu_{\mathrm{UV}}]:\; \mathbf{p}(\mu) \in \mathcal{V}_{\mathrm{adm}}(\mu) \;\Big\}. \tag{22.65}$$

Consistency across scales requires *non-empty intersection* after transporting all sections to a common scale:

$$\boxed{\mathcal{V}_{\rm adm}^{\rm global} := \bigcap_{k=0}^{K} \Phi_{\mu_k}^{\mu_\star}\big[\mathcal{V}_{\rm adm}(\mu_k)\big] \neq \varnothing, \quad \{\mu_k\} \text{ grid in } [\mu_{\rm IR}, \mu_{\rm UV}] \ .} \tag{22.66}$$

Monotonicity and thresholds. Positivity cones (Hankel LMIs) are preserved under integrating out heavy states (complete-the-square arguments), and passivity cones are preserved under coarse graining (Bochner positivity). Therefore the projection of $\mathcal{V}_{\rm adm}(\mu)$ to fewer operators is *monotone*: admissibility cannot be lost by *discarding* irrelevant directions, though it can be lost by RG *driving* into a forbidden region if beta functions violate cone tangency.

3. Gauge fixing, field redefinitions, and basis invariance

Gauge-parameter invariance. Let ξ denote gauge-fixing parameters. On the BRST-physical subspace the principal symbol and forward amplitudes are ξ-independent; hence $\mathcal{V}_{\rm adm}$ is *gauge-parameter invariant*.

Field/basis changes. Under linear field redefinitions $F : \mathbf{c} \mapsto \mathbf{c}' = T\mathbf{c}$, Hankel LMIs transform by congruence $H(\mathbf{c}) = H_0 + \sum_r c_r H_r \mapsto H'(\mathbf{c}') = Q^\top H(\mathbf{c}) Q$ for an invertible Q, preserving PSD. Thus admissibility is *basis covariant* (Warsaw/Higgs/gauge bases).

4. Computable inner/outer approximations

Truncation and IR subtractions. At finite derivative order $D_{\max}$ and with t-channel poles subtracted (Sec. 22.2), we obtain a *tractable* LMI:

$$H_i^{(n_{\max})}(\mathbf{c}) \succeq 0, \qquad i \in \mathcal{I}_{\rm chan}, \quad n_{\max} \le n(D_{\max}), \tag{22.67}$$

defining an *outer* approximation $\overline{\mathcal{V}}_{\rm adm}(\mu)$ (superset). Adding $t{<}0$ derivatives and multiple channels yields an *inner* approximation $\underline{\mathcal{V}}_{\rm adm}(\mu)$ sandwiching the exact set:

$$\underline{\mathcal{V}}_{\rm adm}(\mu) \subseteq \mathcal{V}_{\rm adm}(\mu) \subseteq \overline{\mathcal{V}}_{\rm adm}(\mu). \tag{22.68}$$

Anisotropy cone. Cone inclusion $G \preceq_{\rm cone} g_{\rm eff}$ (Sec. 22.1) is implemented as LMIs on $G^{\mu\nu}$ in a tetrad frame: $e^a{}_\mu G^{\mu\nu} e^b{}_\nu \preceq \eta^{ab}$ on the null cone. Linearization in small $\delta\kappa$ reduces to block-PSD constraints.

5. Decisive tests as boundary-saturation problems

Extremal rays and support function. Let $\mathcal{C} \equiv \mathcal{V}_{\rm adm}^{\rm global}$ (transported to a common $\mu_\star$). For an observable linear functional $L(\mathbf{p}) = \ell \cdot \mathbf{p}$ the *admissible range* is

$$L_{\min\,/\,\max} = \min\,/\,\max_{\mathbf{p}\in\mathcal{C}} \ \ell\cdot\mathbf{p}. \tag{22.69}$$

If data demand $L < L_{\min}$ or $L > L_{\max}$, the model is *decisively falsified*. Operationally this is an SDP (semi-definite program) using (22.67) and D-gate LMIs.

Boundary signatures. Saturation of Hankel PSD ($\det H = 0$) implies forward amplitudes with *minimal* curvature; saturation of passivity ($\sigma_{\parallel}{=}0$ or $\eta_{\perp}\eta_{\parallel}{=}\eta_{\times}^2$) implies nearly nondissipative transport. Such saturations yield characteristic signatures (e.g. maximal light-by-light scattering anisotropy at fixed cutoff) that can serve as *decisive tests*.

6. Worked miniature: EW photon sector with anisotropy

Setup. Take (α_1, α_2) from $\Delta\mathcal{L} = (\alpha_1/\Lambda^4)(F^2)^2 + (\alpha_2/\Lambda^4)(F\tilde{F})^2$ and a tiny director deformation $\delta\kappa\, u_\mu u_\nu F^{\mu\alpha}F^{\nu}{}_{\alpha}$.

Admissible slice. From Sec. 22.2:

$$\alpha_1 \ \geq 0, \qquad \alpha_1 + \alpha_2 \ \geq 0, \tag{22.70}$$

and from Sec. 22.1:

$$\kappa_G > 0, \qquad \delta\kappa > -1. \tag{22.71}$$

Thus the admissible polygon in (α_1, α_2) is a convex wedge with edge $\alpha_1{+}\alpha_2{=}0$ and apex at the origin; the anisotropy coordinate adds a half-line $\delta\kappa > -1$, producing a convex prism (cartesian product) at fixed Λ.

7. Data integration, posteriors, and kill criteria

Admissibility-first inference. Let $\Pi(\mathbf{p})$ be a prior and $\mathcal{L}_{\text{data}}(\mathbf{p})$ the global likelihood (Parts **??**, **??**, 18.4). Define the *admissible posterior*

$$\mathsf{Post}(\mathbf{p}) \ \propto \ \mathcal{L}_{\text{data}}(\mathbf{p})\ \Pi(\mathbf{p})\ \mathbf{1}_{\mathcal{C}}(\mathbf{p}), \qquad \mathcal{C} = \mathcal{V}_{\text{adm}}^{\text{global}}, \tag{22.72}$$

i.e. a hard feasibility gate. If $\mathcal{C} = \varnothing$ the model is *ruled out* (kill); if $\mathcal{C} \neq \varnothing$ but data prefer its boundary, targeted *decisive tests* (boundary saturations) are prioritized.

8. Summary (properties of $\mathcal{V}_{\text{adm}}$)

Structural properties.

(P1) Closure/convexity: $\mathcal{V}_{\text{adm}}(\mu)$ is closed and convex (LMI feasible set);
(P2) Gauge invariance: independent of gauge-fixing parameters, BRST exact parts drop out;
(P3) Basis covariance: stable under linear field/operator redefinitions;
(P4) Monotonicity: projections to subsets of operators preserve feasibility;
(P5) RG coherence: $\mathcal{V}_{\text{adm}}^{\text{global}} \neq \varnothing$ iff the admissible tube (22.65) exists;
(P6) Decisiveness: boundary saturations produce concrete, falsifiable signatures.
(22.73)

Conclusion. The admissible volume is the geometric envelope of the spectral program: the intersection of *stability/hyperbolicity*, *causality/analyticity/positivity*, and *thermodynamic passivity* cones, transported consistently along the RG. It furnishes a *prior-free*, *convex*, and *computable* scaffold on which global fits are performed and decisive tests are designed. Empty intersection falsifies the construction; a non-empty yet thin region prescribes a focused strategy for discovery.

Chapter 23

Where the Spectral Model Wins or Fails vs. SM

23.1 Decision tables across observables

Aim. We provide a *decision algebra* to classify, for each observable family, whether the Spectral Model (SpM) *wins*, is *indistinguishable*, or *fails* against the Standard Model (SM), subject to the global consistency gates (S/G/TQ/D) of Part 22. The procedure is likelihood-based, gate-aware, and basis-independent, and it outputs *decision tables* that summarize: (i) the relevant spectral handles, (ii) the mapping to observables, (iii) the statistical decision rule and penalties for extra parameters, and (iv) decisive follow-up tests.

1. Likelihoods, penalties, and decision algebra

Observable vector and theory maps. Let $\mathbf{y} \in \mathbb{R}^N$ collect measured observables with covariance C. The SM prediction is $\mathbf{f}_{\rm SM}(\boldsymbol{\theta}_{\rm SM})$, while the SpM prediction is

$$\mathbf{f}_{\rm SpM}(\mathbf{p}) \;=\; \mathbf{f}_{\rm SM}(\boldsymbol{\theta}_{\rm SM}) \;+\; M\,\mathbf{c} \;+\; \mathbf{n}(\boldsymbol{\varepsilon}, \mathbf{a}_0) \;+\; \Delta_{\rm loop}(\delta\kappa, \ldots), \tag{23.1}$$

where $\mathbf{p}$ is the spectral parameter vector (cf. Eq. (22.60)), M is the linear EFT map from Wilsons $\mathbf{c}$ to low-energy shifts, $\mathbf{n}$ encodes neutrino-NSI/aniso pieces (when relevant), and $\Delta_{\rm loop}$ contains loop-level deformations (e.g. oblique/dipole entries).

Deviance and information criteria. Define the Gaussian deviance

$$\chi^2_{\rm SM} = \Big(\mathbf{y} - \mathbf{f}_{\rm SM}\Big)^{\top} C^{-1} \Big(\mathbf{y} - \mathbf{f}_{\rm SM}\Big), \quad \chi^2_{\rm SpM} = \min_{\mathbf{p}\in\mathcal{C}} \Big(\mathbf{y} - \mathbf{f}_{\rm SpM}(\mathbf{p})\Big)^{\top} C^{-1} \Big(\mathbf{y} - \mathbf{f}_{\rm SpM}(\mathbf{p})\Big), \tag{23.2}$$

with $\mathcal{C}$ the admissible (gate-feasible) set of Sec. 22.4. Let Δk be the effective number of *active* spectral directions in the fit (rank of M over $\mathcal{C}$ restricted to the dataset). We then form

$$\Delta\chi^2 \;:=\; \chi^2_{\rm SM} - \chi^2_{\rm SpM} \;\geq 0, \qquad \Delta{\rm BIC} \;:=\; \underbrace{\chi^2_{\rm SpM} + \Delta k\, \ln N}_{\rm BIC(SpM)} \;-\; \underbrace{\chi^2_{\rm SM}}_{\rm BIC(SM)}\,. \tag{23.3}$$

Negative ΔBIC favors SpM after penalizing complexity; Wilks/Chernoff caveats are handled by Monte Carlo when parameters hit boundaries of $\mathcal{C}$.

Decision rule (per family). For a given observable family $\mathcal{F}$ (with $N_{\mathcal{F}}$ points):

$$\text{Decision}(\mathcal{F}) = \begin{cases} \textsc{Win}, & \Delta\chi^2 \geq \tau_{\text{strong}}, \quad \Delta\text{BIC} \leq 0, \quad \mathbf{p}^\star \in \mathcal{C}, \\ \textsc{Lean Win}, & \tau_{\text{mild}} \leq \Delta\chi^2 < \tau_{\text{strong}}, \quad \Delta\text{BIC} \leq 0, \quad \mathbf{p}^\star \in \mathcal{C}, \\ \textsc{Indist.}, & \Delta\chi^2 < \tau_{\text{mild}} \text{ or } \Delta\text{BIC} > 0 \text{ (no robust gain)}, \\ \textsc{Fail}, & \text{any gate violated or } \Delta\chi^2 < 0 \text{ (overfit/underfit)}. \end{cases} \tag{23.4}$$

We adopt $\tau_{\text{strong}} = 9$ ($\sim 3\sigma$) and $\tau_{\text{mild}} = 4$ as defaults, adjustable per family and correlation structure.

2. Canonical decision table (master view)

Family	Key spectral handles	Map highlights
EW precision (LEP/SLD)	$(\kappa_G,\ c_{HWB}, c_{HD}, \delta\kappa)$	$S,T,U \leftrightarrow (c_{HWB}, c_{HD})$; s_W^2 geometry (Sec.
Higgs rates & self-coupling	$(Z_h,\ \lambda_3,\ c_{HW/HB/HWB}, c_{H\Box}, c_{fH})$	κ-framework $\subset$ SpM EFT; VBS sensitivity
TGC/QGC (diboson)	$(\delta g_{1Z}, \delta\kappa_\gamma, \lambda_Z)$ from c_i	helicity amplitudes $\to$ LMIs (Sec. 9.2)
Flavor/CP & EDMs	$(\tilde{d}_{q,\ell},\ w,\ C_{\not{C}\not{P}}^{(4f)})$	sum-rules & hadronic matrix elements
LFV/rare decays	$(C^{\mu e\gamma}, C^{\mu 3e}, C^{bs\ell\ell})$	RG mixing $\to$ radiative LFV; semi-leptoni
Neutrino LBL (NSI/aniso)	$(\varepsilon_{\alpha\beta},\ a_{\alpha\beta,0})$	Hamiltonian (20.29); series (20.34)
QCD & jets	$(\kappa_G,\ \alpha_s,\ \tau_s)$	event-shape moments; $\alpha_s(Q)$ flow windows
Dark portals	$(g_D/m_{Z'},\ y_\nu y_f/m_S^2)$	NSI map (20.30,20.31)

3. Worked construction (per family)

Linearized pull and gain. In the linear regime $\mathbf{f}_{\text{SpM}} \simeq \mathbf{f}_{\text{SM}} + M\mathbf{c}$ with gate-feasible $\mathbf{c}$, the best-fit gain is

$$\Delta\chi^2 \;=\; \mathbf{r}^\top C^{-1}\mathbf{r} \;-\; \min_{\mathbf{c}\in\mathcal{C}_{\text{pos}}} (\mathbf{r}-M\mathbf{c})^\top C^{-1}(\mathbf{r}-M\mathbf{c}), \qquad \mathbf{r} := \mathbf{y} - \mathbf{f}_{\text{SM}}, \tag{23.5}$$

a convex QP or SDP when $\mathcal{C}_{\text{pos}}$ enforces Hankel LMIs from Sec. 22.2. Active directions (nonzero components at the minimum) determine Δk in (23.3).

Gate-aware profiling. All fits are *constrained* to the admissible cone $\mathcal{C}$ (Sec. 22.4); any candidate point violating S/G/TQ/D gates is discarded prior to likelihood evaluation.

4. Electroweak (master example)

Oblique layer. With (S,T,U) from $(c_{HWB}, c_{HD}, \delta\kappa)$, the Z-pole residuals obey

$$\delta\mathbf{O}_Z \;=\; A_S\,S + A_T\,T + A_U\,U \;+\; \mathcal{O}(\Lambda^{-4}), \tag{23.6}$$

with known coefficient vectors $A_{S,T,U}$. The constrained minimization over (S,T,U) subject to positivity/causality (no superluminal cones, Sec. 17.3) yields $\Delta\chi^2_{\text{oblique}}$.

Weinberg angle from geometry. From Sec. 7.4,

$$\sin^2\theta_W^{\rm geom} \;=\; \frac{\kappa_2}{\kappa_1+\kappa_2} \;+\; \Delta_{\rm th}(\mu), \tag{23.7}$$

with threshold/running correction $\Delta_{\rm th}$. The decision adds a *hard* pull on $\sin^2\theta_W$:

$$\chi^2_{s_W} = \frac{\left(\sin^2\theta_W^{\rm exp} - \sin^2\theta_W^{\rm geom}\right)^2}{\sigma^2_{s_W}}\,. \tag{23.8}$$

A WIN requires that Eqs. (23.6)–(23.8) together drive $\Delta\chi^2 \geq \tau$ with $\Delta\text{BIC} \leq 0$.

5. Condensed decision tables (per dataset block)

Block	Handles	Rule (95% C.L.)	Notes
Z-pole (LEP/SLD)	S, T, s^2_W	$\Delta\chi^2 \geq 4$, $\Delta\text{BIC} \leq 0$	include correlations
WW/WZ diboson	$\delta g_{1Z}, \delta\kappa_\gamma, \lambda_Z$	helicity-bin QP, LMI pass	EFT truncation check
Higgs signal strengths	κ_V, κ_f, $c_{H\Box}$	global $\Delta\text{BIC} \leq 0$	include theory σ
VBS ($VVjj$)	$c_{W,B,WB}$	positivity LMIs + fit gain	unitarity-safe bins only
EDMs (global)	$\tilde{d}_f, w, C^{(4f)}$	joint likelihood gain	hadronic nuisance profiling
LFV ($\mu\to e\gamma, 3e$)	C's	CL_s gain, gates pass	RG mixing included
NSI (LBL)	ε, a_0	convex ND/FD feasibility	magic-baseline diagnostics
Event shapes	α_s, τ_s	$\Delta\chi^2$ on moments	resummation scheme fixed

6. Boundary-driven decisive tests

Positivity edge. Saturation of a forward-positivity LMI (smallest eigenvalue $\to 0^+$) predicts enhanced polarization patterns in $VV\to VV$; dedicated helicity analyses at fixed m_{VV} bins provide a *decisive* discriminator.

Dissipation edge. Near-vanishing $\sigma_\parallel$ or $\eta_\perp\eta_\parallel \simeq \eta_\times^2$ implies nearly nondissipative transport; low-ω line-shape tests in conductive media (Part 18.2) become decisive.

7. Summary

We have established a gate-aware, penalty-conscious decision procedure: compute $\Delta\chi^2$ within the admissible cone $\mathcal{C}$, penalize via ΔBIC, and classify outcomes per Eq. (23.4). The decision tables provide an operational bridge from spectral handles to concrete datasets and decisive tests, and they will be populated with numbers in the global-fit chapter (Sec. 24.1–24.2).

23.2 Stress tests and edge cases

Aim. We codify *stress tests* that push the Spectral Model (SpM) to the boundaries of the admissible cones defined in Sec. 22, and catalog *edge cases* where EFT truncations, kinematic limits, or background deformations can mimic or hide failures. Each test is stated as a computable inequality/equality on amplitudes, spectral densities, RG flows, or transport kernels, accompanied by explicit *kill conditions*.

1. UV growth, partial-wave unitarity, and EFT truncation

Forward growth and Froissart window. For a $2\to2$ channel i with Mandelstam s at $t=0$, polynomial boundedness (22.26) with $N\le 2$ is probed by

$$\mathcal{G}_i(s) \;:=\; \frac{|\widehat{\mathcal{M}}_{ii}(s,0)|}{s^2\log^2(s/s_0)}\,, \qquad \limsup_{s\to\infty}\ \mathcal{G}_i(s) \;<\; \infty \quad \text{(pass)}. \tag{23.9}$$

Here $\widehat{\mathcal{M}}$ has t-channel IR poles subtracted (Sec. 22.2). Divergent lim sup signals EFT misuse or cone/analyticity failure.

Partial waves in longitudinal $VV\to VV$. Let $a_J^{(\lambda)}(s)$ be eigen-partial waves in a helicity block λ. The unitarity circle requires

$$\forall\, J,\lambda: \quad |\operatorname{Re} a_J^{(\lambda)}(s)| \;\le\; \tfrac{1}{2} \quad\Rightarrow\quad \sqrt{s} \;\le\; E_{\text{unit}}^{(\lambda,J)}. \tag{23.10}$$

Define the *operating margin*

$$\mathfrak{m}_{\text{unit}} \;:=\; \min_{\lambda,J}\ \frac{E_{\text{unit}}^{(\lambda,J)}}{E_{\max}^{\text{exp}}}\,, \qquad \mathfrak{m}_{\text{unit}}>1 \text{ (safe)}. \tag{23.11}$$

Saturation at $\mathfrak{m}_{\text{unit}}\downarrow 1$ indicates a *decisive-test* regime; $\mathfrak{m}_{\text{unit}}<1$ is a kill unless form-factor or UV completion parameters are activated and remain gate-feasible.

EFT truncation remainder estimator. For an amplitude expanded as $\mathcal{M}(s,t)=\sum_{n=0}^{N}A_n(s,t)(E/\Lambda)^{2n}+\mathcal{R}_{N+1}$, we adopt the conservative estimator

$$\epsilon_{\text{trunc}} \;:=\; \max_{\text{bins}}\ \frac{|A_N(s,t)|(E/\Lambda)^{2N}}{|\sum_{n=0}^{N-1}A_n(E/\Lambda)^{2n}|+\delta_{\text{exp}}} \quad\Rightarrow\quad \epsilon_{\text{trunc}} \;\le\; \epsilon_\star \text{ (e.g. 0.3)}, \tag{23.12}$$

with δ_{exp} the experimental uncertainty in the bin. We down-weight (or exclude) bins failing (23.12) in decision tables.

2. Positivity-edge saturation and $t<0$ improvements

Hankel PSD edge. Let $H_i(\mathbf{c})\succeq 0$ be the forward Hankel matrix (Sec. 22.2). Define its smallest eigenvalue $\lambda_{\min}^{(i)}$.

$$\lambda_{\min}^{(i)} \;\to\; 0^+ \quad\Longrightarrow\quad \text{edge case with enhanced helicity patterns at fixed } s. \tag{23.13}$$

We log the approaching eigenvectors to predict decisive polarization observables; $\lambda_{\min}^{(i)}<0$ is an immediate kill.

$t<0$ convex lift. Adding derivatives in t (Lehmann ellipse) sharpens bounds: build the block matrix

$$\mathbb{H}_{mn}^{(i)}(t_k)=\partial_s^{2(m+n)}\partial_t^k\,\widehat{\mathcal{M}}_{ii}(0,t_k), \qquad t_k<0, \tag{23.14}$$

and require $\mathbb{H}\succeq 0$ jointly over $\{t_k\}$. Approaching singular values diagnose which operator combinations lie at the boundary.

3. Cone stress: anisotropy and microcausality windows

Cone inclusion under large director deformation. For a deformed co-metric $G^{\mu\nu} = g_{\text{eff}}^{\mu\nu} + \delta\kappa\, u^\mu u^\nu + \beta\, \hat{\chi}^\mu \hat{\chi}^\nu$, the *cone-inclusion* (CI) inequality (22.11) reduces, in a tetrad basis, to

$$1 + \delta\kappa \;\geq\; 0, \qquad 1 + \beta \;\geq\; 0, \qquad (1 + \delta\kappa)(1 + \beta) \;-\; \gamma^2 \;\geq\; 0, \tag{23.15}$$

where γ encodes possible $u - \hat{\chi}$ mixing terms. Violation of any inequality kills the background (microcausality breach). Near-saturation defines the *microcausality window* in which dispersion and positivity still apply but phase velocities become strongly angle dependent.

4. Singular limits: IR poles, custodial alignment, chiral and massless edges

IR pole management. In photons/gluons/gravitons exchange channels, construct

$$\widehat{\mathcal{M}}(s, 0) = \mathcal{M}(s, 0) - \sum_{\text{massless}} \frac{r_{\text{pole}}}{t}\bigg|_{t\to 0}, \tag{23.16}$$

before applying positivity. Failure to subtract correctly leads to spurious $a_2 < 0$ flags; we test numerically by comparing two independent subtraction schemes.

Custodial alignment ($g' \to 0$) and Higgs decoupling. In the ($g' \to 0,\ \alpha_T \to 0$) limit the ρ parameter obeys $\rho = 1 + \mathcal{O}(\alpha_T)$; stress test:

$$\big|\rho - 1\big| \;\leq\; r_\rho \,(\text{e.g. } 10^{-3}) \quad \text{as } g' \to 0 \quad \Rightarrow \quad \text{basis and counterterm implementation consistent.} \tag{23.17}$$

Similarly, in the alignment limit $Z_h \to 1$ and $\lambda_3 \to \lambda_{\text{SM}}$ modulo calculable spectral thresholds (Sec. 8.3); large residuals indicate mis-normalized kinetic terms or hidden operators.

Chiral/massless quark limits. As $m_q \to 0$ the GMOR relation (Sec. 11.2) must be recovered with spectral corrections B_{sp}

$$m_\pi^2 f_\pi^2 \;=\; -2 m_q \langle \bar{q} q \rangle \,\Big[1 + \Delta_{\text{sp}}\Big], \qquad \Delta_{\text{sp}} \;\to\; 0 \text{ as } m_q \to 0. \tag{23.18}$$

Non-vanishing Δ_{sp} at fixed lattice spacing signals improper matching of spectral string tension inputs.

5. Threshold turbulence and heavy-field non-decoupling

Appelquist–Carazzone diagnostic. Across a heavy threshold M_H the decoupled Wilson jump must satisfy

$$\mathbf{c}(\mu \downarrow M_H) - \mathbf{c}(\mu \uparrow M_H) \;=\; \Delta\mathbf{c}_{\text{EFT}}(M_H) \;+\; \mathcal{O}\Big(\frac{\mu^2}{M_H^2}\Big), \tag{23.19}$$

with $\Delta\mathbf{c}_{\text{EFT}}$ computed from the matching functional on the fiber (Sec. 3.2). Large residuals imply hidden non-decoupling (e.g. incomplete custodial multiplets) and trigger a revision path.

6. BRST/ghost sector integrity under deformations

Nilpotency and Nielsen identities. In deformed backgrounds (anisotropy, curved $g_{\rm eff}$) the BRST charge $Q_{\rm B}$ must remain nilpotent on the gauge-fixed functional space:

$$Q_{\rm B}^2 = 0, \qquad \frac{\partial \Gamma}{\partial \xi} \;=\; \langle s(\mathcal{O}_\xi)\rangle \quad \Rightarrow \quad \frac{d\mathcal{O}_{\rm phys}}{d\xi} = 0 \ \text{(Nielsen)}. \tag{23.20}$$

Any observed ξ-dependence of S-matrix elements or physical poles is a kill (gauge-parameter contamination).

7. Transport edge: passivity and attenuation minima

Low-frequency edge. From (22.38) test the smallest eigenvalue of $\omega\,\mathrm{Im}\,\chi^R(\omega)$:

$$\lambda^{\chi}_{\rm min}(\omega) \;\geq\; 0 \quad (\omega > 0), \qquad \lim_{\omega\to 0^+} \frac{\lambda^{\chi}_{\rm min}(\omega)}{\omega} \;\equiv\; \lambda_{\rm DC} \;\geq 0. \tag{23.21}$$

A vanishing $\lambda_{\rm DC}$ locates nearly nondissipative rays (decisive-test targets); negative values kill the transport model.

8. Numerical stress: lattice/continuum and SDP robustness

Lattice/continuum cross-check. For glueball or string-tension inputs (Sec. 12.1, 10.2), verify continuum scaling by the Richardson test on three lattice spacings $a_{1,2,3}$:

$$\mathcal{O}(a) \;=\; \mathcal{O}(0) \;+\; k a^p + \cdots \quad \Rightarrow \quad p \;=\; \frac{\ln \frac{\mathcal{O}(a_1)-\mathcal{O}(a_2)}{\mathcal{O}(a_2)-\mathcal{O}(a_3)}}{\ln \frac{a_1-a_2}{a_2-a_3}}, \quad p \approx 2 \ \text{(Symanzik)}. \tag{23.22}$$

Inconsistent p indicates discretization systematics dominating the stress test.

SDP conditioning. Positivity LMIs are solved by SDP. We monitor the duality gap $g_{\rm dual}$ and minimal slack $\sigma_{\rm min}$:

$$g_{\rm dual} \;\leq\; 10^{-7}, \qquad \sigma_{\rm min} \;\geq\; 10^{-8} \quad \text{(default tolerances)}. \tag{23.23}$$

Violations imply numerical rather than physical failure; the run is repeated with scaled bases/pivoting.

9. Edge-case catalog (checklist)

- **UV-growth**: Eq. (23.9) finite (pass) / diverging (kill).
- **Partial-wave unitarity**: margin $\mathfrak{m}_{\rm unit} > 1$ (safe) / < 1 (kill or upgrade to UV-completion mode).
- **Hankel PSD**: $\lambda^{(i)}_{\rm min} \geq 0$ (pass); $= 0$ (edge signature); < 0 (kill).
- **t<0 blocks**: $\mathbb{H} \succeq 0$ (pass) / $\not\succeq 0$ (kill).
- **Cone inclusion**: inequalities (23.15) (pass) / violated (kill).

- **IR poles**: subtraction (23.16) validated by scheme cross-check.
- **Custodial/chiral limits**: Eqs. (23.17), (23.18) satisfied.
- **Decoupling**: matching jump (23.19) within tolerance.
- **BRST integrity**: Eq. (23.20) numerically verified (no ξ traces).
- **Passivity/attenuation**: Eq. (23.21) holds; $D, \Gamma_s \geq 0$ (Sec. 22.3).
- **EFT truncation**: $\epsilon_{\rm trunc} \leq \epsilon_\star$ (keep bins) / exclude otherwise.

10. Summary

Stress tests expose the precise ways in which SpM predictions can approach or cross the boundaries of the *admissible cones*. Each edge case is equipped with a quantitative diagnostic (growth, partial waves, Hankel/Lehmann blocks, cone inequalities, BRST/KMS tests, decoupling checks) and a clear kill rule. Near-saturation regimes provide *decisive* experimental targets; violations trigger *revision paths* specified in Sec. 23.4.

23.3 Parameter maps and degeneracies

Aim. We formalize the *parameter maps* from spectral variables to observables and provide a differential–geometric framework to diagnose and resolve *degeneracies*. Locally, degeneracies are kernels of a whitened Jacobian (or Fisher) and globally they are submanifolds cut by the S/G/TQ/D gates. We give explicit maps in the electroweak, Higgs, diboson, flavor/NSI, and QCD sectors, and list *design rules* that break the leading degeneracy families.

1. Master map and whitened geometry

Spectral parameter vector. At a fixed scale $\mu_\star$, collect

$$\mathbf{p} \equiv \Big(\kappa_1, \kappa_2, \kappa_G,\ Z_h,\ \lambda_3,\ \delta\kappa;\ \mathbf{c}_{\rm EFT};\ \varepsilon_{\alpha\beta}, a_{\alpha\beta,0};\ \alpha_s, \tau_s;\ \dots\Big), \tag{23.24}$$

where (κ_1, κ_2) are the fiber-metric invariants that fix (g, g') (Sec. 7.1), $\delta\kappa$ parameterizes small director-induced kinetic anisotropy (Sec. 22.1), $\mathbf{c}_{\rm EFT}$ collects Wilson coefficients in a chosen basis, and (ε, a_0) encode NSI/aniso in neutrino matter effects (Sec. 20.2).

Observable map and whitening. Let $\mathbf{y} \in \mathbb{R}^N$ be the stacked vector of measured observables with covariance C. The prediction is $\mathbf{f}(\mathbf{p})$. Define the *whitened residual* and Jacobian

$$\mathbf{r}(\mathbf{p}) \ :=\ C^{-1/2}\Big[\mathbf{y} - \mathbf{f}(\mathbf{p})\Big], \qquad J(\mathbf{p}) \ :=\ C^{-1/2}\,\frac{\partial \mathbf{f}}{\partial \mathbf{p}} \ \in\ \mathbb{R}^{N\times d_p}. \tag{23.25}$$

Near a reference point $\mathbf{p}_0$, the quadratic deviance reads

$$\chi^2(\mathbf{p}) \ \simeq\ \chi^2_0 \ +\ (\delta\mathbf{p})^\top \underbrace{J^\top J}_{\mathcal{I}_F}(\delta\mathbf{p}) \ -\ 2\,(\delta\mathbf{p})^\top J^\top \mathbf{r}_0, \tag{23.26}$$

where $\mathcal{I}_F$ is the *Fisher information*.

Local degeneracies. The *linear* degeneracy space is the kernel of J:

$$\mathcal{N}_0 \;:=\; \ker J(\mathbf{p}_0) \;=\; \Big\{\delta\mathbf{p} \neq 0 \mid J\,\delta\mathbf{p} = 0\Big\}. \tag{23.27}$$

If $\operatorname{rank} J = r$, then $\dim \mathcal{N}_0 = d_p - r$. Equivalently, the smallest singular values of J (or eigenvalues of $\mathcal{I}_F$) diagnose *near-degeneracies*.

Gate-aware tangent cone. Feasibility constraints $\mathbf{p} \in \mathcal{C}$ (Sec. 22.4) restrict displacements to the *tangent cone* $\mathsf{TC}_{\mathbf{p}_0}\mathcal{C}$. The admissible local degeneracy is

$$\mathcal{N}_0^{\mathrm{adm}} \;:=\; \ker J \;\cap\; \mathsf{TC}_{\mathbf{p}_0}\mathcal{C}. \tag{23.28}$$

2. Reparameterization, principal directions, and invariants

Basis changes. Under an invertible linear redefinition $\mathbf{p}' = T\,\mathbf{p}$, one has $J' = J\,T^{-1}$, so $\ker J' = T\,(\ker J)$. Therefore degeneracy *dimension* is basis invariant. We diagonalize $\mathcal{I}_F$ by SVD of $J = U\,\Sigma\,V^\top$ and define *principal parameter combinations* $\pi_i = (V^\top\delta\mathbf{p})_i$ with curvatures σ_i^2.

Nonlinear invariants. Exact reparameterization invariants $I(\mathbf{p})$ satisfy $dI \propto \text{row-span}(J)$-orthogonal; they label *degeneracy leaves* along which $\mathbf{f}$ is constant at the working order.

3. Canonical maps and degeneracy manifolds (worked sectors)

(a) Electroweak: (g, g'), obliques, and m_W/s_W^2

Geometry to couplings. From Secs. 7.1, 7.4:

$$g \;=\; \frac{\sqrt{\kappa_G}}{\sqrt{\kappa_1}}, \qquad g' \;=\; \frac{\sqrt{\kappa_G}}{\sqrt{\kappa_2}}, \qquad \sin^2\theta_W \;=\; \frac{\kappa_2}{\kappa_1+\kappa_2} \;+\; \Delta_{\mathrm{th}}(\mu_\star). \tag{23.29}$$

Linearizing, the differential map is

$$\frac{\delta m_W}{m_W} \;=\; \tfrac{1}{2}\delta\ln\kappa_G - \tfrac{1}{2}c_W^2\,\delta\ln\kappa_1 - \tfrac{1}{2}s_W^2\,\delta\ln\kappa_2 \;+\; \tfrac{1}{2}\,\alpha_T, \quad \delta s_W^2 \;=\; \frac{\kappa_1\,\delta\kappa_2 - \kappa_2\,\delta\kappa_1}{(\kappa_1+\kappa_2)^2} + \delta(\Delta_{\mathrm{th}}), \tag{23.30}$$

supplemented by (S, T) shifts from c_{HWB}, c_{HD} (Sec. 8.4).

EW degeneracy line. At fixed m_Z, G_F the pair $(m_W, \sin^2\theta_{\mathrm{eff}})$ constrains the 2-plane spanned by $\{\delta\ln\kappa_1, \delta\ln\kappa_2\}$ up to a line where

$$\delta m_W = 0 = \delta s_W^2 \;\;\Rightarrow\;\; \frac{\delta\kappa_2}{\kappa_2} \;=\; \frac{\delta\kappa_1}{\kappa_1} \;=\; \delta\ln\kappa_G \;-\; \alpha_T \;+\; \cdots. \tag{23.31}$$

This *EW-geometry line* is lifted by Z-pole asymmetries and W width/polarization observables, which probe orthogonal combinations and by positivity LMIs that bound (c_{HWB}, c_{HD}).

(b) Higgs: (λ_3, Z_h) vs. single/double-Higgs

Map. To leading order in the EFT (Sec. 8.3),

$$\mu_{h\to XX} \simeq 1 + a_X\,\delta Z_h + b_X\,\delta c_X\,, \qquad \sigma(gg\to hh) \simeq \sigma_{\rm SM}\Big(1 + A\,\Delta\lambda_3 + B\,\delta c_g + C\,\delta Z_h\Big), \tag{23.32}$$

where a_X, b_X, A, B, C are calculable channel weights and $\Delta\lambda_3 \equiv (\lambda_3 - \lambda_3^{\rm SM})/\lambda_3^{\rm SM}$.

Degeneracy ridge. Single-Higgs rates alone exhibit a near-flat direction $b_X\delta c_X + a_X\delta Z_h \approx$ const. Adding hh production lifts it partially, but an approximate ridge persists if A/B matches a_X/b_X. Differential m_{hh} and VBS($VVjj$) insertions (Sec. 9.2) break the ridge by accessing independent helicity structures.

(c) Diboson/TGC: $(\delta g_{1Z}, \delta\kappa_\gamma, \lambda_Z)$ vs. four-fermions

Map. In the helicity basis for $q\bar q \to WZ$,

$$\delta\mathcal{A}_{\lambda_1\lambda_2} \;\sim\; \alpha_\lambda\,\frac{s}{m_W^2}\,\delta g_{1Z} \;+\; \beta_\lambda\,\frac{s}{m_W^2}\,\delta\kappa_\gamma \;+\; \gamma_\lambda\,\frac{s}{m_W^2}\,\lambda_Z \;+\; \sum_f \eta_\lambda^f\,\frac{s}{\Lambda^2}C_f^{(4f)}. \tag{23.33}$$

A *contact–TGC degeneracy* arises if the luminosity/acceptance projects onto a combination with $\sum_f \eta^f \approx$ const $\times\,\alpha_\lambda$. Polarization tagging and high-m_{VZ} bins deform the weight matrices and separate the directions; t<0 positivity (Sec. 22.2) also constrains the contact coefficients independently.

(d) Neutrino NSI/aniso: (ε, a_0) vs. (δ, θ_{23})

Map and degeneracy. From Sec. 20.2, the golden channel obeys

$$P_{\mu e} \;\supset\; \mathcal{I}_\delta\,\cos(\Delta+\delta) \;+\; \mathcal{N}\cdot{\rm Re}(\hat\varepsilon_{e\mu},\hat\varepsilon_{e\tau}) \;+\; \tilde{\mathcal{N}}\cdot{\rm Im}(\hat\varepsilon_{e\mu},\hat\varepsilon_{e\tau}) \;+\; \mathcal{A}\cdot(a_{e\mu,0}, a_{e\tau,0}), \tag{23.34}$$

producing the well-known δ–*NSI* degeneracy $\delta_{\rm eff} \simeq \delta + \arg(s_{23}\hat\varepsilon_{e\mu} - c_{23}\hat\varepsilon_{e\tau})$ (Sec. 20.2). Dual-baseline design and $\nu/\bar\nu$ running break it by flipping A and the sign of a_0.

(e) QCD/event shapes: (α_s, τ_s)

Map. Event-shape moments M_n obey $M_n(Q) = \sum_k c_{nk}(\ln Q)\,\alpha_s^k(Q) + d_n\,\tau_s/Q + \cdots$. The slope $dM_n/d\ln Q$ isolates $\beta(\alpha_s)$ while the $1/Q$ tail lifts the α_s–τ_s degeneracy when combining multiple Q points.

4. Degeneracy resolution: design rules

Energy scaling. Exploit different E-scalings: contact/TGC $\sim s/\Lambda^2$ vs. obliques (logarithmic), vs. λ_3 (threshold-sensitive). High-mass bins (with EFT safe cuts) align principal directions and raise the smallest singular value of J.

Angles and polarizations. Use helicity projectors (e.g. W longitudinal fraction, azimuthal correlations in $VVjj$) to separate operator tensors that are degenerate in inclusive rates.

Flavor and CP tags. Leverage flavor-specific observables (e.g. $A_{\rm FB}^b$, LFV ratios) and CP-odd asymmetries (EDM-compatible) to split CP-even/odd mixtures that otherwise share the same modulus response.

Multiple media/baselines. In neutrinos, combine near+far detectors, varying matter densities (reactor vs. LBL), and $\nu/\bar{\nu}$ to orthogonalize $(\delta, \varepsilon, a_0)$.

t<0 **positivity lifts.** Include t<0 derivatives (Sec. 22.2) to correlate different Lorentz structures and eliminate flat directions invisible at $t = 0$.

5. Algorithmic pipeline

(P1) Build the map. Assemble $\mathbf{f}(\mathbf{p})$ and analytic or automatic-differentiation Jacobian J at $\mathbf{p}_0$.

(P2) Gate projection. Compute the tangent cone $\mathsf{TC}_{\mathbf{p}_0}\mathcal{C}$ from S/G/TQ/D LMIs and project J onto it:

$$J_{\rm adm} \;:=\; J\,\Pi_{\mathsf{TC}}, \tag{23.35}$$

with Π_{TC} the projector onto feasible displacements.

(P3) SVD and principal combinations. Compute $J_{\rm adm} = U\,\Sigma\,V^\top$. Report *well-constrained* combinations $\pi_i = (V^\top \delta\mathbf{p})_i$ with large σ_i, and *degenerate* combinations with $\sigma_i \approx 0$.

(P4) Design iteration. Augment $\mathbf{y}$ with candidate observables (new bins, polarizations, baselines) and recompute the minimal singular value $\sigma_{\rm min}$ as a figure of merit; choose additions that maximize $\sigma_{\rm min}$ subject to admissibility (an SDP).

6. Miniature: two-parameter degeneracy visualization

Ellipse and ridge. For $\delta\mathbf{p} = (p_1, p_2)$ with whitened gradient vectors $\mathbf{g}_i$ (columns of J),

$$\Delta\chi^2 \;\simeq\; p_1^2\,\|\mathbf{g}_1\|^2 + p_2^2\,\|\mathbf{g}_2\|^2 + 2p_1p_2\,(\mathbf{g}_1\cdot\mathbf{g}_2), \tag{23.36}$$

so the degeneracy ridge angle is $\tan 2\phi = 2(\mathbf{g}_1\cdot\mathbf{g}_2)/(\|\mathbf{g}_1\|^2 - \|\mathbf{g}_2\|^2)$. Adding an observable with gradient $\mathbf{g}_3$ rotates and shrinks the ellipse; optimal additions maximize the Gram determinant $\det(G)$ of $\{\mathbf{g}_i\}$.

7. Role of gates in sculpting degeneracies

Positivity cone. Hankel/t < 0 LMIs carve out half-spaces in Wilson space, often *cutting* the naive linear ridge and producing a bounded segment rather than a line.

Passivity cone. D-gate PSD conditions on transport exclude unphysical sign choices that could otherwise mimic parameter trade-offs in line-shape fits.

Cone inclusion. C/A-gate cone inequalities limit $\delta\kappa$ and forbid compensations that would otherwise restore m_W via acausal deformations.

8. Summary

Parameter maps $\mathbf{p} \mapsto \mathbf{f}$ are linearized by the whitened Jacobian; degeneracies are kernels intersected with the admissible tangent cone. Across sectors we identified the principal degeneracy manifolds (EW geometry line; Higgs (λ_3, Z_h) ridge; TGC–contact blend; NSI–δ trade; α_s–power-tail) and the concrete *design rules* that break them (energy scaling, polarizations, flavor/CP tags, multi-baseline, t<0 lifts). Algorithmically, SVD within the gate-projected space provides basis-invariant principal directions and a quantitative target $(\sigma_{\min})$ for experiment optimization and global-fit stability.

23.4 Kill criteria and revision paths

Aim. We formalize *kill criteria*—non-negotiable conditions that decisively falsify a spectral realization—and we design *revision paths*—minimal, ordered modifications that can restore feasibility without sacrificing the founding axioms (local isometries on $\mathcal{F}$, BRST structure, anomaly consistency, S/G/TQ/D gates). The logic is *gate-first*: feasibility is tested within the admissible cones (Sec. 22.4); only then does statistical evidence decide between SpM and SM (Sec. 23.1).

1. Kill criteria: formal definition

Feasibility emptiness (hard kill). Let $\mathcal{C} = \mathcal{V}_{\text{adm}}^{\text{global}}$ be the global admissible cone transported to a common scale $\mu_\star$ (Eq. (22.66)).

$$\boxed{\mathcal{C} = \varnothing \quad \Longrightarrow \quad \textsc{Kill (inconsistent construction)}} \tag{23.37}$$

This occurs if any S/G/TQ/D gate (Secs. 22.1–22.3) is violated along the RG tube (Eq. (22.65)).

Gate breach at any scale (hard kill). There exists $\mu \in [\mu_{\text{IR}}, \mu_{\text{UV}}]$ such that at least one of:

$$\begin{aligned}
&\textbf{(S)} \;\; \mathsf{K}_{tt}(\mu) \not\succ 0 \quad \text{or} \quad \text{principal symbol not hyperbolic},\\
&\textbf{(C/A/TQ)} \;\; \exists i: \; H_i(\mathbf{c}(\mu)) \not\geq 0 \;\; \text{or} \;\; t\text{<}0 \text{ block } \mathbb{H} \not\geq 0,\\
&\textbf{(D)} \;\; \omega \, \mathrm{Im}\, \chi^R(\omega;\mu) \not\geq 0 \text{ for some } \omega > 0 \text{ or } \{\sigma, \eta, \zeta\} \text{ violating PSD},
\end{aligned} \tag{23.38}$$

with notation as in Secs. 22.2–22.3.

Causality cone violation (hard kill). For co-metric deformation $G^{\mu\nu}$ the cone-inclusion inequalities (e.g. Eq. (23.15)) fail:

$$G \not\preceq_{\text{cone}} g_{\text{eff}} \quad \Longrightarrow \quad \textsc{Kill (microcausality breach)}. \tag{23.39}$$

BRST/anomaly integrity (hard kill).

$$Q_{\text{B}}^2 \neq 0 \text{ on the gauge-fixed functional space} \quad \text{or} \quad \mathcal{I}_{d+2} \text{ mismatch across UV/IR} \;\Longrightarrow\; \textsc{Kill}, \tag{23.40}$$

where $\mathcal{I}_{d+2}$ is the anomaly polynomial (Chs. 4, 6); a single un-matched global/Witten anomaly is fatal.

Data-driven decisive kill. With admissibility enforced ($\mathbf{p} \in \mathcal{C}$),

$$\boxed{\Delta\chi^2 \equiv \chi^2_{\rm SM} - \chi^2_{\rm SpM} < 0 \quad \text{and} \quad \Delta\mathrm{BIC} > 0 \quad \Longrightarrow \quad \textsc{Kill (overfit/penalized)}} \tag{23.41}$$

persistent across cross-validation splits or blinded subsets.

2. Operational kill checklist (with measurable proxies)

- **Unitarity margin** $\mathfrak{m}_{\rm unit}$ (Eq. (23.11)): $\mathfrak{m}_{\rm unit} < 1 \Rightarrow$ Kill (unless a gate-feasible UV form factor is specified and validated).
- **Positivity edge** (Eq. (23.13)): $\lambda^{(i)}_{\rm min} < 0 \Rightarrow$ Kill.
- **Passivity** (Eq. (23.21)): $\lambda_{\rm DC} < 0 \Rightarrow$ Kill.
- **Custodial/chiral limits** (Eqs. (23.17), (23.18)): violation at the per-mille (EW) or controlled chiral level $\Rightarrow$ Kill.
- **Decoupling** (Eq. (23.19)): residual jump $\gg \mu^2/M_H^2 \Rightarrow$ Kill.

3. Revision paths: minimal surgeries

Principles. A revision is *admissible* iff it preserves: (i) local isometries of $(\mathcal{F}, h)$ and their gauging; (ii) BRST nilpotency and gauge-parameter independence of physical observables; (iii) anomaly matching; (iv) S/G/TQ/D gates. We order surgeries from least to most invasive and attach *proof obligations*.

(R1) Reparameterization and retuning (least invasive)

$$\text{Action:} \quad \mathbf{p} \mapsto \mathbf{p}' \text{ within } \mathcal{C} \text{ (basis change } T\text{, priors, thresholds, nuisance re-profiling).} \tag{23.42}$$

Obligations: $\mathcal{C}$ unchanged under congruence; all gates re-checked; no new operators.

(R2) Threshold realignment & decoupling repair

$$\text{Action:} \quad \text{adjust heavy spectra } \{M_H\} \text{ and matching } \Delta\mathbf{c}_{\rm EFT}(M_H) \text{ to restore Eq. (23.19).} \tag{23.43}$$

Obligations: preserve anomaly inflow (Ch. 5); maintain cone inclusion and positivity.

(R3) Add higher-derivative operators (EFT lift)

$$\text{Action:} \quad \Delta\mathcal{L} = \sum_{d>6} \frac{c^{(d)}}{\Lambda^{d-4}} \mathcal{O}^{(d)} \quad \text{with} \quad H_i(\mathbf{c}) \succeq 0 \ (t{<}0 \text{ blocks included}). \tag{23.44}$$

Obligations: re-check partial-wave unitarity window; ensure SK/KMS consistency for dissipative extensions.

(R4) Portal completion (UV form-factorization)

Action: introduce gate-feasible mediators (e.g. Z', S) with couplings constrained by forward positivity (23.45) to unitarize high-energy growth (Sec. 19.3). *Obligations:* anomaly matching (mixing with hypercharge consistent), cone inclusion for kinetic mixings, BRST-closed gauge fixing.

(R5) Fiber-geometry deformation

Action: $h_{AB} \mapsto h_{AB} + \delta h_{AB}, \quad \kappa_{1,2} \mapsto \kappa_{1,2} + \delta\kappa_{1,2} \quad \Rightarrow \quad (g, g', \theta_W)$ shift per Eq. (23.29). (23.46)

Obligations: preserve isometries G; recompute moment maps and the Ehresmann connection; re-verify oblique bounds and EW data (Sec. 7.4).

(R6) Symmetry extension/breaking pattern update (most invasive)

Action: $G \to G' \supseteq G$ or G/G'' with new coset structure; Higgs potential/topology updated. (23.47)

Obligations: full anomaly re-check ('t Hooft matching), WZW/CS re-derivation, BRST sector rebuild, gate re-evaluation.

4. Kill-to-revision map (triage table)

Failure symptom	Diagnosis (gate)	Minimal revision path
$\lambda_{\min}^{(i)} < 0$ (forward)	Positivity (TQ)	(R3) add operators with t<0 lift; (R4) portal UV c
$\mathfrak{m}_{\text{unit}} < 1$ in $VV \to VV$	Unitarity/S-gate	(R4) mediator completion; (R3) form factors; rebir
$\omega\ \mathrm{Im}\,\chi^R \not\geq 0$	D-gate passivity	(R1) reframe; (R3) SK-consistent noise kernel; (R5
Cone inclusion violated	C/A-gate	(R5) shrink $\delta\kappa, \beta, \gamma$; (R4) decouple offending mixin
$Q_{\mathrm{B}}^2 \neq 0$ or ξ-trace in S-matrix	BRST/gauge	(R1) gauge-fixing functional; (R6) rebuild ghost/B
Anomaly mismatch	Topological	(R2) threshold realignment; (R6) symmetry conten
Decoupling residuals & thresholds	RG/decoupling	(R2) heavy spectrum & matching fix; (R3) higher
$\Delta\chi^2 < 0$ & $\Delta\mathrm{BIC} > 0$	Data decision	(R1) reparameterize; if persistent ⇒ abandon chan

5. Proof obligations and re-certification

Any revision ($\mathcal{R}$) must pass the re-certification suite:

(i) S/G/TQ/D gates re-verified; (ii) BRST nilpotency & Nielsen identities; (iii) anomaly polyno

(23.48)

Numerically: SDP duality gap $g_{\text{dual}} \leq 10^{-7}$, slack $\sigma_{\min} \geq 10^{-8}$ (Eq. (23.23)); partial-wave margins $\mathfrak{m}_{\text{unit}} > 1$; transport PSD satisfied.

6. Algorithmic protocol (gate-first decision)

Pseudocode. Given data $\mathbf{y}$ and maps $\mathbf{f}$:

$$
\begin{aligned}
&\texttt{(1) Build } \mathcal{C} \text{ from LMIs (S/G/TQ/D) at } \{\mu_k\} \text{ and transport to } \mu_\star;\\
&\texttt{(2) If } \mathcal{C} = \varnothing \;\Rightarrow\; \textsc{Kill};\\
&\texttt{(3) Solve } \min_{\mathbf{p}\in\mathcal{C}} \chi^2(\mathbf{p}); \text{ compute } \Delta\chi^2,\ \Delta\mathrm{BIC};\\
&\texttt{(4) If } \Delta\chi^2 < 0 \;\&\; \Delta\mathrm{BIC} > 0 \;\Rightarrow\; \textsc{Kill}; \text{ else classify (Win/Lean/Indist.)};\\
&\texttt{(5) If edge saturation} \;\Rightarrow\; \text{schedule decisive tests (Sec. 23.1)};\\
&\texttt{(6) If kill} \;\Rightarrow\; \text{apply minimal } \mathcal{R} \in \{\mathrm{R1}\ldots\mathrm{R6}\} \text{ and re-certify (23.48)}.
\end{aligned}
\tag{23.49}
$$

7. Summary

Kill criteria are *gate breaches* (S/G/TQ/D, BRST, anomalies, cone inclusion) or *decisive data penalties* within the admissible region. When failure is not fundamental, a graded set of *revision paths*—from benign retuning to symmetry/geometry surgery—can restore feasibility, provided all proof obligations are met. This closes the loop between theory-space exploration, gate-aware statistics, and an actionable roadmap for either discovery or principled revision.

Chapter 24

Definitive Signatures and Near-Term Roadmap

24.1 Shortlist of decisive discriminants

Aim. We compile a *minimal, orthogonal* set of *decisive discriminants*—observables or combinations thereof whose measurement, at gate-feasible kinematics, can *separate* the Spectral Model (SpM) from the Standard Model (SM) without recourse to model-dependent priors. Each discriminant $\mathcal{D}_i$ is (i) BRST/gauge-parameter independent, (ii) anchored in the S/G/TQ/D cones (Part 22), (iii) expressed as a low-dimensional statistic with an analytic map to spectral parameters, and (iv) equipped with a sign/shape prediction whenever the spectral geometry fixes it. We denote by $\mathbf{p}$ the spectral parameter vector at a common scale $\mu_\star$ (cf. Eq. (23.24)).

1. Electroweak geometry test: $(m_W, \sin^2\theta_W)$ doublet

Construction. From Sec. 7.4, the weak angle is geometrically fixed by the fiber-metric invariants (κ_1, κ_2),

$$\sin^2\theta_W^{\text{geom}} \;=\; \frac{\kappa_2}{\kappa_1+\kappa_2} \;+\; \Delta_{\text{th}}(\mu_\star), \qquad m_W^{\text{geom}} \;=\; \frac{g(\kappa_1,\kappa_G)\,v}{2}\Big[1+\tfrac{1}{2}\,\alpha_T+\cdots\Big], \tag{24.1}$$

with Δ_{th} a small, computable threshold and α_T the custodial oblique (Sec. 8.4). Define the whitened residual vector

$$\mathbf{r}_{\text{EW}} \;:=\; \begin{pmatrix} \big(\sin^2\theta_W^{\text{exp}} - \sin^2\theta_W^{\text{geom}}\big)/\sigma_{s_W} \\ \big(m_W^{\text{exp}} - m_W^{\text{geom}}\big)/\sigma_{m_W} \end{pmatrix}, \qquad \mathcal{D}_{\text{EW}} \;:=\; \|\mathbf{r}_{\text{EW}}\|_2^2, \tag{24.2}$$

and the *decisive rule*

$$\boxed{\mathcal{D}_{\text{EW}} \;\geq\; \tau_{\text{EW}} \quad\Rightarrow\quad \textsc{Decisive}; \qquad \tau_{\text{EW}} \simeq 9 \;\; (3\sigma)\;.} \tag{24.3}$$

Because (κ_1, κ_2) also enter TGC/QGC positivity (via $H_i \succeq 0$), (24.2) is *gate-stable*.

2. Longitudinal VBS positivity edge: helicity shape at high m_{VV}

Construction. In $V_LV_L \to V_LV_L$ ($V = W, Z$) the forward positivity LMIs (Sec. 22.2) imply, after IR-pole subtraction,

$$\partial_s^2 \widehat{\mathcal{M}}_{LL\to LL}(0,0) \;=\; \alpha_1 \;\geq 0, \qquad \partial_s^2\Big[\widehat{\mathcal{M}}_{++\to++}(0,0) + \widehat{\mathcal{M}}_{+-\to+-}(0,0)\Big] \;=\; \alpha_1 + \alpha_2 \;\geq 0, \tag{24.4}$$

with (α_1, α_2) the photon-like quartic coefficients (Sec. 22.2). Projecting LHC $VVjj$ data onto helicity-sensitive angles φ^* and large dijet invariant mass m_{jj}, define the signed asymmetry

$$\mathcal{A}_{\rm hel} \;:=\; \frac{N(\cos 2\varphi^* > 0) - N(\cos 2\varphi^* < 0)}{N_{\rm tot}}\bigg|_{m_{VV}\in[\rm TeV]}, \qquad \mathcal{D}_{\rm VBS} \;:=\; \frac{\mathcal{A}_{\rm hel} - \mathcal{A}_{\rm hel}^{\rm SM}}{\sigma(\mathcal{A}_{\rm hel})}. \tag{24.5}$$

Near the positivity edge ($\det H \to 0^+$) the SpM predicts a *definite sign* for $\mathcal{A}_{\rm hel}$ (fixed by the eigenvector of H), whereas generic EFT deformations allow both signs. Hence one-sided $\mathcal{D}_{\rm VBS} \gtrless \tau$ is decisive and *directional*.

3. Higgs structural resonance: hh threshold lineshape and λ_3

Construction. In Sec. 8.3 the structural resonance induces a correlated shift $(\Delta\lambda_3, \delta Z_h)$ with sign fixed by spectral stiffness variations. Consider the differential di-Higgs spectrum near threshold, $x \equiv m_{hh}/(2m_h) - 1$:

$$\frac{d\sigma}{dx} \;\simeq\; \frac{d\sigma_{\rm SM}}{dx}\Big[1 + A(x)\,\Delta\lambda_3 + B(x)\,\delta Z_h + \cdots\Big], \qquad A(x) \text{ changes sign at } x \approx 0.2, \tag{24.6}$$

while in SpM $B(x)$ and $A(x)$ have *locked* relative sign. Define the two-bin statistic

$$\mathcal{D}_{hh} \;:=\; \frac{\Big[\sigma(x<x_0) - \sigma(x>x_0)\Big] - \Big[\sigma(x<x_0) - \sigma(x>x_0)\Big]_{\rm SM}}{\sqrt{\sigma_{\rm stat}^2 + \sigma_{\rm syst}^2}}, \qquad x_0 \simeq 0.2, \tag{24.7}$$

which is largely insensitive to smooth K-factor systematics. A sign-flip relative to SM at fixed total $\sigma(hh)$ is a *spectral hallmark*.

4. EDM correlation triple: $(d_e, d_n, d_{\rm Hg})$ on a spectral ray

Construction. If CPV originates from a single spectral phase texture (Sec. 14.2), the dipole/Wilson set lies on a one-dimensional ray after RG running:

$$\Big(\tilde{d}_e,\ \tilde{d}_q,\ w,\ C_{\not{CP}}^{(4f)}\Big) \;=\; \lambda_{\not{CP}}\,\mathbf{v}_{\rm sp}(\mu_\star), \qquad \lambda_{\not{CP}} \in \mathbb{R}. \tag{24.8}$$

Hadronic/nuclear matrix elements K_i map this to EDMs, $d_i = K_i \cdot \mathbf{v}_{\rm sp}\,\lambda_{\not{CP}}$. Form the null-combination residual

$$\mathcal{R}_{\rm EDM} \;:=\; \frac{d_n - \alpha\, d_e - \beta\, d_{\rm Hg}}{\sqrt{\sigma_{d_n}^2 + \alpha^2\sigma_{d_e}^2 + \beta^2\sigma_{d_{\rm Hg}}^2}}, \qquad \alpha, \beta \text{ chosen from } K_i \text{ to cancel } \lambda_{\not{CP}}, \tag{24.9}$$

so that $\mathcal{D}_{\rm EDM} := |\mathcal{R}_{\rm EDM}|$ must be $\ll 1$ in SpM but need not vanish in generic multi-source CPV. A significant nonzero value is *decisive against* the one-phase spectral origin.

5. Neutrino NSI/aniso isolator at the magic baseline

Construction. At $L_{\rm magic}$ (Sec. 20.2) the α-interference term in $P_{\mu e}$ cancels, isolating NSI/aniso pieces:

$$\Xi_{\rm magic} \;:=\; \left.\frac{P_{\mu e}-\bar{P}_{\mu e}}{P_{\mu e}+\bar{P}_{\mu e}}\right|_{L=L_{\rm magic}} \;\simeq\; \mathfrak{c}_1\,{\rm Im}(\hat{\varepsilon}_{e\mu},\hat{\varepsilon}_{e\tau}) \;+\; \mathfrak{c}_2\,(a_{e\mu,0},a_{e\tau,0}), \tag{24.10}$$

with calculable weights $\mathfrak{c}_{1,2}$ and negligible dependence on δ. Define $\mathcal{D}_{\rm LBL} := |\Xi_{\rm magic}|/\sigma(\Xi_{\rm magic})$. A nonzero $\Xi_{\rm magic}$ consistent with forward-positivity (Sec. 20.2) points to spectral portals; conversely, a signal *incompatible* with the convex NSI set $\mathcal{A}_{\rm NSI}$ (Eq. (20.32)) is a *kill*.

6. Photon-sector anisotropy: sidereal cavity beat & birefringence

Construction. Director-induced kinetic deformation (Sec. 22.1) generates a tiny refractive-index anisotropy $\delta c/c \simeq \delta\kappa\,\cos^2\theta$. In orthogonal high-Q cavities rotating with Earth, the beat frequency shows a sidereal modulation

$$\mathcal{A}_{\rm sid} \;=\; \frac{\Delta\nu}{\nu} \;\approx\; \frac{1}{2}\,\delta\kappa\,(\cos^2\theta_1-\cos^2\theta_2)\,, \tag{24.11}$$

while astrophysical birefringence bounds the same $\delta\kappa$ combination through polarization rotation $\Delta\psi$. Define the joint statistic $\mathcal{D}_\gamma := \sqrt{(\mathcal{A}_{\rm sid}/\sigma)^2+(\Delta\psi/\sigma_\psi)^2}$ with a *common-sign* prediction (no π-ambiguity) in SpM.

7. QCD string–glueball relation (spectral tension)

Construction. The spectral string tension τ_s (Sec. 10.2) predicts a relation for the scalar glueball,

$$\frac{m_{0^{++}}}{\sqrt{\tau_s}} \;=\; r_{\rm sp} \;+\; \Delta_{\rm latt} + \mathcal{O}\!\left(\tau_s^{-1/2}\right), \tag{24.12}$$

with $r_{\rm sp}$ fixed by composite Φ–χ^μ dynamics (Sec. 12.1) and $\Delta_{\rm latt}$ a controlled continuum extrapolation. The discriminant

$$\mathcal{D}_{\rm glue} \;:=\; \frac{m_{0^{++}}/\sqrt{\tau_s}-r_{\rm sp}}{\sqrt{\sigma_m^2/\tau_s+\sigma_{\tau_s}^2\,m_{0^{++}}^2/(4\tau_s^3)+\sigma_{\rm latt}^2}} \tag{24.13}$$

tests nonperturbative matching; sizable deviation with validated lattice systematics would *falsify* the strong-sector embedding.

8. Anomaly quantization: WZW/CS coefficient integrity

Construction. WZW and Chern–Simons functionals (Ch. 5) fix quantized coefficients $k_{\rm WZW}, k_{\rm CS}$ tied to spectral winding/Hopf indices. In low-energy EFTs, these appear as exact normalizations of anomalous couplings (e.g. $\pi^0\to\gamma\gamma$) once mixing angles are accounted for. Define the anomaly-resolved pull

$$\mathcal{D}_{\rm anom} \;:=\; \frac{\left(g^{\rm exp}_{\pi^0\gamma\gamma}-g^{\rm sp}_{\pi^0\gamma\gamma}\right)}{\sigma_g} \;\oplus\; \cdots\,, \tag{24.14}$$

aggregating channels with robust hadronic control; integral mismatches are *fatal*.

9. Orthogonality and convexity

Near-orthogonality. The gradients $\{\nabla_{\mathbf{p}}\mathcal{D}_i\}$ at the SpM fiducial point are nearly orthogonal when whitened by experimental covariance and projected to the admissible tangent cone (Sec. 23.3). We quantify this by the Gram matrix $G_{ij} = \langle \nabla\mathcal{D}_i, \nabla\mathcal{D}_j \rangle$. A well-conditioned G ensures that joint evidence adds in quadrature without hidden degeneracies.

Gate-stability. Each $\mathcal{D}_i$ is built from forward-safe, IR-subtracted amplitudes or from protected ratios, so that S/G/TQ/D LMIs act as *simple convex constraints*. This prevents *spurious decisiveness* from kinematic regions where EFT truncation or cone inclusion would fail.

10. Decision thresholding and multiple testing

Global rule. For a set $\{\mathcal{D}_i\}_{i=1}^m$ with small correlations we define

$$\mathcal{T}^2 := \sum_{i=1}^{m} \mathcal{D}_i^2 , \qquad \textsc{Decisive} \iff \mathcal{T}^2 \geq \tau_{\text{glob}}, \tag{24.15}$$

with τ_{glob} chosen to control the global Type-I error under trials (Bonferroni/Sidak or exact MC). Because each $\mathcal{D}_i$ is *sign- or shape-definite* in SpM (items 2, 3, 6), one-sided tests further enhance power.

11. Summary

We identified eight decisive discriminants spanning EW geometry, VBS helicity positivity, Higgs threshold shape, EDM correlation rays, neutrino NSI/aniso at the magic baseline, photon-sector anisotropy, QCD string–glueball matching, and anomaly quantization. They form a near-orthogonal, gate-stable basis of tests with clear sign/shape predictions. Their combined statistic (24.15) enables a robust discovery/falsification program, and the next section (Sec. 24.2) aligns these with concrete timelines and facilities.

24.2 Timeline-aligned forecasts

Aim. We develop a *gate-aware* forecasting framework that projects the *decisiveness* of the shortlist observables in Sec. 24.1 onto concrete time horizons. The machinery is built on (i) Asimov likelihoods and Fisher information, (ii) convex projection to the admissible cones (S/G/TQ/D), (iii) realistic scaling with exposure, efficiencies, and systematic floors, and (iv) *decision triggers* ("go/no-go") that activate revision paths (Sec. 23.4) if decisiveness is not reached on schedule.

1. Forecast machinery: Asimov, Fisher, and gate projection

Signal prototypes and Asimov deviance. Let $\mathbf{y}$ denote the stacked observable vector, $C(\mathcal{E})$ its covariance at exposure $\mathcal{E}$ (luminosity, live-time, statistics), and $\Delta\mathbf{f}(\mathbf{p}_\star) := \mathbf{f}_{\text{SpM}}(\mathbf{p}_\star) - \mathbf{f}_{\text{SM}}$ the *spectral signal prototype* at a fiducial point $\mathbf{p}_\star$ within the admissible

cone $\mathcal{C}$. The *Asimov* (median) deviance improvement expected for the spectral hypothesis is

$$\Delta\chi^2_{\mathrm{A}}(\mathcal{E}) \;=\; \Delta\mathbf{f}^{\top}\, C^{-1}(\mathcal{E})\,\Delta\mathbf{f}\,, \qquad C(\mathcal{E}) \;=\; C_{\mathrm{stat}}(\mathcal{E}) \oplus C_{\mathrm{syst}} \oplus C_{\mathrm{th}}, \tag{24.16}$$

with $C_{\mathrm{stat}} \propto \mathcal{E}^{-1}$ and $(C_{\mathrm{syst}}, C_{\mathrm{th}})$ exposure-independent floors.

Fisher matrix and principal directions. Linearizing predictions around $\mathbf{p}_\star$,

$$J \;:=\; \left.\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right|_{\mathbf{p}_\star}, \qquad \mathcal{I}_F(\mathcal{E}) \;=\; J^{\top}\, C^{-1}(\mathcal{E})\, J, \qquad \Sigma_p(\mathcal{E}) \;\simeq\; \mathcal{I}_F^{-1}, \tag{24.17}$$

and the *projected* Fisher within feasibility is

$$\mathcal{I}_F^{\mathrm{adm}} \;=\; (J\Pi_{\mathsf{TC}})^{\top}\, C^{-1}(\mathcal{E})\,(J\Pi_{\mathsf{TC}}), \qquad \Pi_{\mathsf{TC}}:\ \text{projector onto the tangent cone } \mathsf{TC}_{\mathbf{p}_\star}\mathcal{C}. \tag{24.18}$$

Decisiveness statistic. For m near-orthogonal discriminants $\{\mathcal{D}_i\}$ (Sec. 24.1), define the combined Asimov statistic

$$\mathcal{T}^2(\mathcal{E}) \;:=\; \sum_{i=1}^{m} \mathcal{D}^2_{i,\mathrm{A}}(\mathcal{E}) \;\simeq\; \Delta\mathbf{f}^{\top}\, W(\mathcal{E})\,\Delta\mathbf{f}, \qquad W \;=\; \sum_i \mathbf{w}_i\,\mathbf{w}_i^{\top}, \quad \mathbf{w}_i := \frac{\nabla\mathcal{D}_i}{\|\nabla\mathcal{D}_i\|}, \tag{24.19}$$

where gradients are whitened by $C^{-1/2}$ and projected to $\mathsf{TC}_{\mathbf{p}_\star}\mathcal{C}$. A threshold $\mathcal{T}^2 \geq \tau_{\mathrm{glob}}$ (Sec. 24.1) defines DECISIVE.

2. Horizons and scaling laws

Horizon bins. We partition the near-term roadmap into exposure-driven bins:

$$\mathsf{H}_0:\ \text{present}, \qquad \mathsf{H}_1:\ \text{short-term (}\sim\text{few years)}, \qquad \mathsf{H}_2:\ \text{mid-term (}\sim\text{5–8 years)}, \qquad \mathsf{H}_3:\ \text{longer-ter} \tag{24.20}$$

with channel-specific $\mathcal{E}$ growth models (luminosity, live-time, detector upgrades).

Stat–syst transition. For any scalar discriminant $\mathcal{D}$ with signal scale S and variance $\sigma^2(\mathcal{E}) = \sigma^2_{\mathrm{stat}}(\mathcal{E}) + \sigma^2_{\mathrm{floor}}$,

$$\mathcal{D}_{\mathrm{A}}(\mathcal{E}) \;\simeq\; \frac{S}{\sqrt{\sigma^2_{\mathrm{stat}}(\mathcal{E}) + \sigma^2_{\mathrm{floor}}}}\,, \qquad \mathcal{E}_{\mathrm{knee}}:\ \sigma_{\mathrm{stat}}(\mathcal{E}_{\mathrm{knee}}) = \sigma_{\mathrm{floor}}, \tag{24.21}$$

identifies the "knee" beyond which systematic control, not exposure, drives progress.

3. Channel-by-channel forecasts (analytic scalings)

(A) EW geometry: $(m_W, \sin^2\theta_W)$. Let $\Delta_{\mathrm{EW}} = (\Delta s^2_W, \Delta m_W)$ be the spectral shifts (Eq. (24.1)). Assuming uncorrelated errors (for illustration),

$$\mathcal{D}^2_{\mathrm{EW,A}}(\mathcal{E}) \;\simeq\; \frac{(\Delta s^2_W)^2}{\sigma^2_{s_W}(\mathcal{E})} \;+\; \frac{(\Delta m_W)^2}{\sigma^2_{m_W}(\mathcal{E})}, \qquad \sigma^2_{s_W} = a_s^2\,\mathcal{E}^{-1} + \sigma^2_{s,\mathrm{floor}},\ \sigma^2_{m_W} = a_m^2\,\mathcal{E}^{-1} + \sigma^2_{m,\mathrm{floor}}. \tag{24.22}$$

Solve $\mathcal{D}^2_{\mathrm{EW,A}}(\mathcal{E}) = \tau_{\mathrm{EW}}$ to obtain the *minimal decisive exposure* $\mathcal{E}^{\mathrm{dec}}_{\mathrm{EW}}$.

(B) Longitudinal VBS helicity asymmetry. With selection efficiency ϵ, purity for LL fraction P_{LL}, and background-subtracted counts $N \simeq \epsilon\, P_{LL}\, \sigma_{LL}(\mathcal{E})\, \mathcal{E}$ in high-m_{VV} bins, the asymmetry variance is

$$\mathrm{Var}(\mathcal{A}_{\rm hel}) \;\simeq\; \frac{1-\mathcal{A}_{\rm hel}^2}{N} \;+\; \sigma_{\rm syst}^2, \qquad \mathcal{D}_{\rm VBS,A} \;\simeq\; \frac{|\mathcal{A}_{\rm hel}^{\rm SpM} - \mathcal{A}_{\rm hel}^{\rm SM}|}{\sqrt{(1-\bar{\mathcal{A}}^2)/N + \sigma_{\rm syst}^2}}. \tag{24.23}$$

Near the positivity edge (Sec. 24.1) the sign of the numerator is fixed, enabling one-sided tests.

(C) Di-Higgs threshold shape. For the two-bin statistic $\mathcal{D}_{hh}$ (Eq. (24.7)) with bin efficiencies $(\epsilon_<, \epsilon_>)$,

$$\mathcal{D}_{hh,\rm A}^2 \;\simeq\; \frac{\left[\epsilon_< \,\sigma_<^{\rm SpM} - \epsilon_> \,\sigma_>^{\rm SpM} - (\mathrm{SM})\right]^2}{\epsilon_< \,\sigma_<^{\rm SM}\,\mathcal{E} + \epsilon_> \,\sigma_>^{\rm SM}\,\mathcal{E} + \Sigma_{\rm syst}^2}, \qquad \sigma_{\lessgtr}^{\rm SpM} = \sigma_{\lessgtr}^{\rm SM}\,[1 + A_{\lessgtr}\Delta\lambda_3 + B_{\lessgtr}\delta Z_h]. \tag{24.24}$$

(D) EDM correlation triple. Let $\mathbf{d} = (d_e, d_n, d_{\rm Hg})$, with covariance $V = V_{\rm exp} \oplus V_{\rm th}$ (theory floor). For the null-combination residual $\mathcal{R}_{\rm EDM}$ in Eq. (24.9),

$$\mathcal{D}_{\rm EDM,A} \;=\; \frac{|\mathbf{n}^\top \mathbf{d}^{\rm SpM}|}{\sqrt{\mathbf{n}^\top V\, \mathbf{n}}}, \qquad \mathbf{n} = (1, -\alpha, -\beta), \quad \alpha, \beta \text{ chosen to cancel } \lambda_{\not CP}. \tag{24.25}$$

Progress is dominated by reducing $V_{\rm th}$ (hadronic/nuclear matrix-element floors) once $V_{\rm exp}$ is small.

(E) Neutrino magic-baseline isolator. For $\Xi_{\rm magic}$ (Eq. (24.10)) with appearance counts $(N, \bar{N})$,

$$\mathrm{Var}(\Xi_{\rm magic}) \;\simeq\; \frac{4\,N\,\bar{N}}{(N+\bar{N})^3} \;+\; \sigma_{\rm flux}^2 + \sigma_{\rm xs}^2 + \sigma_{\rm det}^2, \qquad \mathcal{D}_{\rm LBL,A} \;\simeq\; \frac{|\Xi_{\rm magic}^{\rm SpM}|}{\sqrt{\mathrm{Var}(\Xi_{\rm magic})}}. \tag{24.26}$$

A second energy bin (on/off resonance) suppresses common-mode systematics and lifts the δ–NSI degeneracy.

(F) Photon-sector anisotropy. Sidereal cavity beat (Eq. 24.11) with cavity quality factor Q and integration time T yields

$$\sigma(\mathcal{A}_{\rm sid}) \;\simeq\; \frac{1}{Q}\sqrt{\frac{\tau_c}{T}} \;\oplus\; \sigma_{\rm env}, \qquad \mathcal{D}_{\gamma,\rm A} \;\simeq\; \frac{|\mathcal{A}_{\rm sid}^{\rm SpM}|}{\sqrt{Q^{-2}(\tau_c/T) + \sigma_{\rm env}^2}} \;\oplus\; \frac{|\Delta\psi^{\rm SpM}|}{\sigma_\psi}, \tag{24.27}$$

where τ_c is the cavity coherence time; the two terms are statistically independent.

(G) Glueball/string matching. With three lattice spacings $a_{1,2,3}$ and Symanzik scaling exponent $p \approx 2$, the forecast variance for $m_{0^{++}}/\sqrt{\tau_s}$ at $a \to 0$ scales as

$$\sigma_{\rm cont}^2 \;\simeq\; \sum_{i=1}^{3} \left(\frac{\partial \widehat{R}}{\partial \mathcal{O}_i}\right)^2 \sigma_i^2 \;+\; \kappa\, a_{\rm min}^{2p}, \qquad \widehat{R} := \frac{m_{0^{++}}}{\sqrt{\tau_s}}, \quad \kappa \text{ captures truncation bias,} \tag{24.28}$$

leading to $\mathcal{D}_{\rm glue,A}$ via Eq. (24.13).

(H) Anomaly quantization. For an anomaly-protected coupling $g_{\rm anom}$ with tiny experimental error, decisiveness is immediate once mixing angles entering $g^{\rm sp}_{\rm anom}$ are known better than the theory floor; forecasts focus on the mixing-angle program rather than exposure.

4. Minimal decisive exposure and scheduling

Closed-form exposure target. When $C(\mathcal{E}) = C_0/\mathcal{E} \oplus C_{\rm floor}$ and $\Delta\mathbf{f}$ lies mostly in the statistical subspace, solve

$$\Delta\chi^2_{\rm A}(\mathcal{E}) \;=\; \Delta\mathbf{f}^\top\!\left(\frac{\mathcal{E}}{C_0^{-1}} + C_{\rm floor}^{-1}\right)\!\Delta\mathbf{f} \;\stackrel{!}{=}\; \tau \quad\Rightarrow\quad \mathcal{E}_{\rm dec} \;\simeq\; \frac{\tau - \Delta\mathbf{f}^\top C_{\rm floor}^{-1}\Delta\mathbf{f}}{\Delta\mathbf{f}^\top C_0^{-1}\Delta\mathbf{f}}\,, \tag{24.29}$$

whenever the numerator is positive (else the floor dominates and systematics must be reduced).

Decision calendar. Let $\mathsf{t}(\mathcal{E})$ convert exposure to wall time. Define *decision times* $\mathsf{t}_i^{\rm dec}$ by $\mathcal{D}_{i,\rm A}(\mathcal{E}(\mathsf{t}_i^{\rm dec})) = \tau_i$, and the global time $\mathsf{t}^{\rm dec}_{\rm glob}$ by $\mathcal{T}^2(\mathcal{E}(\mathsf{t}^{\rm dec}_{\rm glob})) = \tau_{\rm glob}$. Missed milestones trigger the revision paths (R1)–(R6) in Sec. 23.4.

5. Multi-channel resource allocation (convex surrogate)

Budgeted maximization. Let $\mathbf{x}$ be the vector of channel allocations (e.g. run fractions, bin emphases), $\sum_i x_i \le 1$, $x_i \ge 0$. A convex surrogate for the combined decisiveness is

$$\max_{\mathbf{x}\in\Delta}\; \mathcal{T}^2(\mathbf{x}) \;=\; \Delta\mathbf{f}^\top\left[\sum_i x_i\, W_i\right]\Delta\mathbf{f} \quad \text{s.t.}\;\; x_i \ge 0,\; \sum_i x_i \le 1,\; \text{gate LMIs feasible for each } i, \tag{24.30}$$

with W_i the whitened weight matrix of channel i (including efficiencies and floors). The KKT conditions give the optimal *frontier mix* $\mathbf{x}^\star$; practically, allocate until the marginal gain per unit cost equalizes across channels.

6. Risk register and mitigation levers

Systematic floors. If $\sigma_{\rm floor}$ dominates in Eq. (24.21), the only path to decisiveness is *floor reduction*: calibration (EW), polarization unfolding (VBS), theory inputs (EDM matrix elements), or environmental stability (photon anisotropy). Mitigation enters as a multiplicative reduction $\sigma_{\rm floor} \to r\,\sigma_{\rm floor}$ with $r < 1$, immediately lowering $\mathcal{E}_{\rm dec}$ via Eq. (24.29).

EFT truncation. Bins failing the truncation criterion $\epsilon_{\rm trunc} \le \epsilon_\star$ (Eq. (23.12)) are dropped or modeled with gate-feasible form factors (R3/R4), else decisiveness may be *spurious*.

Gate proximity. Near-saturation of positivity or cone LMIs enhances sensitivity (fixed sign/shape), but also increases susceptibility to modeling error; require numerical *slack* thresholds (e.g. $\sigma_{\min} \ge 10^{-8}$ in SDPs) before claiming decisiveness.

7. Illustrative milestone table (symbolic)

Horizon	Primary lever	Target statistic	Trigger
H_1 (short)	EW (m_W, s_W^2), LBL Ξ_{magic}	$\mathcal{D}_{\text{EW}} \gtrsim 2$, $\mathcal{D}_{\text{LBL}} \gtrsim 2$	proceed to VBS focus if n
H_2 (mid)	VBS helicity, EDM triple	$\mathcal{D}_{\text{VBS}} \gtrsim 3$, $\mathcal{D}_{\text{EDM}} \lesssim 1$ (ray test)	di-Higgs shape push
H_3 (long)	hh threshold shape, glueball/string	$\mathcal{D}_{hh} \gtrsim 3$, $\mathcal{D}_{\text{glue}} \gtrsim 2$	global $\mathcal{T}^2 \geq \tau_{\text{glob}}$

8. Summary

The forecasts are *analytics-first*: Asimov deviance (24.16), Fisher projection (24.18), and channel scalings (24.22)–(24.28) yield closed-form decisive exposures (24.29) and a budget-optimal allocation (24.30). A milestone calendar sets clear triggers for discovery or for activating revision paths. Because each discriminant is gate-stable and sign/shape-definite where applicable, the roadmap is robust to modeling choices and prioritizes the most *decisive* measurements over merely precise ones.

24.3 Interfaces to Volumes V and VI

Aim. We formalize the *interfaces* that connect the particle-physics construction of Volume IV to:

- **Volume V: Spectral Quantum Foundations** (Damascus interpretation, GKSL-consistent open dynamics, entanglement/no-signalling threads, spectral quantum control);
- **Volume VI: Spectral Matter Science** (spectral chemistry, condensed-matter/topological phases, constrained mobility, viscoelastic spectral transitions, prelude to spectral biology).

Interfaces are specified as RG-covariant data passed at a pivot scale $\mu_\star$, together with gate certificates (S/G/TQ/D). The guiding principle is: *push* minimal geometric/effective data downstream; *pull* only feedback observables with controlled back-reaction on the particle sector.

1. Pivot scale and exported artifacts

Parameter/state hand-off. At the common pivot $\mu_\star$, Volume IV exports the particle-side state

$$\mathbf{p}_{\text{IV}}(\mu_\star) \;:=\; \Big(\kappa_{1,2},\ \kappa_G;\ g, g', g_3;\ Z_h, \lambda_3;\ \mathbf{c}_{\text{EFT}};\ \varepsilon_{\alpha\beta},\ a_{\alpha\beta,0};\ \alpha_s, \tau_s;\ \dots \Big), \tag{24.31}$$

and the constitutive bundle of the spectral medium

$$\mathfrak{C}_{\text{IV}} \;:=\; \Big(g_{\text{eff}},\ u^\mu,\ \chi^\mu;\ \delta\kappa_{IJ}^{\mu\nu};\ \mathbb{K}_{\text{trans}}(\omega, \mathbf{k}) \Big), \tag{24.32}$$

where $\mathbb{K}_{\text{trans}}$ denotes the frequency–momentum Kubo kernels (conductivities, viscosities, noise spectra). Each artifact is accompanied by admissible-cone projectors and slack certificates for the S/G/TQ/D gates.

RG tube. We also pass an *RG tube*

$$\mathcal{T}_{\mathrm{RG}} := \Big\{\mathbf{p}(\mu),\, \mathfrak{C}(\mu)\Big\}_{\mu\in[\mu_{\mathrm{IR}},\,\mu_{\mathrm{UV}}]}, \qquad \text{with threshold/decoupling maps and gate-preserving matching.} \tag{24.33}$$

2. Interface to Volume V: Spectral Quantum Foundations

(V1) GKSL generators from spectral noise. System–medium couplings $H_{\mathrm{int}} = \sum_A S_A \otimes B_A$ with medium correlators $S_{AB}(\omega) = \int dt\, e^{i\omega t}\langle B_A(t) B_B(0)\rangle_{\mathcal{S}}$ induce a Lindblad generator

$$\dot\rho \;=\; -i[H_{\mathrm{sys}} + H_{\mathrm{LS}}, \rho] + \sum_\alpha \gamma_\alpha\Big(L_\alpha \rho L_\alpha^\dagger - \tfrac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\}\Big), \qquad \gamma_\alpha \;=\; \mathcal{W}_\alpha\Big[S_{AB}(\omega)\Big], \tag{24.34}$$

where the maps $\mathcal{W}_\alpha$ are KMS/Kubo-consistent functionals of $\mathbb{K}_{\mathrm{trans}}$ and of (ρ, η, χ^μ). Volume IV provides $\{L_\alpha, \gamma_\alpha, H_{\mathrm{LS}}\}$ as explicit functions/tables of the exported kernels.

(V2) Damascus: Born weights as stationary energy shares. For pointer projectors $\{\Pi_i\}$ stabilized by large (ρ, η), the long-time energy partition obeys

$$p_i \;=\; \lim_{\tau\to\infty} \frac{\mathrm{Tr}\Big(\Pi_i H_{\mathrm{sys}} \rho(\tau)\Big)}{\sum_j \mathrm{Tr}\Big(\Pi_j H_{\mathrm{sys}} \rho(\tau)\Big)} \;=\; \langle\psi|\Pi_i|\psi\rangle \;+\; \delta p_i, \qquad |\delta p_i| \;\le\; \frac{C_i}{\Gamma_{\mathrm{lock}}\,\tau}, \tag{24.35}$$

with locking rate Γ_{lock} computable from the γ_α and spectral gaps of $\mathbb{K}_{\mathrm{trans}}$. Volume V consumes $(\Gamma_{\mathrm{lock}}, C_i)$ to bound controlled deviations from the Born limit.

(V3) No-signalling and entanglement threads. Local GKSL structure and cone inclusion imply $[\mathcal{L}_A, \mathcal{L}_B] = 0$ for spacelike-separated generators, forbidding superluminal signalling. Thread bipartitions are specified by a graph G_χ induced by χ^μ-mediated couplings; operator-norm bounds of L_α certify Tsirelson-limited correlations:

$$\langle \mathsf{CHSH}\rangle \;\le\; 2\sqrt{2}, \qquad \text{with equality if and only if the thread norms saturate } \|L_\alpha\| \text{ constraints.} \tag{24.36}$$

(V4) Spectral quantum control (colored Zeno). Control fields $u(t)$ shape filter functions $F(\omega)$, yielding Zeno/decoherence rates

$$\Gamma_{\mathrm{ctrl}} \;\simeq\; \int_{-\infty}^{+\infty} d\omega\, \Big|F(\omega; u)\Big|^2\, S_{\mathrm{eff}}(\omega), \qquad S_{\mathrm{eff}} \text{ extracted from } \mathbb{K}_{\mathrm{trans}}. \tag{24.37}$$

Volume IV supplies admissible-control cones (passivity/positivity preserved) and gradient rules $\partial\Gamma_{\mathrm{ctrl}}/\partial u$.

3. Interface to Volume VI: Spectral Matter Science

(VI1) Spectral chemistry: selection rules and matrix elements. Atomic/molecular resonators are modeled by effective Hamiltonians $H_{\mathrm{at}}(\mathcal{F}, h; g_{\mathrm{eff}})$. Fiber isometries $G_{\mathcal{F}}$ induce Wigner–Eckart selection rules for transition operators $\mathcal{O}^{(q)}$:

$$\langle\psi_f||\mathcal{O}^{(q)}||\psi_i\rangle \;=\; \frac{\langle\psi_f|\mathcal{O}^{(q)}|\psi_i\rangle}{C^{\lambda_f\, m_f}_{\lambda_i\, m_i;\, q}} \quad\Rightarrow\quad \Delta\lambda \in \mathrm{Rep}\Big(G_{\mathcal{F}}\Big), \;\; \Delta m = q, \tag{24.38}$$

with reduced elements computed from overlap integrals on $\mathcal{F}$:

$$\mathcal{M}_{fi} \;=\; \int_{\mathcal{F}} d\mu_h(y)\; \psi_f^*(y)\, \mathcal{O}^{(q)}(y)\, \psi_i(y)\,. \tag{24.39}$$

Volume IV exports the necessary basis functions and quadrature rules.

(VI2) Bonds as synchronization. Inter-resonator phases $\{\varphi_i\}$ obey Kuramoto-type locking

$$\dot{\varphi}_i = \omega_i + \sum_j K_{ij} \sin(\varphi_j - \varphi_i), \qquad K_{ij} \;\propto\; \int_{\mathcal{F}} d\mu_h\, \psi_i^* \psi_j, \tag{24.40}$$

with stability/feasibility constrained by S/G/D gates (positive dissipation; bounded response). Volume VI uses (K_{ij}) to construct bonding maps and vibrational spectra.

(VI3) Topological and constrained-mobility phases. Band structures inherit Berry curvature $\mathcal{F}_k$ from $\mathcal{F}$-geometry; Chern numbers

$$C \;=\; \frac{1}{2\pi} \int_{\mathrm{BZ}} \mathrm{Tr}\, \mathcal{F}_k \tag{24.41}$$

classify topological phases. Internal anisotropy (director χ^μ) can realize rank-2 conservation laws leading to constrained mobility (fracton-like):

$$\partial_t \rho \;+\; \partial_i \partial_j J^{ij} \;=\; 0, \tag{24.42}$$

with admissible cones ensuring causality and passivity.

(VI4) Viscoelastic spectral transitions. From $\mathbb{K}_{\mathrm{trans}}$ we deliver complex moduli

$$G^*(\omega) \;=\; G'(\omega) + i\, G''(\omega), \qquad G''(\omega) \;\geq\; 0 \;\; \text{(passivity)}, \tag{24.43}$$

and memory kernels for constitutive laws; Volume VI builds phase diagrams and critical dynamics from these inputs.

(VI5) Prelude to spectral biology. We quantify mesoscale coherent resonance by an order parameter

$$R\, e^{i\Phi} \;:=\; \frac{1}{N} \sum_{j=1}^{N} e^{i\varphi_j}, \qquad R \in [0,1], \tag{24.44}$$

and by decoherence-free subspaces given by the null space of the exported Lindbladian $\mathcal{L}$; Volume VI explores regimes where R is stabilized by spectral synchronization and protected dissipation.

4. Mathematical contracts and gate preservation

Variational primitives. Downstream variational problems receive: (i) $S_{\mathrm{eff}}[g_{\mathrm{eff}}, \mathcal{A}, \mathcal{S}]$ to the requested derivative order; (ii) Ward/Slavnov identities; (iii) forward-positivity LMIs $H_i(\mathbf{c}) \succeq 0$ and cone-inclusion inequalities with certified slack.

Gate transport. All artifacts carry machine-checkable certificates (eigenvalue floors of Hankel blocks, BRST nilpotency checks, passivity integrals). Gates remain valid under RG evolution using the tube $\mathcal{T}_{\mathrm{RG}}$, with threshold maps that preserve feasibility.

5. Data/likelihood interoperability

Portable products. Volume IV delivers: (i) tabulated RG solutions; (ii) kernel tables $\mathbb{K}_{\text{trans}}$; (iii) basis/overlap data on $\mathcal{F}$; (iv) EFT Wilsons and covariances; (v) admissible-cone projectors. Volumes V/VI return *likelihood pulls* (e.g. GKSL rates, control performance metrics, spectroscopic lines, transport coefficients) that are *projected* onto the admissible tangent cone before any retuning of $\mathbf{p}_{\text{IV}}$.

6. Summary

The interfaces are minimal, RG-covariant, and gate-certified. Volume V consumes noise/response and symmetry data to develop Damascus, GKSL dynamics, entanglement threads, and quantum control. Volume VI consumes overlaps, Berry data, and transport kernels to build spectral chemistry, topological/constrained phases, viscoelastic transitions, and coherent-resonance indicators. A bidirectional likelihood loop maintains S/G/TQ/D integrity across the full program.

24.4 Closing synthesis

Aim. We condense Volume IV into a compact set of *axioms, maps, and gates* that jointly define the Spectral Model (SpM) at particle-physics scales, delineate its SM limit, and codify its falsifiable predictions and revision paths. The synthesis is *geometry-first*, *gate-aware*, and *forecast-aligned*.

1. Axioms and primitive data

(A1) Spectral medium. A macroscopic state $\mathcal{S} = \{\Phi, \rho, \eta, \chi^{\mu}, T\}$ on (M, g_{eff}) (Vol. I) with hyperbolic dynamics and well-posed Kubo kernels (D-gate).

(A2) Fiber geometry. A Riemannian fiber $(\mathcal{F}, h_{AB})$ with isometry group G, section $y : M \to \mathcal{F}$, and Killing fields ξ_I (Ch. 1).

(A3) Gauging by horizontality. An Ehresmann connection $\mathcal{A}_{\mu} \in \mathfrak{g}$ making gradients horizontal; curvature $\mathcal{F}_{\mu\nu}$ and Yang–Mills dynamics arise from spectral stiffness and fluctuations (Secs. 1.2–1.4).

(A4) BRST/anomalies. Gauge fixing via a BRST-exact term with nilpotent Q_{B}; anomaly polynomial $\mathcal{I}_{d+2}$, inflow, and WZW/CS functionals consistent and quantized (Chs. 2, 4, 5).

(A5) Gates. S/G/TQ/D gates enforced locally and along the RG tube (Parts 22, 17); feasibility encoded as convex cones and LMIs.

2. Master effective action and SM limit

Action. Up to the desired derivative order and with BRST-exact gauge fixing,

$$\begin{aligned} S_{\rm eff} \;=\; \int d^4x\,\sqrt{-g_{\rm eff}}\,\Big[&-\frac{\kappa_G}{4}\,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{\kappa_y}{2}\,h_{AB}\,\mathcal{D}_\mu y^A \mathcal{D}^\mu y^B + \mathcal{L}_{\rm Higgs}(v,\lambda_3,Z_h) \\ &+\; \mathcal{L}_{\rm f}(\text{overlaps} \Rightarrow Y_{u,d,e,\nu}) + \mathcal{L}_{\rm portals}(\varepsilon,a_0) + \mathcal{L}_{\rm WZW} + \mathcal{L}_{\rm CS} + \sum_{d>4}\frac{c^{(d)}}{\Lambda^{d-4}}\mathcal{O}^{(d)}\Big] \\ &+\; S^{\rm BRST}_{\rm gf+gh} \;+\; S_{\rm inflow}[\partial X]\;, \end{aligned} \tag{24.45}$$

subject to the gates summarized below.

SM limit (*projection*). There exists a geometric regime where

$$(\mathcal{F},h)\;\Rightarrow\; G_{\rm SM}=SU(3)\times SU(2)\times U(1), \qquad (\kappa_1,\kappa_2,\kappa_G)\;\mapsto\;(g',g,g_3), \qquad \text{portals}\to 0,\;\; c^{(d>4)}\to 0, \tag{24.46}$$

and all WZW/CS coefficients reduce to their SM-topology values. In this limit $S_{\rm eff}$ reproduces the renormalizable SM with controlled higher-dimensional tails.

3. Geometry⇒phenomenology dictionary (selected maps)

$$(\kappa_1,\kappa_2)\;\xrightarrow{\text{Sec. 7.4}}\;\sin^2\theta_W=\frac{\kappa_2}{\kappa_1+\kappa_2}+\Delta_{\rm th},\quad m_W=\tfrac{1}{2}\,g(\kappa_1,\kappa_G)\,v\,(1+\tfrac{1}{2}\alpha_T+\cdots), \tag{24.47}$$

$$\text{fiber sub-isometries}\;\xrightarrow{\text{Part IV}}\;SU(3)_c \text{ with } \tau_s(\mathcal{S})\Rightarrow \text{confinement scales}, \tag{24.48}$$

$$\text{topological indices}\;\xrightarrow{\text{Part V}}\;\text{generations, flavor labels, EDM-ray structure}, \tag{24.49}$$

$$\text{overlaps on } \mathcal{F}\;\xrightarrow{\text{Ch. 14}}\;Y_{u,d,e,\nu}\;\Rightarrow\;\text{CKM/PMNS}, \tag{24.50}$$

$$\text{portals } (Z',S)\;\xrightarrow{\text{Sec. 19.3, Sec. 20.2}}\;\varepsilon_{\alpha\beta},\;a_{\alpha\beta,0}\;\Rightarrow\;\text{LBL NSI/aniso.} \tag{24.51}$$

4. Gates: a compact ledger

Local gates (per scale).

(S) : $\mathsf{K}_{tt}\succ 0$, hyperbolic principal symbols;
(C/A/TQ) : $H_i(\mathbf{c})\succeq 0$ (forward LMIs), IR poles subtracted, $t{<}0$ blocks included;
(D) : $\omega\,\mathrm{Im}\,\chi^R(\omega)\succeq 0$ (passivity), σ,η,ζ PSD;
BRST/anom. : $Q_{\rm B}^2=0$, $\mathcal{I}^{\rm UV}_{d+2}=\mathcal{I}^{\rm IR}_{d+2}$, WZW/CS quantized. (24.52)

RG transport. Gates are preserved along the RG tube via cone-transport and threshold matching (Secs. 3.2, 3.3); feasibility at all μ is necessary for viability.

5. Decisive signatures and decision rule

Shortlist. The orthogonal discriminants $\{\mathcal{D}_i\}$ (Sec. 24.1) cover: EW geometry $(m_W, \sin^2\theta_W)$, VBS helicity-positivity edge, di-Higgs threshold shape $(\Delta\lambda_3, \delta Z_h)$, EDM correlation ray, neutrino magic-baseline isolator, photon anisotropy, glueball/string matching, and anomaly quantization.

Combined test. With whitened, gate-projected gradients,

$$\mathcal{T}^2 = \sum_i \mathcal{D}_i^2 \quad \xrightarrow{\text{Sec. 24.2}} \quad \textsc{Decisive} \;\Leftrightarrow\; \mathcal{T}^2 \geq \tau_{\text{glob}}, \tag{24.53}$$

and missed milestones trigger the graded revision paths (R1–R6) of Sec. 23.4.

6. Falsifiability and minimal revision

Hard kills. Any breach of (24.52), anomaly/BRST integrity, or cone inclusion is fatal; data-level over-penalization within the cone ($\Delta\text{BIC} > 0$) warrants rejection (Sec. 23.4).

Soft repairs. When failure is not fundamental, minimal surgeries—retuning within $\mathcal{C}$, threshold realignment, gate-feasible higher-d lifts, portal completions, or gentle fiber-metric deformations—restore feasibility, subject to re-certification.

7. Outlook and unifying statement

Program loop.

$$\boxed{\text{Geometry on } \mathcal{F} \;\Rightarrow\; \text{Gauge \& flavor structure} \;\Rightarrow\; \text{Positivity/causality gates} \;\Rightarrow\; \text{Decisive signatures}} \tag{24.54}$$

Interfaces to Volumes V/VI pass $(\mathbf{p}_{\text{IV}}, \mathfrak{C}_{\text{IV}})$ and certified gates to cosmology and pre-geometry, enabling a bidirectional, gate-preserving inference pipeline.

One-line synthesis.

$$\boxed{\text{SpM} = \big(\mathcal{F}, h;\ \mathcal{S};\ \text{horizontality}\big) \;+\; \text{BRST/anomaly integrity} \;+\; \text{S/G/TQ/D gates} \;\Longrightarrow\; \text{SM at a lin}} \tag{24.55}$$

Summary. Volume IV establishes that once local isometries of $(\mathcal{F}, h)$ are gauged by horizontality, the entire particle sector—EW, QCD, flavor/CP, and neutrinos—follows from a small set of geometric invariants and their RG evolution, tightly constrained by universal gates. The model is predictive by construction, falsifiable by decisive, gate-stable signatures, and interoperable with cosmology and pre-geometry through minimal, certified interfaces.

Appendix A: Fiber Geometry and Connection Calculus

Aim. This appendix collects the geometric technology underlying Volume IV: principal bundles, Ehresmann connections, curvature and Bianchi identities, associated bundles and covariant derivatives, the mechanical (metric) connection on orbit bundles, and the calculus for gauged maps $y : M \to (\mathcal{F}, h)$ used throughout the text. We work on a smooth spacetime manifold (M, g_{eff}), a compact Lie group G with Lie algebra $\mathfrak{g}$ (basis $\{T_I\}$, structure constants $f^K{}_{IJ}$), and a Riemannian fiber $(\mathcal{F}, h_{AB})$ with an isometric G-action generated by Killing fields $\{\xi_I\}$.

24.5 Principal bundles and Ehresmann connections

Principal G-bundle. A principal G-bundle is a smooth manifold P with a free, right G-action $R_g : P \to P$ and a projection $\pi : P \to M$ such that P is locally $U \times G$. The *vertical* subspace at $p \in P$ is $V_pP := \ker d\pi_p$. Fundamental vertical vectors are

$$\zeta_X(p) \;:=\; \tfrac{d}{dt}\Big|_{t=0} \big(p\, e^{tX}\big), \qquad X \in \mathfrak{g}, \tag{24.56}$$

and satisfy $[\zeta_X, \zeta_Y] = \zeta_{[X,Y]}$.

Ehresmann connection. An Ehresmann connection is a G-equivariant horizontal distribution $H \subset TP$ with $TP = H \oplus V$ and $(R_g)_* H = H$. Equivalently, a *connection 1-form*

$$\omega \in \Omega^1(P, \mathfrak{g}), \qquad \omega(\zeta_X) = X, \quad (R_g)^*\omega = \mathrm{Ad}_{g^{-1}}\omega, \tag{24.57}$$

whose kernel defines $H = \ker \omega$. The *curvature* is

$$\Omega := d\omega + \tfrac{1}{2}[\omega, \omega] \in \Omega^2(P, \mathfrak{g}), \qquad (R_g)^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega. \tag{24.58}$$

The *Bianchi identity* is $D\Omega := d\Omega + [\omega, \Omega] = 0$.

Local form. On a local section $s : U \subset M \to P$ the gauge potential and field strength are the pullbacks

$$\mathcal{A} := s^* omega \in Omega^1(U, \mathfrak{g}), \qquad \mathcal{F} := s^*\Omega = d\mathcal{A} + \tfrac{1}{2}[\mathcal{A}, \mathcal{A}]. \tag{24.59}$$

Under a change of section by $g : U \to G$, $s \mapsto s \cdot g$, one has

$$\mathcal{A} \mapsto \mathcal{A}' = g\, \mathcal{A}\, g^{-1} - (dg)\, g^{-1}, \qquad \mathcal{F} \mapsto \mathcal{F}' = g\, \mathcal{F}\, g^{-1}. \tag{24.60}$$

24.6 Associated bundles and covariant derivatives

Associated bundle. For a left G-space $(\mathcal{F}, h)$, the associated bundle is $E := P \times_G \mathcal{F} \xrightarrow{\pi_E} M$. A section $y \in \Gamma(E)$ corresponds locally (on U) to a map $y : U \to \mathcal{F}$ defined by a trivialization.

Covariant exterior derivative. For a $\mathfrak{g}$-module V (representation R) and a V-valued form Ψ, define

$$D\Psi \;:=\; d\Psi + R(\mathcal{A}) \wedge \Psi. \tag{24.61}$$

For adjoint-valued forms, $R = \mathrm{ad}$ and $D\mathcal{F} = 0$ (Bianchi).

Gauged derivative of y. Let $\xi_I = \xi_I^A(y)\,\partial_{y^A}$ be the Killing fields generating the G-action on $\mathcal{F}$. The *fiber-covariant* derivative used in Vol. IV is

$$\mathcal{D}_\mu y^A \;:=\; \partial_\mu y^A - \xi_I^A(y)\,\mathcal{A}_\mu^I, \qquad \mathcal{D}y \;=\; dy - \xi_I(y)\,\mathcal{A}^I, \tag{24.62}$$

transforming as a tensor under local G.

24.7 Mechanical (metric) connection on orbit bundles

Orbit metric. The isometric G-action on $(\mathcal{F}, h)$ induces the *locked inertia tensor* (orbit metric)

$$\gamma_{IJ}(y) \;:=\; h_{AB}(y)\,\xi_I^A(y)\,\xi_J^B(y), \qquad \gamma_{IJ} = \gamma_{JI},\;\; \gamma_{IJ} \text{ positive definite on } \mathfrak{g}. \tag{24.63}$$

Let γ^{IJ} be its inverse.

Horizontal projector and mechanical connection. For a map $y : U \to \mathcal{F}$, decompose dy into horizontal (orthogonal to orbits) and vertical parts. The unique $\mathcal{A}_\mu^I$ that makes $\mathcal{D}_\mu y$ horizontal (orthogonal to all ξ_I) is

$$\boxed{\mathcal{A}_\mu^I \;=\; \gamma^{IJ}(y)\,h_{AB}(y)\,\xi_J^A(y)\,\partial_\mu y^B} \qquad \Longleftrightarrow \qquad h_{AB}\,\xi_I^A\,\mathcal{D}_\mu y^B = 0. \tag{24.64}$$

This is the *mechanical connection*. It is the L^2-orthogonal projection of dy onto the orbit directions. In bundle language, (24.64) coincides with the Ehresmann connection whose horizontal spaces are h-orthogonal to G-orbits.

Pure-gauge curvature for orbit pullbacks. If $\mathcal{A}$ is *defined* by (24.64) from a single-valued y, its curvature obeys

$$\mathcal{F}_{\mu\nu}^I \;=\; -\gamma^{IJ}\,h_{AB}\,\xi_J^A\left(\nabla_C \xi_K^B\right)\mathcal{D}_{[\mu}y^C\,\mathcal{D}_{\nu]}y^K \;\;+\;\; (\text{orbit torsion terms}). \tag{24.65}$$

If y takes values in a single group orbit (coset model with flat orbit connection), $\mathcal{F}$ may vanish; dynamical gauge curvature arises once $\mathcal{A}$ acquires independent degrees of freedom beyond the induced mechanical piece (cf. Vol. IV, Sec. V4-1.2).

24.8 Curvature, Bianchi identities, and Cartan structure

Structure equations (local). In a local trivialization,

$$\mathcal{F} \;=\; d\mathcal{A} + \tfrac{1}{2}[\mathcal{A},\mathcal{A}], \tag{24.66}$$

$$D\mathcal{F} \;=\; d\mathcal{F} + [\mathcal{A},\mathcal{F}] \;=\; 0 \qquad \text{(Bianchi)}. \tag{24.67}$$

Gauge-covariant Lie derivative. For a vector field X on M,

$$\mathcal{L}_X^{(D)}\mathcal{A} \;:=\; i_X\mathcal{F} + D(i_X\mathcal{A}), \qquad \mathcal{L}_X^{(D)}\mathcal{F} \;:=\; i_X D\mathcal{F} + D(i_X\mathcal{F}) \;=\; D(i_X\mathcal{F}), \tag{24.68}$$

useful for symmetry and Noether analyses.

24.9 Gauged sigma model geometry on $(\mathcal{F}, h)$

Levi–Civita on $\mathcal{F}$. Let $\Gamma^A{}_{BC}$ be the Levi–Civita connection of h. For a section y,

$$(\mathcal{D}_\mu\mathcal{D}_\nu y)^A \;:=\; \partial_\mu(\mathcal{D}_\nu y^A) - \Gamma^A{}_{BC}\,\mathcal{D}_\mu y^B\,\mathcal{D}_\nu y^C - \xi_I^A\,\mathcal{F}^I_{\mu\nu}. \tag{24.69}$$

Antisymmetrizing yields the *Ricci identity*

$$\mathcal{D}_{[\mu}\mathcal{D}_{\nu]}y^A \;=\; -\,\xi_I^A\,\mathcal{F}^I_{\mu\nu} \;+\; \tfrac{1}{2}\,R^A{}_{BCD}(y)\,\mathcal{D}_{[\mu}y^B\,\mathcal{D}_{\nu]}y^C\,(\nabla y)^D, \tag{24.70}$$

with $R^A{}_{BCD}$ the Riemann tensor of $(\mathcal{F}, h)$. In particular, gauge curvature obstructs the integrability of horizontal lifts.

Energy density and stress tensor. The gauged sigma model Lagrangian

$$\mathcal{L}_y \;=\; \tfrac{\kappa_y}{2}\,g_{\text{eff}}^{\mu\nu}h_{AB}\,\mathcal{D}_\mu y^A\,\mathcal{D}_\nu y^B - V_{\text{int}}(y) \tag{24.71}$$

gives

$$T^{(y)}_{\mu\nu} \;=\; \kappa_y\,h_{AB}\,\mathcal{D}_\mu y^A\,\mathcal{D}_\nu y^B - g_{\text{eff},\mu\nu}\,\mathcal{L}_y. \tag{24.72}$$

24.10 Moment maps, Killing fields, and Noether currents

Killing data. The G-action is generated by Killing fields ξ_I obeying

$$\mathcal{L}_{\xi_I}h = 0, \qquad [\xi_I,\xi_J] = f^K{}_{IJ}\,\xi_K, \qquad \nabla_A\xi_{BI} + \nabla_B\xi_{AI} = 0. \tag{24.73}$$

Moment maps (Kähler case). If $(\mathcal{F}, h)$ carries a compatible symplectic form ω, the *moment map* $\mu : \mathcal{F} \to \mathfrak{g}^*$ obeys

$$d\mu_I \;=\; i_{\xi_I}\omega, \qquad \{\mu_I,\mu_J\} \;=\; f^K{}_{IJ}\,\mu_K. \tag{24.74}$$

In the purely Riemannian setting, $\gamma_{IJ} = h(\xi_I,\xi_J)$ substitutes for μ_I in the construction of the mechanical connection (24.64).

Noether current. Varying $\mathcal{L}_y$ under $\delta y^A = \epsilon^I \xi_I^A$ and $\delta \mathcal{A}_\mu = -D_\mu \epsilon$ yields

$$J_I^\mu \;=\; \kappa_y \, h_{AB} \, \xi_I^A \, \mathcal{D}^\mu y^B, \qquad D_\mu J_I^\mu \;=\; 0, \tag{24.75}$$

and the Yang–Mills equation $\kappa_G D_\nu \mathcal{F}_I^{\nu\mu} = J_I^\mu + \cdots$.

24.11 Variational calculus for Yang–Mills and gauge fixing

Yang–Mills action and EOM.

$$S_{\mathrm{YM}} \;=\; -\frac{\kappa_G}{4} \int_M \sqrt{-g_{\mathrm{eff}}}\, \mathrm{Tr}\, \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}, \qquad \delta S_{\mathrm{YM}} \;=\; \kappa_G \int_M \sqrt{-g_{\mathrm{eff}}}\, \mathrm{Tr}\, (D_\mu \mathcal{F}^{\mu\nu}\, \delta \mathcal{A}_\nu), \tag{24.76}$$

so $D_\mu \mathcal{F}^{\mu\nu} = \kappa_G^{-1}\, J^\nu$ after coupling to currents.

BRST-ready gauge fixing (linear covariant). With gauge-fixing functional $\mathcal{G}(\mathcal{A}) = \nabla^\mu \mathcal{A}_\mu$ and Faddeev–Popov operator $\mathcal{M} = -\nabla^\mu D_\mu$, the gauge-fixed action reads

$$S_{\mathrm{gf+gh}} \;=\; \int_M \sqrt{-g_{\mathrm{eff}}}\, \mathrm{Tr}\Big[\frac{1}{2\xi}\, \mathcal{G}^2 - \bar{c}\, \mathcal{M}\, c \Big], \tag{24.77}$$

compatible with the BRST structure of Ch. V4-2.

24.12 Chern–Weil forms and descent (quick reference)

Invariant polynomials. For an Ad-invariant symmetric polynomial $\mathcal{P}$ on $\mathfrak{g}$, the Chern–Weil form

$$\mathcal{I}_{2k} \;:=\; \mathcal{P}(\underbrace{\mathcal{F}, \ldots, \mathcal{F}}_{k}) \qquad (2k\text{-form}) \tag{24.78}$$

is closed and its de Rham cohomology class is independent of the connection. In particular, $\mathrm{Tr}\, \mathcal{F} \wedge \mathcal{F}$ yields Pontryagin classes. Descent equations $\mathcal{I}_{2k} = d\mathcal{Q}_{2k-1}$ and $\delta \mathcal{Q}_{2k-1} = d\mathcal{Q}^{(1)}_{2k-2}$ under gauge variation produce anomaly and WZW functionals (Vol. IV, Part V4-2).

24.13 Parallel transport, holonomy, and Ambrose–Singer

Parallel transport. Given a curve $\gamma : [0,1] \to M$, the parallel transporter U_γ solves

$$\frac{d}{dt} U_\gamma(t) \;=\; -\,\dot{\gamma}^\mu(t)\, \mathcal{A}_\mu(\gamma(t))\, U_\gamma(t), \qquad U_\gamma(0) = \mathbf{1}. \tag{24.79}$$

Holonomy around small loops measures $\mathcal{F}$ to leading order.

Ambrose–Singer theorem. The Lie algebra of the holonomy group at a point is generated by the set of $\mathcal{F}_x(X, Y)$ and its parallel transports along curves from x (the curvature determines holonomy).

24.14 Coordinate expressions and useful identities

Local indices. With $\mathcal{A}_\mu = \mathcal{A}_\mu^I T_I$ and $[T_I, T_J] = f^K{}_{IJ} T_K$,

$$\mathcal{F}^I_{\mu\nu} = \partial_\mu \mathcal{A}^I_\nu - \partial_\nu \mathcal{A}^I_\mu + f^I{}_{JK}\, \mathcal{A}^J_\mu \mathcal{A}^K_\nu, \tag{24.80}$$

$$(D_\mu \mathcal{F}^{\mu\nu})^I = \nabla_\mu \mathcal{F}^{\mu\nu I} + f^I{}_{JK}\, \mathcal{A}^J_\mu \mathcal{F}^{\mu\nu K}, \tag{24.81}$$

$$\mathcal{D}_\mu y^A = \partial_\mu y^A - \xi^A_I\, \mathcal{A}^I_\mu, \qquad J^\mu_I = \kappa_y\, h_{AB}\, \xi^A_I \mathcal{D}^\mu y^B. \tag{24.82}$$

Bianchi and Jacobi.

$$D_{[\mu}\mathcal{F}_{\nu\rho]} = 0, \qquad f^M{}_{I[J} f^I{}_{KL]} = 0. \tag{24.83}$$

Hodge and YM in forms. With Hodge star $\star$ for g_{eff},

$$S_{\text{YM}} = -\frac{\kappa_G}{2} \int_M \operatorname{Tr}(\mathcal{F} \wedge \star \mathcal{F}), \qquad D \star \mathcal{F} = \kappa_G^{-1} \star J. \tag{24.84}$$

24.15 Remarks on positivity and cone inclusions (quick link)

Forward limit. Forward-positivity (Part V4-17) constrains the eigenvalues of Hankel-like matrices built from scattering amplitudes $M^{(n)}(t = 0)$; these constraints are *representation-independent* and hence compatible with the geometric construction here.

Cone inclusion for kinetic tensors. Anisotropic kinetic deformations $\delta\kappa^{\mu\nu}_{IJ}$ must satisfy g_{eff}-cone inclusion to preserve hyperbolicity; in index-free form

$$\forall\, \xi \neq 0: \quad \xi_\mu \xi_\nu \left(\kappa_G\, \delta^I{}_J\, g^{\mu\nu}_{\text{eff}} + \delta\kappa^{\mu\nu I}{}_J \right) \succ 0. \tag{24.85}$$

24.16 Minimal "cheat sheet"

Connection: $\omega \in \Omega^1(P, \mathfrak{g}), \quad (R_g)^* \omega = \mathrm{Ad}_{g^{-1}} \omega, \quad \omega(\zeta_X) = X;$ (24.86)

Curvature: $\Omega = d\omega + \frac{1}{2}[\omega, \omega], \quad D\Omega = 0;$ (24.87)

Local forms: $\mathcal{A} = s^* \omega, \mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}], \mathcal{A}' = g\mathcal{A}g^{-1} - (dg)g^{-1};$ (24.88)

Gauged map: $\mathcal{D}y = dy - \xi_I(y)\, \mathcal{A}^I, \quad \mathcal{A}^I_\mu = \gamma^{IJ} h_{AB} \xi^A_J \partial_\mu y^B$(mechanical); (24.89)

YM + source: $D_\mu \mathcal{F}^{\mu\nu} = \kappa_G^{-1}\, J^\nu, \quad J^\nu_I = \kappa_y\, h_{AB}\, \xi^A_I \mathcal{D}^\nu y^B.$ (24.90)

Summary. The Ehresmann connection encodes horizontality; its local representative is the gauge potential $\mathcal{A}$ with curvature $\mathcal{F}$ obeying Bianchi identities. The metric on $\mathcal{F}$ selects a canonical (mechanical) connection that orthogonally projects gradients of y off the group orbits, yielding the covariant derivative used in the main text. Characteristic classes and descent enter through Chern–Weil theory. These ingredients form the geometric backbone for all gauged constructions in Volume IV.

Appendix B: BRST/FP Derivations in the Spectral Setting

Aim. This appendix supplies the technical derivations underlying Chapter 2: (i) the Faddeev–Popov (FP) construction on (M, g_{eff}) with fields valued in the fiber-induced gauge group G, (ii) the BRST algebra and its nilpotency on the full field content (gauge, matter on $\mathcal{F}$, and Nakanishi–Lautrup auxiliaries), (iii) Slavnov–Taylor (ST) identities and the Zinn–Justin equation, (iv) Nielsen identities (gauge-parameter dependence), (v) Kugo–Ojima quartet/physical subspace, and (vi) spectral-specific variants (background field method, R_ξ gauges in the structural/Higgs phase, and allowed anisotropy deformations consistent with the S/G/TQ/D gates).

Throughout we use the conventions of Appendix 24.4. Gauge indices $I, J, \dots$ are adjoint; space–time is (M, g_{eff}); the Lie algebra is $\mathfrak{g}$ with $[T_I, T_J] = f^K{}_{IJ} T_K$ and $\mathrm{Tr}(T_I T_J) = \delta_{IJ}$.

24.17 Faddeev–Popov construction on (M, g_{eff})

Gauge orbits and gauge fixing functional. Let Φ collectively denote all dynamical fields that transform under G:

$$\Phi = (\mathcal{A}_\mu, y, \psi, \bar\psi, \dots), \qquad \delta_\epsilon \mathcal{A}_\mu = D_\mu \epsilon, \quad \delta_\epsilon y^A = -\xi_I^A(y)\,\epsilon^I, \quad \delta_\epsilon \psi = R(\epsilon)\psi, \ \dots$$

Choose a local gauge-fixing functional $\mathcal{G}^I(\mathcal{A}, \Phi_{\text{matter}})$ (e.g. linear covariant $\mathcal{G}^I = \nabla^\mu \mathcal{A}^I_\mu$). The FP identity inserts unity in the path integral:

$$1 = \Delta_{\text{FP}}[\mathcal{A}, \Phi] \int \mathcal{D}g \ \delta(\mathcal{G}[\mathcal{A}^g, \Phi^g]), \qquad \mathcal{A}^g = g\mathcal{A}g^{-1} - (dg)g^{-1}. \tag{24.91}$$

Determinant and ghost action. Linearizing the gauge variation $\epsilon(x) \in \mathfrak{g}$ around the identity,

$$\delta\mathcal{G}^I(x) = \int d^4x' \, \mathcal{M}^I{}_J(x, x')\, \epsilon^J(x'), \qquad \mathcal{M} \equiv \frac{\delta \mathcal{G}}{\delta \epsilon},$$

yields $\Delta_{\text{FP}} = \det \mathcal{M}$. Introducing Grassmann ghosts $c^I, \bar c_I$,

$$\Delta_{\text{FP}} = \int \mathcal{D}\bar c \, \mathcal{D}c \ \exp\Big[- \int_M \sqrt{-g_{\text{eff}}}\ \bar c_I \, \mathcal{M}^I{}_J \, c^J \Big]. \tag{24.92}$$

For the linear covariant gauge $\mathcal{G}^I = \nabla^\mu \mathcal{A}^I_\mu$,

$$\mathcal{M}^I{}_J = -\nabla^\mu D_\mu^{IJ}[\mathcal{A}], \qquad (D_\mu c)^I = \nabla_\mu c^I + f^I{}_{JK} \mathcal{A}^J_\mu c^K. \tag{24.93}$$

Auxiliary field and gauge-fixing term. Introduce the Nakanishi–Lautrup field b^I and parameter $\xi > 0$:

$$S_{\rm gf} = \int_M \sqrt{-g_{\rm eff}}\ {\rm Tr}\left(\frac{1}{2\xi}\,\mathcal{G}\cdot\mathcal{G}\right) \iff S_{\rm gf} = \int_M \sqrt{-g_{\rm eff}}\ {\rm Tr}\left(b\,\mathcal{G} + \frac{\xi}{2}\,b^2\right), \tag{24.94}$$

with equation of motion $b = -\xi^{-1}\,\mathcal{G}$. The full gauge-fixed action reads $S' = S_{\rm cl} + S_{\rm gf} + S_{\rm gh}$, with

$$S_{\rm gh} = \int_M \sqrt{-g_{\rm eff}}\ \bar{c}_I \Big(-\nabla^\mu D^{IJ}_\mu[\mathcal{A}]\Big) c_J \quad \text{(linear covariant gauge, curved background).} \tag{24.95}$$

24.18 BRST algebra and nilpotency

Fields and ghost numbers.

$$\mathrm{gh}(\mathcal{A}_\mu) = 0, \quad \mathrm{gh}(y) = 0, \quad \mathrm{gh}(c) = +1, \quad \mathrm{gh}(\bar{c}) = -1, \quad \mathrm{gh}(b) = 0.$$

BRST is an odd nilpotent differential s with $s^2 = 0$ acting as:

$$s\,\mathcal{A}^I_\mu = (D_\mu c)^I, \qquad s\,c^I = -\tfrac{1}{2} f^I{}_{JK}\,c^J c^K, \qquad s\,\bar{c}_I = b_I, \qquad s\,b_I = 0, \tag{24.96}$$

$$s\,y^A = -\,\xi^A_I(y)\,c^I, \qquad s\,\psi = R(c)\psi, \qquad s\,\bar{\psi} = -\bar{\psi}R(c). \tag{24.97}$$

Nilpotency follows from the Jacobi identity and Killing algebra on $\mathcal{F}$: $s^2\mathcal{A}_\mu = 0$, $s^2 c = 0$, $s^2 y^A = 0$ (since $[\xi_I, \xi_J] = f^K{}_{IJ}\xi_K$).

Gauge fixing as a BRST-exact term. Define the gauge-fermion $\Psi = \int_M \sqrt{-g_{\rm eff}}\ \bar{c}_I \left(\mathcal{G}^I + \frac{\xi}{2} b^I\right)$. Then

$$S_{\rm gf} + S_{\rm gh} = s\,\Psi = \int_M \sqrt{-g_{\rm eff}}\ \left[b_I \mathcal{G}^I + \tfrac{\xi}{2}\, b_I b^I - \bar{c}_I\,\frac{\delta \mathcal{G}^I}{\delta \mathcal{A}^J_\mu}(D_\mu c)^J - \bar{c}_I\,\frac{\delta \mathcal{G}^I}{\delta \Phi}\, s\Phi \right], \tag{24.98}$$

ensuring $s(S_{\rm cl} + S_{\rm gf} + S_{\rm gh}) = 0$ if $S_{\rm cl}$ is gauge invariant.

24.19 Background field method (BFM) variant

Split and transformations. Write $\mathcal{A}_\mu = \bar{\mathcal{A}}_\mu + a_\mu$. Demand background gauge invariance and quantum BRST:

$$\delta_{\rm bg}\,\bar{\mathcal{A}}_\mu = D^{(\bar{\mathcal{A}})}_\mu \omega, \quad \delta_{\rm bg}\,a_\mu = [a_\mu, \omega], \qquad \delta_{\rm bg}\,c = [c, \omega], \ \ldots, \tag{24.99}$$

$$s\,a_\mu = D^{(\bar{\mathcal{A}}+a)}_\mu c, \quad s\,\bar{c} = b, \quad s\,b = 0, \quad s\,\bar{\mathcal{A}}_\mu = 0. \tag{24.100}$$

Choose the background-covariant gauge-fixing functional $\mathcal{G}^I = \bar{\nabla}^\mu a^I_\mu$ to obtain a manifestly background-gauge-invariant effective action and simplified ST identities.

24.20 Slavnov–Taylor and Zinn–Justin identities

Sources for BRST variations. Introduce antifields/sources K coupled to BRST variations:

$$S_K = \int_M \sqrt{-g_{\text{eff}}}\,\Big[K^\mu_I\, s\mathcal{A}^I_\mu + K^{(y)}_A\, sy^A + K^{(c)}_I\, sc^I + \cdots\Big]. \tag{24.101}$$

The generating functional $Z[J,K] = \int \mathcal{D}\Phi\, \exp\{-S' + S_K + J\cdot\Phi\}$ is BRST invariant (up to source terms). For the connected $W = \ln Z$ and the 1PI functional Γ (Legendre of W in J), the *Zinn–Justin* equation reads

$$\mathcal{S}(\Gamma) \;\equiv\; \int_M \sqrt{-g_{\text{eff}}}\,\left(\frac{\delta\Gamma}{\delta\mathcal{A}^I_\mu}\frac{\delta\Gamma}{\delta K^\mu_I} + \frac{\delta\Gamma}{\delta y^A}\frac{\delta\Gamma}{\delta K^{(y)}_A} + \frac{\delta\Gamma}{\delta c^I}\frac{\delta\Gamma}{\delta K^{(c)}_I} + \cdots\right) = 0. \tag{24.102}$$

In BV notation this is the master equation $(\Gamma,\Gamma)=0$ w.r.t. the antibracket.

Linearized ST operator and cohomology. Linearizing around the classical action gives the nilpotent operator $\mathcal{B}_\Gamma$ acting on functionals $\mathcal{F}$,

$$\mathcal{B}_\Gamma\mathcal{F} = \int \sqrt{-g_{\text{eff}}}\,\left(\frac{\delta\Gamma}{\delta\Phi}\frac{\delta\mathcal{F}}{\delta K} + \frac{\delta\Gamma}{\delta K}\frac{\delta\mathcal{F}}{\delta\Phi}\right), \qquad \mathcal{B}_\Gamma^2 = 0.$$

Possible anomalies correspond to nontrivial cohomology classes $H^1(\mathcal{B}_\Gamma)$ (ghost number 1). Chapter 4 shows their matching/cancellation within the spectral model.

24.21 Nielsen identities: gauge-parameter dependence

Master identity. Promote ξ to a spurion with BRST partner χ: $s\,\xi = \chi$, $s\,\chi = 0$. Extend the gauge fermion $\Psi(\xi)$ accordingly. BRST invariance implies, for the 1PI effective action,

$$\frac{\partial\Gamma}{\partial\xi} = -\int_M \sqrt{-g_{\text{eff}}}\,\left(\frac{\delta\Gamma}{\delta\phi_i}\,\mathcal{C}_i\right), \qquad \mathcal{C}_i \equiv \left\langle \frac{\partial(s\phi_i)}{\partial\xi}\right\rangle, \tag{24.103}$$

where ϕ_i runs over all fields. At extrema of Γ (on-shell), observable poles (masses) and S-matrix elements are ξ-independent; off-shell quantities (e.g. effective potential V_{eff}) obey Nielsen differential relations used in Sec. 8.4.

24.22 Kugo–Ojima quartet and unitarity

Physical subspace. Define the nilpotent BRST charge Q_{B} (Noether current of BRST). States are graded by ghost number. Physical states are BRST cohomology classes at ghost number zero:

$$\mathcal{H}_{\text{phys}} \;\cong\; \ker Q_{\text{B}} \big/ \operatorname{im} Q_{\text{B}}. \tag{24.104}$$

Unphysical polarizations of $\mathcal{A}_\mu$, the ghosts $(c,\bar c)$, and b organize into *quartets* $(|u\rangle, |v\rangle, Q_{\text{B}}|u\rangle, Q_{\text{B}}|v\rangle)$ with zero norm in the cohomology. With BRST-invariant regularization and anomaly freedom, the S-matrix is unitary on $\mathcal{H}_{\text{phys}}$.

24.23 Spectral-specific gauges and structural phase (R_ξ)

Linear covariant (curved) and background gauges. On $(M, g_{\rm eff})$ use $\mathcal{G}^I = \nabla^\mu \mathcal{A}^I_\mu$ or the BFM choice $\mathcal{G}^I = \bar\nabla^\mu a^I_\mu$ (Sec. 24.19). Both preserve hyperbolicity if $\xi > 0$ and the principal symbol is $g_{\rm eff}$-compatible (G-gate).

R_ξ in the Higgs/structural phase. For a structural order parameter $\mathcal{H}$ (doublet in the EW sector), define

$$\mathcal{G}^a = \nabla^\mu W^a_\mu - \xi\, g\, \mathrm{Im}\left(\mathcal{H}^\dagger \tfrac{\sigma^a}{2}\varphi\right), \quad \mathcal{G}' = \nabla^\mu B_\mu - \xi\, g'\, \mathrm{Im}\left(\mathcal{H}^\dagger \tfrac{Y}{2}\varphi\right), \tag{24.105}$$

with φ the Goldstone fluctuation around $\langle\mathcal{H}\rangle$. This cancels gauge–Goldstone mixing at quadratic order and yields ghost masses $\propto \xi m^2_{W,Z}$.

Anisotropy-compatible deformations. Small kinetic anisotropies $\delta\kappa^{\mu\nu}_{IJ}(\chi)$ (Part 22) enter as

$$\Delta\mathcal{L}_{\rm YM} = -\frac{1}{4}\,\delta\kappa^{\mu\nu}_{IJ}\,\mathcal{F}^I{}_{\mu\alpha}\mathcal{F}^J{}_\nu{}^\alpha,$$

leaving BRST intact when $\delta\kappa$ is G-invariant and independent of the gauge potential under s; hyperbolicity requires cone inclusion (Appendix 24.4, Eq. (24.85)).

24.24 Algebraic renormalization and cohomology

Strategy. Assume a regulator preserving BRST (or implement appropriate Slavnov counterterms). Perturb $\Gamma \to \Gamma + \hbar\,\Gamma^{(1)} + \cdots$ with $(\Gamma,\Gamma) = 0$. At order $\hbar^n$,

$$\mathcal{B}_{\Gamma^{(0)}}\,\Gamma^{(n)} = \Delta^{(n)}, \qquad \mathcal{B}_{\Gamma^{(0)}}\Delta^{(n)} = 0. \tag{24.106}$$

If $H^1(\mathcal{B}_{\Gamma^{(0)}}) = 0$ (no anomaly in the relevant sector), then $\Delta^{(n)} = \mathcal{B}_{\Gamma^{(0)}}\hat\Gamma^{(n)}$ and can be removed by counterterms. Invariant counterterms correspond to $H^0(\mathcal{B}_{\Gamma^{(0)}})$ and match the local operators listed in Part 17.

24.25 Worked derivations (details)

(D1) Ghost action in linear covariant gauge (curved). Vary $\mathcal{G}^I = \nabla^\mu\mathcal{A}^I_\mu$ under $\delta_\epsilon\mathcal{A}^I_\mu = (D_\mu\epsilon)^I$:

$$\delta\mathcal{G}^I = \nabla^\mu(D_\mu\epsilon)^I = -(\nabla^\mu D_\mu)^I{}_J\epsilon^J,$$

which identifies $\mathcal{M} = -\nabla^\mu D_\mu$ and produces (24.95). Connections on $(M, g_{\rm eff})$ act only on world indices; D_μ acts on gauge indices.

(D2) Nilpotency on y^A. Using $s\,y^A = -\xi^A_I c^I$ and $s\,c^I = -\frac{1}{2}f^I{}_{JK}c^Jc^K$,

$$s^2y^A = -(s\xi^A_I)\,c^I + \xi^A_I\tfrac{1}{2}f^I{}_{JK}c^Jc^K.$$

But $s\xi^A_I = (\partial_B\xi^A_I)\,sy^B = -(\partial_B\xi^A_I)\,\xi^B_Jc^J$. Using $[\xi_J,\xi_K]^A = \xi^B_J\partial_B\xi^A_K - \xi^B_K\partial_B\xi^A_J = f^I{}_{JK}\xi^A_I$ shows $s^2y^A = 0$.

(D3) Zinn–Justin equation. BRST invariance of the functional measure and of $S' - S_K$ implies $\int \mathcal{D}\Phi\, s \exp\{-S' + S_K + J \cdot \Phi\} = 0$. After Legendre transform one obtains (24.102).

24.26 Quick reference ("cheat sheet")

$$\textbf{BRST:}\quad s\mathcal{A}_\mu = D_\mu c,\quad sc = -\tfrac{1}{2}[c,c],\quad s\bar{c} = b,\quad sb = 0,\quad sy = -\xi(c)\cdot y; \tag{24.107}$$

$$\textbf{GF/ghost:}\quad S_{\rm gf} + S_{\rm gh} = s\int \bar{c}\,(\mathcal{G} + \tfrac{\xi}{2} b),\quad \mathcal{G} = \nabla^\mu \mathcal{A}_\mu \text{ (or BFM)}; \tag{24.108}$$

$$\textbf{FP operator:}\quad \mathcal{M} = -\nabla^\mu D_\mu,\quad S_{\rm gh} = \int \bar{c}\,\mathcal{M}\,c; \tag{24.109}$$

$$\textbf{ST/ZJ:}\quad \mathcal{S}(\Gamma) = 0,\quad \mathcal{B}_\Gamma^2 = 0; \tag{24.110}$$

$$\textbf{Nielsen:}\quad \partial_\xi \Gamma = -\int (\delta\Gamma/\delta\phi_i)\,\mathcal{C}_i \;\Rightarrow\; \text{on-shell observables are } \xi\text{-independent}; \tag{24.111}$$

$$\textbf{Unitarity:}\quad \mathcal{H}_{\rm phys} \simeq \ker Q_{\rm B}/{\rm im} Q_{\rm B} \;\Rightarrow\; S^\dagger S = \mathbf{1} \text{ on } \mathcal{H}_{\rm phys}. \tag{24.112}$$

Summary. The FP determinant, BRST algebra, and ST/ZJ/Nielsen identities are established on $(M, g_{\rm eff})$ with fields transforming under fiber-induced G. Nilpotency and cohomology guarantee unitarity and gauge-parameter independence of observables, while the presented gauges (linear covariant, BFM, R_ξ) respect the S/G/TQ/D gates and are compatible with spectral anisotropies when cone-inclusion constraints are enforced.

Appendix C: Anomaly Polynomial and Descent

Aim. We collect the invariant-polynomial technology underlying all anomaly statements in Volume IV: the anomaly polynomial for chiral matter, the descent algorithm producing consistent anomalies, the relation to covariant anomalies via Bardeen–Zumino (BZ) currents, and the inflow/cancellation mechanism. We adopt index-free differential-form notation with wedge products implicit.

24.27 Conventions and normalizations

Tilde convention. Introduce the *integral* (dimensionless) curvature forms

$$\widetilde{F} \;:=\; \frac{i}{2\pi}\,F, \qquad \widetilde{R} \;:=\; \frac{i}{2\pi}\,R, \tag{24.113}$$

so that characteristic classes built from $\widetilde{F}$, $\widetilde{R}$ have integer periods on closed cycles. Group traces $\mathrm{tr}_{\mathcal{R}}$ act in the representation $\mathcal{R}$ of the gauge group G; for Lorentz curvature we use the trace in the vector/tangent representation (kept abstract via Pontryagin classes).

Characteristic series. The Chern character and the $\widehat{A}$-genus are

$$\mathrm{ch}_{\mathcal{R}}(\widetilde{F}) \;=\; \mathrm{tr}_{\mathcal{R}}\exp\!\Big(\widetilde{F}\Big) = \dim(\mathcal{R}) + \mathrm{tr}_{\mathcal{R}}\widetilde{F} + \tfrac{1}{2}\mathrm{tr}_{\mathcal{R}}\widetilde{F}^2 + \tfrac{1}{6}\mathrm{tr}_{\mathcal{R}}\widetilde{F}^3 + \cdots, \tag{24.114}$$

$$\widehat{A}(\widetilde{R}) \;=\; 1 - \tfrac{1}{24}\,p_1(\widetilde{R}) + \tfrac{1}{5760}\Big(7\,p_1(\widetilde{R})^2 - 4\,p_2(\widetilde{R})\Big) + \cdots, \tag{24.115}$$

where $p_k(\widetilde{R})$ are Pontryagin classes viewed as polynomials in $\widetilde{R}$ (no explicit normalization beyond (24.113) is needed below).

24.28 Anomaly polynomial: general statement

Spin-$\frac{1}{2}$ chiral fermion. For a left-handed Weyl fermion in even spacetime dimension $d = 2n$, transforming in a representation $\mathcal{R}$ of G, the *anomaly polynomial* in degree $2n{+}2$ is

$$\boxed{\;I_{2n+2}\big[\mathcal{R}\big] \;=\; \Big[\;\widehat{A}\big(\widetilde{R}\big)\;\mathrm{ch}_{\mathcal{R}}\big(\widetilde{F}\big)\;\Big]_{2n+2}.\;} \tag{24.116}$$

The subscript $2n{+}2$ denotes taking the homogeneous component of total form-degree $2n{+}2$. The overall sign flips for opposite chirality.

Explicitly in $d = 4$. For $d = 4$ ($n = 2$), one finds

$$\boxed{I_6\big[\mathcal{R}\big] \;=\; \frac{1}{6}\,\mathrm{tr}_{\mathcal{R}}\big(\widetilde{F}^3\big) \;-\; \frac{1}{24}\,p_1(\widetilde{R})\,\mathrm{tr}_{\mathcal{R}}\big(\widetilde{F}\big)\,.} \tag{24.117}$$

The first term sources the (non-Abelian/Abelian) gauge anomaly (ABJ), the second is the mixed gauge–gravitational anomaly. For a $U(1)$ with charges q_i in $\mathcal{R}$, $\mathrm{tr}_{\mathcal{R}}\widetilde{F}^3 = \sum_i q_i^3\,\widetilde{F}^3$ and $\mathrm{tr}_{\mathcal{R}}\widetilde{F} = \sum_i q_i\,\widetilde{F}$.

Other spins/fields. For (Weyl) gravitini or self-dual p-forms the polynomial is modified by replacing $\widehat{A}$ with the appropriate index-theory genus (e.g. Hirzebruch L-genus). We do not need those variants in Volume IV.

24.29 Descent: from I_{2n+2} to consistent anomalies

Transgression (Chern–Simons) form. Given a closed invariant polynomial $P(F, R)$ of degree $n+1$, there exists a $(2n+1)$-form $I^{(0)}_{2n+1}$ such that

$$I_{2n+2} \;=\; d\,I^{(0)}_{2n+1}\,, \qquad I^{(0)}_{2n+1} \;=\; (n+1)\int_0^1 dt\; P\Big(A_t, R_t;\, \underbrace{F_t, \ldots, F_t}_{n}\Big), \tag{24.118}$$

where $A_t = tA$, $F_t = dA_t + A_t^2$, and similarly for the spin connection if R appears. For the purely gauge piece $P(F) = \mathrm{tr}\,\widetilde{F}^{\,n+1}$ this reproduces the standard Chern–Simons transgression.

Descent equations. Under an infinitesimal gauge/Lorentz variation $\delta_\alpha A = D\alpha$, $\delta_\Lambda \omega = D\Lambda$, one has

$$\delta\, I^{(0)}_{2n+1} \;=\; d\,I^{(1)}_{2n}(\alpha, \Lambda), \qquad \Rightarrow \qquad \mathcal{A}_{\mathrm{cons}}(\alpha, \Lambda) \;=\; 2\pi i \int_{M_{2n}} I^{(1)}_{2n}(\alpha, \Lambda). \tag{24.119}$$

$\mathcal{A}_{\mathrm{cons}}$ is the *consistent* anomaly (Wess–Zumino consistent). For mixed gauge–gravitational terms one applies the same transgression to the Lorentz connection.

4D example (gauge part). For the gauge piece in $d = 4$,

$$I_6^{\mathrm{gauge}} = \frac{1}{6}\,\mathrm{tr}_{\mathcal{R}}\big(\widetilde{F}^3\big) = d\,I_5^{(0)}\,, \qquad I_4^{(1)}(\alpha) = \frac{1}{2}\,\mathrm{tr}_{\mathcal{R}}\Big(\alpha\,\widetilde{F}^2\Big), \tag{24.120}$$

so $\mathcal{A}^{\mathrm{gauge}}_{\mathrm{cons}} = 2\pi i \int \frac{1}{2}\,\mathrm{tr}_{\mathcal{R}}(\alpha\,\widetilde{F}^2)$. Adding the mixed term from (24.117) yields $I_4^{(1)}$ with an extra $-\frac{1}{24}\,\mathrm{tr}_{\mathcal{R}}(\alpha)\,p_1(\widetilde{R})$.

24.30 Covariant vs. consistent anomalies and BZ currents

Relation. The *covariant* anomaly $\mathcal{A}_{\mathrm{cov}}$ differs from $\mathcal{A}_{\mathrm{cons}}$ by a local variation of the Bardeen–Zumino polynomial Δ_{BZ}:

$$\mathcal{A}_{\mathrm{cov}} \;=\; \mathcal{A}_{\mathrm{cons}} \;+\; \delta \int_{M_{2n}} \Delta_{\mathrm{BZ}}[A, \omega]. \tag{24.121}$$

Δ_{BZ} is chosen so that the corresponding current transforms covariantly. In $d = 4$, the difference is fixed by requiring $J^\mu_{\mathrm{cov}} \propto \epsilon^{\mu\nu\rho\sigma} F_{\nu\rho} F_{\sigma\cdot}$.

Practical use. For EFT matching and WZW/Chern–Simons constructions (Vol. IV, Chs. 5, 4), it is often convenient to work with consistent anomalies (from descent) and add BZ polynomials when covariant currents are required.

24.31 Anomaly inflow and cancellation

Bulk inflow. Let X_{2n+1} be a manifold with boundary $\partial X_{2n+1} = M_{2n}$. A bulk action

$$S_{\mathrm{CS}}[A,\omega] \;=\; 2\pi i \int_{X_{2n+1}} I^{(0)}_{2n+1}(A,\omega) \tag{24.122}$$

varies on the boundary as

$$\delta S_{\mathrm{CS}} \;=\; -\,2\pi i \int_{M_{2n}} I^{(1)}_{2n}(\alpha,\Lambda) \;=\; -\,\mathcal{A}_{\mathrm{cons}}(\alpha,\Lambda). \tag{24.123}$$

Thus a suitable bulk Chern–Simons functional cancels the boundary consistent anomaly—*anomaly inflow*. In the spectral setting, such inflow terms can arise from integrating out massive fiber-bound modes.

Global anomalies. The polynomial I_{2n+2} also controls certain global (large-gauge) anomalies through index-theory integrality conditions on closed $(2n+2)$-manifolds, but global/Witten anomalies are treated in Vol. IV, Ch. 4.

24.32 Worked 4D cases and quick diagnostics

Abelian $U(1)$. For left-handed fermions with charges $\{q_i\}$,

$$I_6 \;=\; \frac{1}{6}\Big(\sum_i q_i^3\Big)\,\widetilde{F}^3 \;-\; \frac{1}{24}\Big(\sum_i q_i\Big)\,p_1(\widetilde{R})\,\widetilde{F}. \tag{24.124}$$

Cancellation requires $\sum q_i^3 = 0$ (gauge) and $\sum q_i = 0$ (mixed). This reproduces the standard hypercharge constraints when applied generation-wise.

Non-Abelian G. With generators T^a in $\mathcal{R}$ and symmetric tensor $d^{abc} = \frac{1}{2}\mathrm{tr}_{\mathcal{R}}(T^{(a}T^bT^{c)})$,

$$I_6^{\mathrm{gauge}} \;=\; \frac{1}{6}\,d^{abc}\,\widetilde{F}^a\widetilde{F}^b\widetilde{F}^c, \qquad I_6^{\mathrm{mixed}} \;=\; -\,\frac{1}{24}\,p_1(\widetilde{R})\,\mathrm{tr}_{\mathcal{R}}(\widetilde{F}). \tag{24.125}$$

Simple groups with traceless generators have $\mathrm{tr}_{\mathcal{R}}\widetilde{F} = 0$, so only the cubic symmetric invariant matters.

From I_6 to WZW. On a 5-manifold X_5 with boundary M_4, the Wess–Zumino functional is

$$\Gamma_{\mathrm{WZ}}[U,A] \;=\; 2\pi i \int_{X_5} \Big(I^{(0)}_5(A^U) - I^{(0)}_5(A) \Big)\,, \tag{24.126}$$

whose variation on M_4 reproduces the consistent anomaly; see Vol. IV, Ch. 5.

24.33 Spectral-model remarks

Kinetic anisotropies. Small anisotropic deformations of kinetic tensors (via χ^μ) alter propagators but do not modify characteristic classes; I_{2n+2} depends only on *topological* data (bundles and representations), hence anomaly constraints are insensitive to such deformations as long as BRST and locality are maintained.

Emergent groups and matching. When the gauge algebra emerges from fiber isometries, traces $\mathrm{tr}_{\mathcal{R}}$ are computed in the emergent matter representations (Sec. 6.2). The anomaly polynomial (24.116) provides the master matching condition for admissible fermion content and symmetry breaking patterns in the spectral framework.

24.34 Cheat sheet

$$\textbf{Master formula:}\quad I_{2n+2} = \Big[\widehat{A}(\widetilde{R})\,\mathrm{ch}_{\mathcal{R}}(\widetilde{F})\Big]_{2n+2}; \tag{24.127}$$

$$\textbf{4D fermion:}\quad I_6 = \tfrac{1}{6}\mathrm{tr}_{\mathcal{R}}(\widetilde{F}^3) - \tfrac{1}{24}\,p_1(\widetilde{R})\,\mathrm{tr}_{\mathcal{R}}(\widetilde{F}); \tag{24.128}$$

$$\textbf{Descent:}\quad I_{2n+2} = dI^{(0)}_{2n+1},\quad \delta I^{(0)}_{2n+1} = dI^{(1)}_{2n}\ \Rightarrow\ \mathcal{A}_{\mathrm{cons}} = 2\pi i \int I^{(1)}_{2n}; \tag{24.129}$$

$$\textbf{Covariant vs. consistent:}\quad \mathcal{A}_{\mathrm{cov}} = \mathcal{A}_{\mathrm{cons}} + \delta \int \Delta_{\mathrm{BZ}}; \tag{24.130}$$

$$\textbf{Inflow:}\quad \delta\Big(2\pi i \int_{X_{2n+1}} I^{(0)}_{2n+1}\Big) = -\,\mathcal{A}_{\mathrm{cons}} \text{ on } \partial X_{2n+1}. \tag{24.131}$$

Summary. Anomalies of chiral matter are encoded in a closed $(2n+2)$-form built from *topological* invariants of the gauge/Lorentz bundles. The descent algorithm yields consistent anomalies; BZ polynomials convert to covariant form; and bulk Chern–Simons terms realize inflow/cancellation. These tools provide the precise mathematical backbone for all anomaly statements and WZW/CS constructions used across Volume IV.

Appendix D: Positivity Bounds and Dispersion Relations

Aim. We assemble the analytic/positivity machinery used across Volume IV to constrain effective interactions from first principles: locality ⇒ analyticity, unitarity ⇒ optical-theorem positivity, macro-causality/hyperbolicity ⇒ polynomial boundedness and Froissart–Martin growth, and crossing symmetry. We derive forward and beyond-forward dispersion relations with the correct number of subtractions, formulate IR-safe positivity bounds (in presence of t-channel poles or massless exchanges), extend to spinning states via polarization matrices, and summarize SDP-friendly (semidefinite-program) encodings that we use in global "gate" tests.

24.35 Assumptions and analytic domain

Setup. Consider $2\to2$ scattering of equal-mass external excitations (mass $m\geq0$) on the effective background $(M, g_{\rm eff})$. Let the Lorentz scalars be (s,t,u) with $s+t+u=4m^2$. Denote by $\mathcal{M}(s,t)$ the invariant amplitude (helicity indices are suppressed until Sec. 24.39). We assume:

1. **Locality/analyticity:** For fixed t in an ellipse around $t=0$ (Lehmann–Martin domain), $\mathcal{M}(s,t)$ is analytic in the complex s-plane except for right/left cuts $s\geq s_0$ and $s\leq 4m^2-t-s_0$ with $s_0=4m^2$ (or the appropriate inelastic threshold).

2. **Unitarity:** The optical theorem holds,

$$\operatorname{Im}\mathcal{M}(s+i0,0) \;=\; s\,\sigma_{\rm tot}(s) \;\geq\; 0 \qquad (s\geq s_0). \tag{24.132}$$

3. **Crossing:** For identical external states, $\mathcal{M}(s,t)=\mathcal{M}(u,t)$ with $u=4m^2-s-t$; for non-identical, crossing relates helicity/charge-conjugated channels in the usual way.

4. **Polynomial boundedness:** On the real axis at fixed t in the analytic domain,

$$|\mathcal{M}(s,t)| \;\lesssim\; |s|^N \qquad (|s|\to\infty), \tag{24.133}$$

   with N finite; in gapped, short-range theories one may take $N\leq1$ (Froissart–Martin type).

5. **Hyperbolicity (G-gate):** The principal symbol of the linearized equations respects the $g_{\rm eff}$ light cone, ensuring well-posed propagation and the analyticity domain above (see Part V4-17).

24.36 Forward dispersion relations and subtractions

Subtracted dispersion at $t = 0$. For amplitudes satisfying (24.133), one may write (with $N_{\rm sub}$ subtractions)

$$\mathcal{M}(s,0) \;=\; P_{N_{\rm sub}-1}(s) \;+\; \frac{s^{N_{\rm sub}}}{\pi}\int_{s_0}^{\infty} ds'\,\frac{\operatorname{Im}\mathcal{M}(s',0)}{s'^{N_{\rm sub}}}\left(\frac{1}{s'-s}+\frac{1}{s'+s-4m^2}\right), \tag{24.134}$$

where $P_{N_{\rm sub}-1}$ is a real polynomial of degree $N_{\rm sub}-1$ fixed by local counterterms and crossing. For identical external states in a gapped theory, $N_{\rm sub} = 2$ suffices under Froissart–Martin growth.

Derivative (moment) form and positivity. Expanding (24.134) around $s = 2m^2$ or around $s = 0$ (after shifting variables) and using crossing removes odd moments. At $s = 0$:

$$\frac{d^{2n}}{ds^{2n}}\mathcal{M}(s,0)\Big|_{s=0} \;=\; \frac{2(2n)!}{\pi}\int_{s_0}^{\infty} ds'\,\frac{\operatorname{Im}\mathcal{M}(s',0)}{s'^{2n+1}} \;>\; 0 \qquad (n\geq 1), \tag{24.135}$$

while odd derivatives vanish by crossing. Writing the low-energy EFT expansion

$$\mathcal{M}(s,0) \;=\; \sum_{n\geq 1} c_{2n}\, s^{2n} \;+\; \text{IR poles/cuts}, \tag{24.136}$$

we obtain *forward positivity* for the IR-safe part: $c_{2n} > 0$ after subtracting known IR singularities.

IR-safe dispersion. If massless exchanges generate a Coulomb pole or branch at $t = 0$, define

$$\widetilde{\mathcal{M}}(s,0) \;:=\; \mathcal{M}(s,0) \;-\; \mathcal{M}_{\rm IR}^{\rm known}(s,0), \tag{24.137}$$

where $\mathcal{M}_{\rm IR}^{\rm known}$ is the exactly calculable long-range contribution (e.g. one-photon/one-gluon exchange and soft loops from the *baseline* sector). The dispersion and positivity (24.135) then apply to $\widetilde{\mathcal{M}}$.

24.37 Partial waves, Froissart–Martin, and elliptic domain

Partial-wave expansion (schematic). In the c.m. frame with $z = \cos\theta$,

$$\mathcal{M}(s,t) \;=\; 16\pi\sum_{\ell=0}^{\infty}(2\ell{+}1)\,a_\ell(s)\,P_\ell(z), \qquad t = -2p^2(1-z),\;\; p^2 = \tfrac{1}{4}(s-4m^2). \tag{24.138}$$

Unitarity implies $|a_\ell(s) - i/2| \leq 1/2$ for $s \geq s_0$ and a_ℓ analytic in the Lehmann–Martin ellipse in z at fixed s.

Growth bound and subtractions. Polynomial boundedness (24.133) plus the elliptic analyticity yield the standard Froissart–Martin bound (modulo logarithms) and justify $N_{\rm sub} \leq 2$ for gapped, short-range dynamics. In the presence of tiny spectral anisotropies (via χ^μ), the ellipse persists if the hyperbolicity cone coincides with the $g_{\rm eff}$ cone (Part V4-17).

24.38 Beyond-forward positivity at small t

Subtracted fixed-t dispersion. For t in the ellipse, with $N_{\rm sub}$ as before,

$$\partial_s^{2n}\mathcal{M}(0,t) \;=\; \frac{2(2n)!}{\pi}\int_{s_0}^{\infty} ds'\;\frac{\operatorname{Im}\mathcal{M}(s',t)}{(s')^{2n+1}} \;+\; (\text{crossed cut}), \tag{24.139}$$

which is positive if the crossed contribution equals the direct one by crossing (identical states), or can be bounded using u-channel optical data. Taking $t\downarrow 0^-$ avoids t-channel poles. This yields *beyond-forward* bounds controlling Wilson coefficients of operators that do not contribute in the strictly forward limit.

Weighted (smeared) sum rules. Choose a non-negative weight $w(s')\geq 0$ with adequate UV falloff. Then

$$\int_{s_0}^{\infty} ds'\; w(s')\,\operatorname{Im}\mathcal{M}(s',t) \;\geq\; 0, \tag{24.140}$$

and by transferring w to derivatives at $s=0$ one obtains linear positivity constraints on EFT parameters that are robust to Regge uncertainties (finite-energy sum rules).

24.39 Spinning external states and positivity matrices

Helicity matrix. For spinning states, write the forward amplitude as a Hermitian matrix in external polarization space:

$$\mathcal{M}_{\lambda\lambda'}(s,0) \;=\; \epsilon^{(\lambda)}\cdot\epsilon^{(\lambda')*}\,\mathcal{T}(s,0) \quad\Longrightarrow\quad \operatorname{Im}\mathcal{M}_{\lambda\lambda'}(s,0) \;\succeq\; 0 \qquad (s\geq s_0). \tag{24.141}$$

Therefore, for any polarization vector v,

$$\operatorname{Im}\Big[v^\dagger\mathcal{M}(s,0)\,v\Big] \;\geq\; 0 \quad\Rightarrow\quad \partial_s^{2n}\Big[v^\dagger\mathcal{M}(s,0)\,v\Big]_{s=0} \;>\; 0. \tag{24.142}$$

Scanning v produces a *positivity cone* of quadratic forms, which becomes a set of linear matrix inequalities (LMIs) on Wilson matrices when $\mathcal{M}$ is expanded in the EFT basis.

Transversity and gauge invariance. For gauge bosons, work in a gauge-invariant transversity basis; IR poles from t-channel exchange are subtracted per (24.137) before taking $t\to 0^-$.

24.40 IR subtleties and Coulomb singularities

Pole subtraction. If a t-channel massless exchange contributes $\mathcal{M}_{\rm pole}(s,t)\sim g_t^2\,s^2/t$ (schematically), subtract it and its regulated soft/collinear logs computed within the baseline sector. The residual $\widetilde{\mathcal{M}}$ obeys the same analyticity and dispersion with the same $N_{\rm sub}$.

Known-SM (or baseline) removal. In mixed scenarios, separate $\mathcal{M}=\mathcal{M}_{\rm base}+\mathcal{M}_{\rm new}$ and apply positivity to $\mathcal{M}_{\rm new}$; equivalently, move the known piece to the LHS of the dispersion integrals. This is the procedure used in Parts V4-18–20.

24.41 Convex cones and SDP encodings

Hankel LMIs (forward). Define the Hankel matrix of forward derivatives (about $s = 0$),

$$\left[H^{(k)}\right]_{mn} := \frac{1}{(2m+k)!\,(2n+k)!}\,\partial_s^{2m+k}\partial_s^{2n+k}\,\widetilde{\mathcal{M}}(s,0)\Big|_{s=0}, \qquad k = 0, 1. \tag{24.143}$$

For identical states and after crossing symmetrization, $H^{(0)} \succeq 0$; with mild assumptions $H^{(1)} \succeq 0$ as well. In an EFT with parameters $\{c_i\}$, each entry is linear in $\{c_i\}$, giving LMIs suitable for SDP.

Polarization LMIs (spinning). For a chosen polarization basis, write

$$\mathbb{H}^{(0)}_{ab} := \partial_s^2\Big(\widetilde{\mathcal{M}}_{ab}(s,0)\Big)\Big|_{s=0}, \qquad \forall\, v:\ v^\dagger \mathbb{H}^{(0)} v \ > \ 0, \tag{24.144}$$

or equivalently $\mathbb{H}^{(0)} \succeq 0$ as a matrix LMI in the Wilson matrices.

24.42 Worked EFT bounds (representative)

(E1) Shift-symmetric scalar. With

$$\mathcal{L} \ = \ \tfrac{1}{2}(\partial\phi)^2 \ + \ \frac{\lambda_4}{\Lambda^4}(\partial\phi)^4 \ + \ \frac{\lambda_6}{\Lambda^8}(\partial\phi)^6 + \cdots, \tag{24.145}$$

the forward amplitude for $\phi\phi \to \phi\phi$ reads $\mathcal{M}(s,0) = \frac{8\lambda_4}{\Lambda^4}s^2 + \frac{24\lambda_6}{\Lambda^8}s^3 + \cdots$. Crossing removes odd powers after symmetrization; applying (24.135) gives

$$\lambda_4 \ > \ 0, \qquad \text{and} \quad \lambda_6 \text{ bounded from above/below by beyond-forward constraints at } t < 0. \tag{24.146}$$

(E2) Abelian gauge boson self-interactions. Consider the parity-even basis

$$\mathcal{L}_{F^4} \ = \ \frac{c_1}{\Lambda^4}\,(F_{\mu\nu}F^{\mu\nu})^2 \ + \ \frac{c_2}{\Lambda^4}\,(F_{\mu\nu}\tilde{F}^{\mu\nu})^2. \tag{24.147}$$

Using polarization projectors for $\gamma\gamma \to \gamma\gamma$ (after subtracting the t-channel pole if present) one finds the quadratic forms $v^\dagger\mathbb{H}^{(0)}v$ span the cone

$$c_1 \ \geq \ 0, \qquad c_2 \ \geq \ 0, \qquad c_1 + c_2 \ \geq \ 0, \tag{24.148}$$

the last inequality arising from linear polarizations at 45°.

(E3) Non-Abelian quartic gauge operators. For $SU(2)$ electroweak bosons (in the unbroken phase) with operators $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})^2$ and $\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2$ plus color structures $(\mathrm{Tr}F^2)^2$, forward $WW \to WW$ channels yield LMIs that bound particular linear combinations (e.g. the custodial-singlet direction obeys the same cone as (24.148)).

(E4) Mixed scalar–gauge operators. For $\frac{c_{H\Box}}{\Lambda^2}(H^\dagger H)\Box(H^\dagger H)$, $\frac{c_{HW}}{\Lambda^2}H^\dagger H W_{\mu\nu}W^{\mu\nu}$ etc., forward $\pi\pi \to \pi\pi$ (Goldstone) and $\pi W \to \pi W$ amplitudes give linear inequalities in $\{c_{H\Box}, c_{HW}, \ldots\}$ after Equivalence Theorem matching; these are incorporated in the global gates of Part V4-22.

24.43 Relation to S/G/TQ/D gates and anisotropy

Compatibility. Forward/beyond-forward positivity is a *TQ-gate* (quantum positivity) manifestation. The *G-gate* (hyperbolicity) ensures the analytic domain used above; the *S-gate* (stability) eliminates tachyonic or ghost poles that would spoil dispersion; and *D-gate* (dissipation) demands that any small anisotropy $\delta\kappa^{\mu\nu}$ deformations preserve the $g_{\rm eff}$ cone, keeping partial-wave analyticity intact.

Anisotropic deformations. If kinetic tensors are perturbed by $\delta\kappa^{\mu\nu}$, redo the forward limit in the *co-moving* c.m. frame, or equivalently absorb $\delta\kappa$ into redefinitions of Mandelstam invariants built with $g_{\rm eff} + \delta\kappa/\kappa_G$. Positivity inequalities then hold for the reparametrized amplitude; to linear order, they constrain the same Wilson combinations.

24.44 Cheat sheet for applications

$$\textbf{Forward DR:}\quad \mathcal{M}(s,0) = P_{N_{\rm sub}-1}(s) + \frac{s^{N_{\rm sub}}}{\pi}\int_{s_0}^{\infty} ds' \frac{\operatorname{Im}\mathcal{M}(s',0)}{s'^{N_{\rm sub}}}\Big(\frac{1}{s'-s} + \frac{1}{s'+s-4m^2}\Big); \tag{24.149}$$

$$\textbf{Moments:}\quad \partial_s^{2n}\mathcal{M}(0,0) = \frac{2(2n)!}{\pi}\int_{s_0}^{\infty} ds'\ \frac{\operatorname{Im}\mathcal{M}(s',0)}{s'^{2n+1}}\ > 0 \quad (n \geq 1); \tag{24.150}$$

$$\textbf{IR-safe:}\quad \widetilde{\mathcal{M}} = \mathcal{M} - \mathcal{M}_{\rm IR}^{\rm known} \ \Rightarrow\ \text{same DR/positivity for } \widetilde{\mathcal{M}}; \tag{24.151}$$

$$\textbf{Spinning:}\quad \operatorname{Im}\mathcal{M}_{\lambda\lambda'} \succeq 0 \ \Rightarrow\ \partial_s^{2n}\Big[v^\dagger \mathcal{M} v\Big]_{s=0} > 0 \ \ \forall v; \tag{24.152}$$

$$\textbf{SDP:}\quad H^{(0)} \succeq 0,\ \ \mathbb{H}^{(0)} \succeq 0 \text{ as LMIs linear in EFT Wilsons.} \tag{24.153}$$

Summary. Analyticity, unitarity, crossing, and bounded growth imply a hierarchy of *convex* constraints on EFT data. In practice we (i) subtract known long-range pieces to form $\widetilde{\mathcal{M}}$, (ii) impose forward/beyond-forward positivity via Hankel and polarization LMIs, and (iii) combine them with the S/G/TQ/D gates to carve out the admissible parameter region reported in the main text.

Appendix E: Data Likelihoods and Global Fits

Aim. We present a self-contained toolkit for inference in the spectral framework: likelihood construction (counts, differential spectra, Gaussian measurements), correlated systematics and profiling, convex gates (positivity, hyperbolicity) as feasibility constraints, Fisher-information forecasts and Asimov approximations, global combination across experiments, hypothesis testing (frequentist and Bayesian), model comparison, and reproducibility. The tools are used throughout Volume IV but are collected here for clarity and re-use.

24.45 Data structures and theory maps

Theory parameters. Let $\theta \in \mathbb{R}^{n_\theta}$ denote *physics* parameters (EFT Wilsons, spectral couplings, NSI entries, etc.), and $\xi \in \mathbb{R}^{n_\xi}$ denote *nuisance* parameters (fluxes, efficiencies, calibrations, backgrounds). Hard *gates* (S/G/TQ/D, anomaly cancellation, cone/positivity LMIs) define a feasible set

$$\mathcal{C} \;:=\; \{\; (\theta,\xi) \;\big|\; \text{all structural constraints hold} \;\}. \tag{24.154}$$

Experiment-level predictions. Each dataset D_k provides observables $y^{(k)}$ predicted by a map

$$\mu^{(k)}(\theta,\xi) \;=\; \mathsf{F}^{(k)}\Big[\mathcal{O}^{(k)}(\theta)\Big] \;+\; \mathsf{B}^{(k)}(\xi), \tag{24.155}$$

where $\mathsf{F}^{(k)}$ folds parton-level or Hamiltonian-level quantities (cross sections, event shapes, oscillation probabilities) with detector response, acceptances, and binning; $\mathsf{B}^{(k)}$ collects backgrounds and calibration shifts. For template morphing in continuous parameters, use a differentiable interpolant (polynomial/spline or Gaussian-process emulator), cf. Sec. 24.51.

24.46 Likelihood "zoo" and nuisance models

Poisson counts (binned). For independent bins b,

$$\mathcal{L}_{\mathrm{P}}(y|\theta,\xi) \;=\; \prod_b \frac{\mu_b(\theta,\xi)^{\,y_b}\, e^{-\mu_b(\theta,\xi)}}{y_b!}\,. \tag{24.156}$$

Gaussian with covariance. For measurements $y \in \mathbb{R}^m$ with covariance Σ,

$$\mathcal{L}_{\mathrm{G}}(y|\theta,\xi) \;=\; \frac{1}{\sqrt{(2\pi)^m \det\Sigma}}\, \exp\Big[-\tfrac{1}{2}\big(y-\mu(\theta,\xi)\big)^{\top}\Sigma^{-1}\big(y-\mu(\theta,\xi)\big)\Big]. \tag{24.157}$$

Log-normal and multiplicative systematics. For scale systematics $s_j > 0$ with prior $\ln s_j \sim \mathcal{N}(0, \tau_j^2)$, implement $\mu_b \to s_j^{A_{bj}} \mu_b$ with A_{bj} the sensitivity matrix and add

$$\pi_{\mathrm{LN}}(s) = \prod_j \frac{1}{s_j \tau_j \sqrt{2\pi}} \exp\left[-\frac{(\ln s_j)^2}{2\tau_j^2} \right]. \tag{24.158}$$

Nuisance pulls (Gaussian). For additive linearized systematics $\mu \to \mu + A\,\xi$ with prior $\xi \sim \mathcal{N}(0, V)$,

$$\pi_{\mathrm{pull}}(\xi) \;=\; \frac{1}{\sqrt{(2\pi)^{n_\xi} \det V}} \exp\!\Big[-\tfrac{1}{2}\, \xi^\top V^{-1} \xi \Big]. \tag{24.159}$$

Joint likelihood. For datasets $k = 1, \ldots, K$ with priors,

$$\mathcal{L}_{\mathrm{joint}}(y|\theta,\xi) \;=\; \Big[\prod_{k=1}^{K} \mathcal{L}_k\Big(y^{(k)}\Big|\theta,\xi\Big) \Big] \;\times\; \pi(\theta)\; \pi(\xi) \;\times\; \mathbf{1}_{\mathcal{C}}(\theta,\xi), \tag{24.160}$$

where $\mathbf{1}_{\mathcal{C}}$ is the indicator of feasibility (zero outside $\mathcal{C}$). When experiments share nuisances, ξ is common; otherwise split as $(\xi_{\mathrm{shared}}, \xi_k)$.

24.47 Profiling, marginalization, and analytic pulls

Profiling (frequentist). Define the profile log-likelihood

$$\ell_{\mathrm{prof}}(\theta) \;:=\; \sup_{\xi \in \mathcal{C}_\xi(\theta)} \ln \mathcal{L}_{\mathrm{joint}}(y|\theta,\xi). \tag{24.161}$$

Analytic solution for linear-Gaussian sector. For a Gaussian piece with $y \sim \mathcal{N}(\mu(\theta) + A\xi, \Sigma)$ and prior $\xi \sim \mathcal{N}(0, V)$, the maximizing pull is

$$\hat{\xi}(\theta) \;=\; \Big(A^\top \Sigma^{-1} A + V^{-1} \Big)^{-1} A^\top \Sigma^{-1} \Big(y - \mu(\theta) \Big), \tag{24.162}$$

yielding the profiled quadratic form

$$-2\,\ell_{\mathrm{G,prof}}(\theta) \;=\; \Big(y-\mu(\theta)\Big)^\top \Sigma_{\mathrm{eff}}^{-1} \Big(y-\mu(\theta)\Big) + \mathrm{const}, \qquad \Sigma_{\mathrm{eff}}^{-1} \;:=\; \Sigma^{-1} - \Sigma^{-1} A \Big(A^\top \Sigma^{-1} A + V^{-1} \Big)^{-1} A^\top \Sigma^{-1}. \tag{24.163}$$

This is the Schur-complement reduction used for fast scans.

Marginalization (Bayesian). The posterior is

$$p(\theta,\xi|y) \;\propto\; \mathcal{L}_{\mathrm{joint}}(y|\theta,\xi)\,, \qquad p(\theta|y) = \int_{\mathcal{C}_\xi(\theta)} d\xi\; p(\theta,\xi|y). \tag{24.164}$$

For linear-Gaussian ξ the integral is analytic (same Schur structure). Otherwise, use HMC/NUTS or nested sampling with feasibility projection onto $\mathcal{C}$.

24.48 Convex gates as feasibility constraints

Positivity/analyticity LMIs. Let $W(\theta)$ collect EFT Wilson coefficients. Forward/beyond-forward positivity (App. 24.34) yields linear matrix inequalities

$$\mathbb{H}_r(W(\theta)) \succeq 0 \qquad \forall r \in \mathcal{R}, \tag{24.165}$$

where r indexes polarization/projector choices. The feasible set $\mathcal{C}$ includes (24.165) plus hyperbolicity and stability cones (S/G gates). Optimization in (24.161) is then a *conic* problem (QP/SDP) when $\mu(\theta)$ is affine.

Example (NSI block). For LBL neutrinos, impose $\varepsilon \succeq 0$ (vector exchange) and cone LMIs on a_0; parameterize $\varepsilon = LL^\dagger$ to enforce positivity in sampling; add spectral priors restricting operator signs per App. 24.34.

24.49 Asymptotics, test statistics, and intervals

Profile likelihood ratio. For hypothesis $H_0 : \theta = \theta_0$ define

$$q(\theta_0) := -2\Big[\, \ell_{\rm prof}(\theta_0) - \ell_{\rm prof}(\hat\theta) \,\Big], \qquad \hat\theta = \arg\max_{\theta\in\mathcal{C}_\theta} \ell_{\rm prof}(\theta). \tag{24.166}$$

Under standard regularity and away from boundaries, $q(\theta_0) \xrightarrow{d} \chi^2_d$ with $d = \dim\theta$ (Wilks). For one-sided bounded parameters ($\theta \geq 0$), use Chernoff mixtures, e.g. $\frac{1}{2}\delta_0 + \frac{1}{2}\chi^2_1$.

Asimov data and expected sensitivity. For an Asimov dataset $y_A = \mu(\theta^\star, \hat\xi(\theta^\star))$,

$$q_A(\theta_0) \approx (\theta^\star - \theta_0)^\top F(\bar\theta)\,(\theta^\star - \theta_0), \qquad F_{ij} := -\mathbb{E}\Big[\partial_i\partial_j \ell_{\rm prof}(\theta)\Big]_{\bar\theta}, \tag{24.167}$$

yielding expected $Z = \sqrt{q_A}$ for discovery/exclusion forecasts.

CL_s and look-elsewhere. For exclusions with small signals and bounded rates, use $\mathrm{CL}_s = \frac{p(q\geq q_{\rm obs}|s+b)}{p(q\geq q_{\rm obs}|b)}$. Global significance with multiple search regions can be approximated by trials factors or Euler-characteristic methods for smooth random fields.

24.50 Fisher matrix and linear response

General expression. For independent Gaussian pieces with covariance $\Sigma(\theta)$,

$$F_{ij} = \frac{\partial\mu^\top}{\partial\theta_i}\,\Sigma^{-1}\,\frac{\partial\mu}{\partial\theta_j} + \tfrac{1}{2}\,\mathrm{Tr}\left[\,\Sigma^{-1}\frac{\partial\Sigma}{\partial\theta_i}\,\Sigma^{-1}\frac{\partial\Sigma}{\partial\theta_j}\,\right]. \tag{24.168}$$

With Poisson counts, the Gaussian approximation at large counts gives $\Sigma = \mathrm{diag}(\mu)$; for exact Poisson use the information equality $\mathbb{E}[-\partial^2\ell] = \mathbb{E}[(\partial\ell)(\partial\ell)^\top]$.

Linear EFT map. If $\mu(\theta) \approx \mu_0 + J\,\theta$ near $\theta = 0$, then

$$F \approx J^\top\Sigma^{-1}J + F_{\rm prior}, \qquad \mathrm{cov}(\hat\theta) \approx F^{-1}. \tag{24.169}$$

This is used for quick projections and for building Gaussian proposal covariances in MCMC.

24.51 Emulators, morphing, and interpolation

Template morphing. Given grids $\{\theta^{(r)}\}$ with simulated templates $T^{(r)}$, construct

$$\widehat{T}(\theta) \;=\; \sum_r w_r(\theta)\, T^{(r)}, \qquad \sum_r w_r(\theta) = 1,\; w_r \geq 0, \tag{24.170}$$

using barycentric polynomials or simplex interpolation in a triangulation of the parameter domain. Enforce differentiability to allow gradient-based optimization.

Gaussian-process (GP) emulation. For expensive theory maps, train a GP on residuals $\delta(\theta) = \mu(\theta) - \mu_{\text{fast}}(\theta)$ with kernel $k(\theta, \theta')$; propagate emulator uncertainty by augmenting $\Sigma \to \Sigma + \Sigma_{\text{GP}}$ in (24.157).

24.52 Model comparison and posterior predictive checks

Bayes factors and information criteria. For models $\mathcal{M}_1, \mathcal{M}_2$ with evidences $\mathcal{Z}_i = \int d\theta\, d\xi\; \mathcal{L}\,\pi$, compare via $B_{12} = \mathcal{Z}_1/\mathcal{Z}_2$. In large-sample limits, use WAIC/LOO-CV or BIC $\simeq -2\ell_{\max} + k \ln N$ as quick proxies.

Posterior predictive. Draw $\tilde{y} \sim p(y|\theta, \xi)$ with $(\theta, \xi) \sim p(\theta, \xi|y)$ and compare test statistics $T(\tilde{y})$ to $T(y)$. Tension across datasets is quantified by partial posteriors or by shifted-parameter analyses.

24.53 Global combination and correlation handling

Shared nuisances and block structure. Let $y = (y^{(1)}, \ldots, y^{(K)})$ and $\xi = (\xi_{\text{sh}}, \xi_1, \ldots, \xi_K)$. If $y^{(k)}|\xi_{\text{sh}}$ are conditionally independent,

$$\ln \mathcal{L}_{\text{joint}} \;=\; \sum_k \ln \mathcal{L}_k\Big(y^{(k)}\Big|\theta, \xi_{\text{sh}}, \xi_k\Big) \;-\; \tfrac{1}{2}\,\xi_{\text{sh}}^{\top} V_{\text{sh}}^{-1} \xi_{\text{sh}} \;-\; \sum_k \tfrac{1}{2}\,\xi_k^{\top} V_k^{-1} \xi_k. \tag{24.171}$$

Profiling yields a linear system with block-Schur complements analogous to (24.163).

Covariance-level combination. For Gaussian summaries $(\hat{\mu}_k, \Sigma_k)$ of the same observable,

$$\hat{\mu} \;=\; \Big(\sum_k \Sigma_k^{-1}\Big)^{-1} \Big(\sum_k \Sigma_k^{-1} \hat{\mu}_k\Big), \qquad \Sigma \;=\; \Big(\sum_k \Sigma_k^{-1}\Big)^{-1}. \tag{24.172}$$

This is the BLUE (best linear unbiased estimator) used for combined electroweak fits.

24.54 Quality metrics, coverage, and robustness

Goodness-of-fit. For binned Poisson, use saturated-likelihood deviance

$$G^2 \;=\; 2 \sum_b \left[\, \mu_b - y_b + y_b \ln \frac{y_b}{\mu_b} \,\right], \tag{24.173}$$

asymptotically χ^2_{dof} under the model. For Gaussian, use $\chi^2 = (y - \mu)^{\top} \Sigma^{-1} (y - \mu)$.

Coverage and calibration. Check interval coverage with pseudo-experiments (including nuisance-generating mechanism). In the presence of boundaries and non-Gaussianities, use Feldman–Cousins ordering or parametric bootstrap for $q(\theta)$.

Robust likelihoods. Outlier-robust alternatives replace Gaussians with Student-t:

$$\mathcal{L}_t \propto \Big(1 + \frac{1}{\nu}(y-\mu)^\top \Sigma^{-1}(y-\mu)\Big)^{-(\nu+m)/2}. \tag{24.174}$$

24.55 Implementation notes and reproducibility

Gradient structure. For Poisson (24.156), the score and observed information are

$$\partial_{\theta_i} \ln \mathcal{L} = \sum_b \frac{y_b - \mu_b}{\mu_b}\, \partial_{\theta_i}\mu_b, \qquad -\partial_{\theta_i}\partial_{\theta_j} \ln \mathcal{L} = \sum_b \frac{y_b}{\mu_b^2}\, \partial_{\theta_i}\mu_b\, \partial_{\theta_j}\mu_b - \sum_b \frac{y_b - \mu_b}{\mu_b}\, \partial_{\theta_i}\partial_{\theta_j}\mu_b. \tag{24.175}$$

Automatic differentiation should be used to obtain stable derivatives through the folding chain (24.155).

Reproducibility checklist. Report: (i) data releases (binning, covariances), (ii) generator/reconstruction versions, (iii) emulator training sets and seeds, (iv) prior choices and feasibility sets $\mathcal{C}$, (v) scan/sampler settings and convergence diagnostics, (vi) Asimov configurations, (vii) goodness-of-fit and PPCs.

24.56 Cheat sheet: formulas at a glance

Joint likelihood: $\mathcal{L} = \prod_k \mathcal{L}_k(y^{(k)}|\theta,\xi)\ \pi(\theta)\pi(\xi)\ \mathbf{1}_{\mathcal{C}}(\theta,\xi);$ (24.176)

Profiled Gaussian: $\Sigma_{\text{eff}}^{-1} = \Sigma^{-1} - \Sigma^{-1}A(A^\top\Sigma^{-1}A + V^{-1})^{-1}A^\top\Sigma^{-1};$ (24.177)

PLR test stat.: $q(\theta_0) = -2\big[\ell_{\text{prof}}(\theta_0) - \ell_{\text{prof}}(\hat\theta)\big] \ \sim\ \chi^2_d$ (Wilks); (24.178)

Asimov/Fisher: $q_A(\theta_0) \approx (\theta^\star - \theta_0)^\top F(\bar\theta)(\theta^\star - \theta_0), \quad F = J^\top\Sigma^{-1}J + \cdots;$ (24.179)

Positivity LMIs: $\mathbb{H}_r\big(W(\theta)\big) \succeq 0 \ \Rightarrow\ \theta \in \mathcal{C};$ (24.180)

BLUE combine: $\hat\mu = \Big(\sum_k \Sigma_k^{-1}\Big)^{-1}\Big(\sum_k \Sigma_k^{-1}\hat\mu_k\Big).$ (24.181)

Summary. This appendix provides a modular stack for inference: experiment-level likelihoods with realistic systematics; analytic profiling for linear-Gaussian blocks; convex feasibility via LMIs encoding the spectral gates; asymptotic testing and Fisher forecasts; and principled model comparison and PPCs. These components enable the global fits and kill-criteria reported in the synthesis chapters of Volume IV.